On Certified Isotopic Approximation of Space Curves

by

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Abstract

The approximation of implicitly defined curves or surfaces is a problem of interest for many fields. As a result, this problem has been explored using algebraic, geometric, and numerical methods. Amongst these, a numerical method called Marching Cubes Algorithm ([4]) has been the primary choice in implementations because of its efficiency and implementability, even though a guarantee for topological correctness was not generally present.

Research in this area has largely focused on approximations of $n-1$ dimensional manifolds in n dimensional Euclidean space. These are called co-dimension 1 manifolds, defined as the zero sets of single equations in n variables. Plantinga and Vegter (2004) [8] derived the first algorithms with guaranteed topological correctness using interval arithmetic and adaptive subdivision for $n = 2, 3$. Faster variants of such algorithms were described by Yap et al. (2009, 2014) [10] [11]. Galehouse (2008) [9] succeeded in producing such algorithms for all n.

This thesis addresses the problem of computing isotopic approximations of co-dimension 2 manifolds, i.e., $n-2$ dimensional manifolds in n dimensional Euclidean space. Such manifolds are the intersection of the zero sets of two equations in n variables. The first interesting case is $n = 3$, i.e., the problem of computing an isotopic approximation of a space curve in 3D. We work on devising new algorithms by extending the previous interval techniques in co-dimension 1. Moreover, we implement and visualize such curves in order to verify their practical efficiency.

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1 Introduction

1.1 Problem Statement

We present work on the problem of constructing an ambient isotopic estimation S' to a space curve S defined by two implicit surfaces given by $f_1 = 0$ and $f_2 = 0$ in a bounding box $B_0 \subseteq \mathbb{R}^3$ where functions $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ are smooth and non-singular in B_0. We use the notation $\boldsymbol{f} = (f_1, f_2)$ - for which the traced curve can be defined as the set of points given by $S = \boldsymbol{f}^{-1}(\mathbf{0}) = \{\boldsymbol{p} \in \mathbb{R}^3, \boldsymbol{f}(\boldsymbol{p}) = \mathbf{0}\} = f_1^{-1}(0) \cap f_2^{-1}(0)$.

An important result of non-singularity in B_0 is the truth of the following statement:

$$\forall \boldsymbol{p} \in B_0.\ (f_1(\boldsymbol{p}) = 0 \implies \nabla f_1(\boldsymbol{p}) \neq 0) \wedge (f_2(\boldsymbol{p}) = 0 \implies \nabla f_2(\boldsymbol{p}) \neq 0)$$

This result, alongside the smoothness condition, plays a central role in our arguments for the guaranteed halting of our algorithm steps, as it is presented in part 3 of this report.

1.1.1 Assumptions

1. We assume that $S = \boldsymbol{f}^{-1}(\mathbf{0})$ is a 1-dimensional curve.
2. We assume that box functions (point convergent interval functions) for $f_1, f_2, \nabla f_1, \nabla f_2$ exist and are denoted by $\square f_1, \square f_2, \square \nabla f_1, \square \nabla f_2$.

3. We assume that the surfaces given by $f_1 = 0$ and $f_2 = 0$ never intersect tangentially in B_0. $(\forall \boldsymbol{p} \in B_0, \forall c \in R.\ f_1(\boldsymbol{p}) = f_2(\boldsymbol{p}) = 0 \implies \nabla f_1(\boldsymbol{p}) \neq c\nabla f_2(\boldsymbol{p}))$

4. We assume that the traced curve and the implicit surfaces given by $f_1 = 0$ and $f_2 = 0$ intersect the bounding box B_0 transversally or not at all.

The softening or removal of our 4th assumption can presumably be achieved by adequately processing the boundary of B_0 in future research. Steps to ensure the prevention of problems of similar origin in the tracing of curves in 2D space have been previously presented by Lin and Yap ([10])

1.2 Relevant Background

Our work on this problem builds upon concepts from differential geometry and interval arithmetic. The following sections (1.2.1-1.2.4) can be seen for the explanation of core constructs and principles utilized by our approach.

1.2.1 Implicit Manifolds and Curves

For arbitrary $m, n \in \mathbb{Z}^+$, an implicit manifold is a manifold defined as the set of points $\boldsymbol{p} \in \mathbb{R}^m$ satisfying the system of equations $\boldsymbol{f}(\boldsymbol{p}) = (f_1(\boldsymbol{p}), f_2(\boldsymbol{p}), ..., f_n(\boldsymbol{p})) = 0^n = \mathbf{0}$ where $f_1, f_2, ..., f_n$ each map $\mathbb{R}^m$ to $\mathbb{R}$. An implicit curve is an implicit manifold with topological dimension 1.

1.2.2 Isotopy

Two manifolds $S, S' \subseteq \mathbb{R}^3$ are **ambient isotopic** if there exists a continuous map

$$\gamma : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$$

such that:

1. For each $t \in [0,1]$, the map $\gamma_t : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\gamma_t(p) = \gamma(t,p)$ is a homeomorphism;
2. γ_0 is the identity function;
3. $\gamma_1(S) = S'$.

Such a map γ is also called an **ambient isotopy** from S to S'. It is an ε-**ambient isotopy** if, in addition, we have $\|\gamma_0(p) - \gamma_1(p)\| \leqslant \varepsilon$ for all $p \in S$. Note that ε-ambient isotopy of S and S' implies that the Hausdorff distance between S and S' is $\leqslant \varepsilon$.

If the continuous map

$$\gamma : [0,1] \times S \to \mathbb{R}^3$$

satisfies (i)-(iii), then we say that γ is an **isotopy** from S to S'; we also say S is **isotopic** to S'. Clearly ambient isotopy implies isotopy, which in turn implies homeomorphism. The difference between ambient isotopy and (plain) isotopy is that the former requires a simultaneous transformation of the complementary space $\mathbb{R}^3 \setminus S$. But Hirsch [[2]] shows that, conversely, an isotopy can be extended to an ambient isotopy in case S is a smooth manifold.

1.2.3 Interval Arithmetic

Interval arithmetic is a tool to derive information about the range of functions over their domain. For this, operations are carried out on intervals rather than single values to achieve bounds on the global maximum and minimum of the function over the evaluated domain.

More formally, interval arithmetic and its methods are described by Ratschek and Ronke [3] as follows:

Let $X \subseteq \mathbb{R}$ be an arbitrary compact interval (i. e. $X = [a, b]$ such that $a, b \in R$ and $a \leq b$) and let $f : X \to \mathbb{R}$ be a **continuous** function. Let us also denote the range of f on its domain as $\bar{f}(X) = \{f(x) | x \in X\}$.

Then, one can define inner ($\Psi(X)$) and outer ($F(X)$) estimations of $\bar{f}(X)$ as estimations satisfying the following criterion:

$$\Psi(X) \subseteq \bar{f}(X) \subseteq F(X)$$

We are particularly interested in functions outputting outer estimations ($F(X)$), which are known to always exist for continuous f as denoted by Moore (1966) [1]. These functions are often referred to as inclusion functions ([3]) and will be referred to as such here.

These inclusion functions are also often directly computable, specifically for rational functions. An outer bound $F(X)$ for $\bar{f}(X)$ (for a continuous rational function $f : X \to \mathbb{R}, X \in I$ where I is the set of compact intervals in R) can be directly calculated by replacing the variables in an arithmetic expression of f with the domains of the respective variables and using interval arithmetic operations defined as follows ([3]):

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b][c, d] = [min(ac, ad, bc, bd), max(ac, ad, bc, bd)]$$

$$[a, b]/[c, d] = [a, b][1/b, 1/c] \text{ if } 0 \notin [c, d]$$

This provides a direct way to evaluate functions over intervals in algorithms. However, the exact defining arithmetic expression used in this evaluation affects the tightness and convergence

of the calculated outer bound ($F(X)$), and thus a good representation should be utilized for the efficient use of interval arithmetic. The standard-centered form and Krawczyk's form are two commonly used expression forms with particularly good convergence rates. The formulations of these forms, as well as the further properties of interval arithmetic operations, can be seen in the work of Ratschek and Ronke [3], but are not explicitly discussed here.

1.2.4 Boxes and Box Functions:

Building upon interval arithmetic but abstracting away the specific evaluated arithmetic expressions of interval functions, the work discussed in this thesis research builds upon the properties of two constructs: **boxes** and **box functions**

Boxes: An n-dimensional box B is defined as a Cartesian product of n compact intervals denoted as $B = I_1 \times I_2 \times ...I_n$. One important point of interest for such a box is the mid-point of the box denoted by $m(B)$, which has the coordinates $(m(I_1), m(I_2), ...m(I_n))$.

The evaluation of a function f with n interval arguments on such a box is then defined with the following equality:

$$f(B) = f(I_1, ..., I_n)$$

Box Functions: A inclusion function $\square f$ is called a box function for f if, in addition to being an inclusion function, it is a point convergent, i.e., for any strictly decreasing sequence $B_0 \subset B_1 \subset ...$ of boxes that converges to a point p, we have $\square f(B_i) \to f(p)$ as $i \to \infty$. ([10])

These are the main mathematical constructs of interest in our current research and allow us to discuss methods utilizing arbitrary implementations of box functions.

1.3 Related Work

Our work builds upon previous research on the certified isotopic approximations of co-dimension 1 manifolds and root isolation algorithms. Amongst the algorithms produced by previous studies, the following algorithms serve as the main foundations of our approach:

1.3.1 The Plantinga-Vegter (PV) Algorithm ([8])

Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ be an implicit function, and $B \subseteq \mathbb{R}^2$ be a square bounding box. Also, let $\square g$ denote a convergent inclusion function for g for all functions g. The 2D PV algorithm defines a procedure that generates a guaranteed topologically correct piecewise linear estimation for $S = F^{-1}(0)$ where S is a regular curve (0 is a regular value of F as the gradient ∇F is non-zero at every point of the curve.).

This procedure defined by the algorithm starts by initializing a quadtree T on this box (initially with only one node), and subdivides the boxes until either a predicate ensuring the discarding of the box ($0 \notin F(C)$) or a stopping condition ($\langle \square \nabla F(C), \square \nabla F(C) \rangle > 0$) is satisfied where C is the two-dimensional interval defining the box. This result is then refined to make the subdivision quadtree balanced (ensuring that boxes that are adjacent are the same size or have a 1/2 ratio in between).

The termination of this process is certain under the assumptions on the inputs (as proven by Long Lin & Chee Yap in [10]). Furthermore, the inner product constraint puts an upper limit of $\pi/2$ radians for the angles between the gradients of F in any terminal box and furthermore implies the local parameterizability of the implicit curve in either the x or y direction.

This local parametarizability implies that S intersects at most two edges of each cell C and

that there cannot be any self-intersections. With this, within any such cell C, if there are two intersection points along the edges, the part of the curve inside C can be seen to be isotopic to a line segment. And thus, an approximation of the implicit curve S can be constructed by linear lines between the centers of box edges with different signs that will be isotopic to S, showing the correctness of the algorithm with the given restrictions. Moreover, even if any given box has more than two intersection points along the edges, it can be shown that the produced result is still globally isotopic to the original curve even though it might not be isotopic to the approximation in each box.

This method is particularly powerful, as it forgoes local isotopy (at each subdivided box) to be able to produce a globally isotopic approximation while being guaranteed to halt.

The 3D PV algorithm builds upon this base, providing a mesh construction method in 3D to give certified isotopic approximations of surfaces in 3D.

1.3.2 Miranda Algorithm ([13])

Given an initial bounding box B_0, the Miranda algorithm provides guaranteed isolation of simple zeros of a function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ for some $n \in \mathbb{Z}^+$ in B_0. For this, an existence test EC (proving the existence of at least one root in a box) and a Jacobian test JC (proving the existence of at most one root in a box) are utilized alongside an exclusion predicate defined as $C_0(B) = \mathbf{0} \notin \boldsymbol{f}(B)$) with the following algorithm:

```
SIMPLE ISOLATE(f, B_0)
    Output: sequence of isolating boxes for roots in B_0
    Q ← {B_0}
    While Q ≠ ∅
        B ← Q.pop()
        If C_0(B) continue;        ◂ discard B and repeat loop
        If EC(B) ∧ JC(B)           ◂ B has a unique root
            output B and continue;
        If w(N(B) ∩ B) < w(B)      ◂ if contraction succeeds
            Q.push(B)
        else
            Q.push(subdivide(B))
```

Figure 1.1: The Miranda algorithm as presented in [13]

This algorithm is guaranteed to produce correct outputs and can be shown to always halt for the proposed EC and JC tests.

The proposed EC test (the MK Test) is of particular interest to us, as it is a predicate proving the existence of a root of f when it holds. This test utilizes a preconditioning phase (in which the range of f is multiplied by the inverse Jacobian matrix of f evaluated at the midpoint of B. Then, the value of $J_f(m(B))^{-1}f_i$ is evaluated at the opposite ends of the evaluation box in the i'th dimension. If all such evaluations show that f_i is negative towards the negative direction in the i'th axis and positive on the opposite side, the test is said to succeed as this proves the existence of at least one root in the box (as an extension of the Poincare-Miranda Theorem, as explained in [13]).

This test is formally presented as follows in the [13] paper:

ABSTRACT MK TEST

Input: f and box B

Output: `true` iff $\mathtt{MK}_f(B)$ succeeds

1. $C \leftarrow J_f(\boldsymbol{m}(B))$, Jacobian matrix at $\boldsymbol{m}(B)$
   If C^{-1} does not exist, return `false`.
2. Construct a "preconditioned version" $\boldsymbol{g}$:
   $\boldsymbol{g} \leftarrow C^{-1} f = (g_1(\boldsymbol{x}), \ldots, g_n(\boldsymbol{x}))$
3. Apply the Simple Miranda Test to $2B$ for $\boldsymbol{g}$:
   For $i \leftarrow 1, \ldots, n$:
   If $g_i(2B_i^+) \leq 0$ or $g_i(2B_i^-) \geq 0$, (*)
   return `false`
4. Return `true`.

Figure 1.2: The MK test, as presented in [13]

A numerical version of this abstract test proven to work under the constraints of finite-precision arithmetic is also provided here, which serves as a foundational tool utilized by our proposed approach.

2 | Preliminaries and Theory

2.1 Box Predicates

We introduce several predicates (also called "conditions") on boxes $B \subseteq \mathbb{R}^3$:

- (Exclusion Condition C_0)

  The condition $C_0(B) = C_0^{\mathbf{f}}(B)$ holds iff $C_0^{f_1}(B) \vee C_0^{f_2}(B)$ where

$$C_0^{f_j}(B) \equiv \left[0 \notin f_j(B)\right] \quad (j = 1, 2). \tag{2.1}$$

  If $C_0^{\mathbf{f}}(B)$ holds, then the curve $\mathbf{f}^{-1}(\mathbf{0})$ does not intersect B (so B can be excluded in our search).

- (Inclusion Condition C_1)

  We write $D_i f$ for the function $\frac{\partial f}{\partial x_i}$ $(i = 1, 2.3)$. The condition $C_1(B) = C_1^{\mathbf{f}}(B)$ holds iff $C_1^{f_1}(B) \wedge C_1^{f_2}(B)$ where

$$C_1^{f_j}(B) \equiv \left[0 \notin \sum_{i=1}^{3} (D_i f_j(B))^2\right] \tag{2.2}$$

  Note that the expression of the right is the natural set extension of the usual arithmetic operations, with $S^2 := \{st : s, t \in S\}$ (not $S^2 = \{s^2 : s \in S\}$). Condition $C_1(B)$ implies that the angle between the gradients at any two points in B is at most 90 degrees; this implies that B does not contain any any closed surface of $f_1^{-1}(0)$ or $f_2^{-1}(0)$. However B may still

contain a closed loop of $f_1^{-1}(0) \cap f_2^{-1}(0) = \mathbf{f}^{-1}(\mathbf{0})$. The next condition will address this.

- (Jacobian Conditions JC_i)

  First let

$$D\mathbf{f}(\mathbf{x}) := \begin{bmatrix} (f_1)_x(\mathbf{x}) & (f_1)_y(\mathbf{x}) & (f_1)_z(\mathbf{x}) \\ (f_2)_x(\mathbf{x}) & (f_2)_y(\mathbf{x}) & (f_2)_z(\mathbf{x}) \end{bmatrix} \tag{2.3}$$

  denote a 2×3 matrix where $(f_1)_i = \frac{\partial f_1}{\partial x_i}$ and similarly for $(f_2)_i$ $(i = x, y, z)$. The Jacobian condition at a point $\mathbf{p} \subseteq \mathbb{R}^3$ is when the matrix $D\mathbf{f}$ is full-rank[1] when evaluated at $\mathbf{x} := \mathbf{p}$.

  Next, we define the corresponding condition for a box B. Let $\Delta_i\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}$ $(i = 1, 2, 3)$ denote the determinant of the 2×2 matrix obtained by deleting the ith column of $D\mathbf{f}(\mathbf{x})$, multiplied by $(-1)^{i+1}$. Thus

$$\Delta_1\mathbf{f} := \det \begin{bmatrix} (f_1)_y(\mathbf{x}) & (f_1)_z(\mathbf{x}) \\ (f_2)_y(\mathbf{x}) & (f_2)_z(\mathbf{x}) \end{bmatrix}, \ \Delta_2\mathbf{f} := -\det \begin{bmatrix} (f_1)_x(\mathbf{x}) & (f_1)_z(\mathbf{x}) \\ (f_2)_x(\mathbf{x}) & (f_2)_z(\mathbf{x}) \end{bmatrix}, \ \Delta_3\mathbf{f} := \det \begin{bmatrix} (f_1)_x(\mathbf{x}) & (f_1)_y(\mathbf{x}) \\ (f_2)_x(\mathbf{x}) & (f_2)_y(\mathbf{x}) \end{bmatrix}. \tag{2.4}$$

  The **Jacobian condition** at B is defined by

$$JC^{\mathbf{f}}(B) \equiv \left[\bigvee_{i=1}^{3} JC_i^{\mathbf{f}}(B) \right] \tag{2.5}$$

  where

$$JC_i^{\mathbf{f}}(B) \equiv \left[0 \notin \Delta_i\mathbf{f}(B) \right] \qquad (i = 1, 2, 3). \tag{2.6}$$

  Thus $JC^{\mathbf{f}}(B)$ implies that $D\mathbf{f}(\mathbf{p})$ is full-rank for each $\mathbf{p} \in B$. But the main geometric conclusion is seen in Lemma 2.3 in the proceeding sections.

[1] For algebraic curves, a birational correspondence between an irreducible algebraic space curve C and an irreducible plane curve $\mathcal{P}$ with the same genus as C depends on a similar full-rank condition (see [5]).

2.2 Jacobian Condition in Planar Curve Intersection

Temporarily, let us define a pair of planar curves $\mathbf{f} = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$. We prove the following lemma (to be used in proceding sections):

Lemma 2.1 (2D Jacobian Condition)**.** *Let $B \subseteq \mathbb{R}^2$ and $JC_3^{\mathbf{f}}(B)$ holds, i.e.,*

$$0 \notin J_{\mathbf{f}}(B) \coloneqq \det \begin{bmatrix} D_x f(B) & D_y f(B) \\ D_x g(B) & D_y g(B) \end{bmatrix}. \tag{2.7}$$

Then $|Zero(f, g) \cap B| \leqslant 1$.

Proof. Let $\mathbf{a}, \mathbf{b} \in B$ be two distinct zeros of $\mathbf{f}$. Define $L : \mathbb{R} \to \mathbb{R}^2$ where $L(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$. Consider the function $F(t) \coloneqq f(L(t))$: by an application of the Chain Rule, we have

$$F'(t) = \frac{dF}{dt} = \nabla f(L(t) * (\mathbf{b} - \mathbf{a}) \tag{2.8}$$

where $*$ denotes dot product. From $F(0) = f(L(0)) = f(\mathbf{a}) = 0$ and $F(1) = f(L(1)) = f(\mathbf{b}) = 0$, the Mean Value Theorem (MVT) for $F(t)$ implies that there exists $\xi \in [0, 1]$ such that

$$0 = F(0) - F(1) = F'(\xi) = \nabla f(L(\xi) * (\mathbf{b} - \mathbf{a}). \tag{2.9}$$

Similarly, if we define $G(t) \coloneqq g(L(t))$, there exists $v \in [0, 1]$ such that

$$0 = G(0) - G(1) = G'(\xi) = \nabla g(L(\xi) * (\mathbf{b} - \mathbf{a}). \tag{2.10}$$

But 2.9 implies that $\nabla f(L(\xi))$ is perpendicular to $\mathbf{b} - \mathbf{a}$. Similarly, $\nabla g(L(v))$ is perpendicular to

$\mathbf{b} - \mathbf{a}$. Thus $\nabla f(L(\xi))$ and $\nabla g(L(v))$ are parallel, i.e.,

$$0 = \det \begin{bmatrix} D_x f(L(\xi)) & D_y f(L(\xi)) \\ D_x g(L(v)) & D_y g(L(v)) \end{bmatrix}.$$

Since $L(\xi), L(v) \in B$, this implies

$$0 \in \det \begin{bmatrix} D_x f(B) & D_y f(B) \\ D_x g(B) & D_y g(B) \end{bmatrix},$$

which contradicts 2.7. □

2.3 Geometric Consequence of the Jacobian Condition

The Jacobian condition on a box $B \subseteq \mathbb{R}^3$ has geometric consequences that can be derived from the Inverse Function Theorem (IFT) of mathematical analysis [7]. Here is a version from [12] which is suitable for our needs:

Suppose we have a system of m functions in $m+n$ variables,

$$\mathbf{f} = (f_1, ..., f_m) : \Omega \subseteq \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \tag{2.11}$$

where Ω is an open set of $\mathbb{R}^m \times \mathbb{R}^n$ and each $f_i = f_i(\mathbf{x}; \mathbf{y})$ is a function in the real variables $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_n)$. Assume $\mathbf{f}$ is C^1, for which we can define:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \left(\frac{\partial f_i}{\partial x_j}\right)_{i=1,j=1}^{m,m} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{y}} = \left(\frac{\partial f_i}{\partial y_k}\right)_{i=1,k=1}^{m,n} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_n} \end{bmatrix}$$

Theorem 2.2 (The Implicit Function Theorem)**.**

Given $\mathbf{f}$ *as in* 2.11*, let* $(\mathbf{a}; \mathbf{b}) \in \Omega \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *satisfy*

$$\mathbf{f}(\mathbf{a}; \mathbf{b}) = \mathbf{0} \text{ and } \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{a}; \mathbf{b}) \text{ is invertible.}$$

Then there exists an open set $X \subseteq \mathbb{R}^m$ *containing* $\mathbf{a}$*, an open set* $Y \subseteq \mathbb{R}^n$ *containing* $\mathbf{b}$*, and an (implicit) function* $\mathbf{g} : Y \to X$ *such that:*

- *For each* $\mathbf{y}^* \in Y$*, there is a unique* $\mathbf{x}^* \in X$ *such that* $\mathbf{f}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{0}$*. In fact,* $\mathbf{x}^* = \mathbf{g}(\mathbf{y}^*)$*. It follows that* $\mathbf{g}(\mathbf{b}) = \mathbf{a}$*.*
- *The function* $\mathbf{g} : Y \longrightarrow X$ *is* C^1 *with Jacobian matrix* $J\mathbf{g}$ *at any* $\mathbf{y}^* \in Y$ *given by*

$$J\mathbf{g}(\mathbf{y}^*) = -\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{g}(\mathbf{y}^*), \mathbf{y}^*)\right]^{-1}_{m\times m} \cdot \left[\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{g}(\mathbf{y}^*), \mathbf{y}^*)\right]_{m\times n}$$

See [12, Theorem 2] for a proof.

An immediate consequence of the IFT theorem is this:

Lemma 2.3 (Implicit Function under Jacobian Condition)**.**
Let the Jacobian condition $JC^{\mathbf{f}}_3(B)$ *hold. If* B *contains a point* $\mathbf{p} = (p_1, p_2, p_3)$ *of the curve* $\mathbf{f}^{-1}(\mathbf{0})$ *then there exists an implicit* C^1 *function*

$$\mathbf{g} : J_{\mathbf{p}} \longrightarrow \mathbb{R}^2 \tag{2.12}$$

where $J_{\mathbf{p}}$ *is an open interval containing* p_3 *satisfying*

(i) $\mathbf{g}(p_3) = (p_1, p_2)$*.*

(i) For all $z \in J_{\mathbf{p}}$*,*

$$\mathbf{f}(\mathbf{g}(z), z) = \mathbf{0}.$$

Proof. The condition $JC^{\mathbf{f}}_3(B)$ implies that $\Delta_3\mathbf{f} = \det \begin{bmatrix} (f_1)_x(\mathbf{x}) & (f_1)_y(\mathbf{x}) \\ (f_2)_x(\mathbf{x}) & (f_2)_y(\mathbf{x}) \end{bmatrix}$ is non-zero at $\mathbf{x} = \mathbf{p}$ (see 2.4). To apply Theorem 2.2 (IFT Theorem), let $m = 1, n = 2$, $\mathbf{f} = (f_1, f_2)$, $\mathbf{x} = (x, y)$ and $\mathbf{y} = (z)$. If $(\mathbf{a}; \mathbf{b}) = (p_1, p_2; p_3) = \mathbf{p}$, then the hypothesis of IFT Theorem is satisfied, namely

$$\mathbf{f}(\mathbf{a}; \mathbf{b}) = \mathbf{0}, \quad \text{and} \quad \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{p}) = \Delta_3\mathbf{f}(\mathbf{p}) \neq 0.$$

We conclude that there exists an implicit function $\mathbf{g} : X \to Y$ where $X \subseteq \mathbb{R}^2$ is an open set containing (p_1, p_2), $Y \subseteq \mathbb{R}$ is an open interval containing p_3, and for all $\mathbf{y} \in Y$, $\mathbf{f}(\mathbf{g}(\mathbf{y}); \mathbf{y}) = \mathbf{0}$. The theorem follows if we rename Y to be $J_{\mathbf{p}}$, and view the range of $\mathbf{g}$ to be $\mathbb{R}^2$. □

In other words, this lemma tells us that the set

$$\{(\mathbf{g}(z), z) : z \in J_{\mathbf{p}}\}$$

(viewed as the graph of the function $\mathbf{g}$) is a parameterization of the curve $\mathbf{f}^{-1}(\mathbf{0})$ in some neighborhood of $\mathbf{p}$. Next, if $\mathbf{g}_i : J_i \to Y_i$ $(i \in I)$ is a collection of such graphs with the property that $J_* := \cup_{i \in I} J_i$ is a connected interval. Then we can define a unique function $\mathbf{g}_* : J^* \to \mathbb{R}^2$ where $\mathbf{g}_*(\mathbf{p}) = \mathbf{g}_i(\mathbf{p})$ for all $\mathbf{p} \in J_*$.

Lemma 2.4 (No Loop under the Jacobian Condition)**.**
If $JC^{\mathbf{f}}(B)$ holds, then B does not contain a closed curve (i.e., loop) of $\mathbf{f}^{-1}(\mathbf{0})$.

Proof. Consider a box B such that $JC^{\mathbf{f}}(B)$ holds. Without loss of generality, assume that $JC^{\mathbf{f}}_3(B)$ holds. Then, Suppose B contains a loop $C \subseteq \mathbf{f}^{-1}(\mathbf{0})$. Pick a point $\mathbf{p} = (p_1, p_2, p_3) \in C$ where p_3 is maximum. Note that such a point exists since C is contained in B. By Lemma 2.3, and $JC^{\mathbf{f}}_3(B)$, there must be an open interval containing p_3 in which C is parameterizable by the third coordinate axis. This contradicts p_3 being the maximum of the third coordinates amongst the points in C when C is non-singular. □

2.4 The Miranda Conditions

So far, we have given three conditions: $C_0(B), C_1(B), JC(B)$. They all amount to the exclusion of $\mathbf{0}$ from various algebraic expressions evaluated on the box B. The next one is slightly different, and may be called "Miranda conditions".

We temporarily consider the general setting of an n-dimensional axes-parallel box $B \subseteq \mathbb{R}^n$: let B_i^- and B_i^+ (for $i = 1, ..., n$) denote the pair of opposite facets[2] that are normal to the ith axis. Moreover, if $\Pi_i(\mathbf{x}) = x_i$ denote the projection to the ith coordinate, then assume that $x_i^- < x_i^+$ where $\Pi_i(B_i^-) = \left\{x_i^-\right\}$ and $\Pi_i(B_i^+) = \left\{x_i^+\right\}$. If $\mathbf{f} = (f_1, ..., f_n) : \mathbb{R}^n \to \mathbb{R}^n$, the following box predicate

$$MT^{\mathbf{f}}(B) \quad \equiv \quad \left[\bigwedge\nolimits_{i=1}^{n} (f_i(B_i^- < 0 < f_i(B_i^+)) \right] \tag{2.13}$$

was called the "simple Miranda test" in [13]. The Miranda Theorem (1940) says that if $MT^{\mathbf{f}}(B)$ holds, then $B \cap \mathbf{f}^{-1}(\mathbf{0})$ is non-empty (e.g., [6]). But in our co-dimension one setting, the number of functions is not n, but $n-1$. If $\mathbf{f} = (f_1, ..., f_{n-1})$, then we define the **Miranda condition** $MK(B) = MK^{\mathbf{f}}(B)$ as

$$MK^{\mathbf{f}}(B) \equiv \left[\bigvee_{i=1}^{n} MK_i^{\mathbf{f}}(B) \right]$$

where $MK_i(B) = MK_i^{\mathbf{f}}(B)$ is given by

$$MK_i^{\mathbf{f}}(B) \equiv \left[\bigwedge_{j=1}^{i-1} (f_j(B_j^-) < 0 < f_j(B_j^+)) \right] \wedge \left[\bigwedge_{j=i+1}^{n} (f_{j-1}(B_j^-) < 0 < f_{j-1}(B_j^+)) \right]. \tag{2.14}$$

Note that $MK_i(B)$ places no restrictions on the pair of faces B_i^- and B_i^+.

[2] I.e., $(n-1)$-dimensional faces of B.

2.5 A Miranda-Type Theorem for Space Curves

For $n = 3$, and $\mathbf{f} = (f_1, f_2)$, $MK_3^{\mathbf{f}}(B)$ is simply

$$MK_3(B) = (f_1(B_1^-) < 0 < f_1(B_1^+)) \wedge (f_2(B_2^-) < 0 < f_2(B_2^+))$$

Now, we present the following theorem:

Theorem 2.5 (Miranda Theorem for Curves)**.**
Let $B \subseteq \mathbb{R}^3$ and the following conditions hold:

$$C_1^{\mathbf{f}}(B) \wedge JC_3^{\mathbf{f}}(B) \wedge MK_3^{\mathbf{f}}(B). \tag{2.15}$$

Then $Zero(f, g) \cap B$ is comprised of a single curve component with endpoints in B_3^- and B_3^+, respectively.

If the given theorem is true, the local curves in a set of boxes B in an initial box B_0 satisfying $C_1^{\mathbf{f}}(B) \wedge JC_3^{\mathbf{f}}(B) \wedge MK_3^{\mathbf{f}}(B)$ can be "stitched together" to curve components in B_0, allowing for an approximation of the traced curve components.

Now, let us consider the following lemma:

Lemma 2.6 (Existence of a curve)**.** *Assume that a box B satisfies $JC_3(B) \wedge MK_3(B)$, then there exists a curve $H : I \to \mathbb{R}^3$ within $\mathbf{f}^{-1}(\mathbf{0})$, which connects the faces B_z^- and B_z^+.*

Proof. Because of the standard 2-dimensional Miranda theorem, we know that there exists a point $\mathbf{p} \in B_z^+$ such that $\mathbf{f}(\mathbf{p}) = \mathbf{0}$. By Lemma 2.2, there exists an open interval $J_{\mathbf{p}}$ containing p_3, and a differentiable function

$$\mathbf{h}_{\mathbf{p}} : J_{\mathbf{p}} \to \mathbb{R}^2 \tag{2.16}$$

such that $\mathbf{h_p}(p_3) = (p_1, p_2)$ and, for all $z \in J_\mathbf{p}$,

$$\mathbf{f}(\mathbf{h_p}(z), z) = \mathbf{0}.$$

Among all open intervals $J_\mathbf{p}$ with that property, let $J = (a, b)$ be one, which minimizes a. Note that the curve $H : J \to \mathbb{R}^3, z \mapsto \mathbf{f}(\mathbf{h_p}(z), z)$ cannot intersect any of B's faces except for B_z^- and B_z^+ since $MK_3(B)$ is satisfied. If $a < \Pi_3(B_z^-)$ then the curve H connects the opposite faces B_z^- and B_z^+ as desired. We will now show that $a \geq \Pi_3(B_z^-)$ leads to a contradiction. We again use the Lemma 2.2 for $\mathbf{q} = (\mathbf{h}(a), a)$ to derive an open interval $J_\mathbf{q}$ together with a differentiable function $\mathbf{h_q}$ similar as before. But this is a contradiction to the maximality of interval J because also the longer interval $J \cup J_\mathbf{q}$ could have been chosen together with the function $\mathbf{h}$:

$$\mathbf{h} : J \cup J_\mathbf{q}, z \mapsto \begin{cases} \mathbf{h_p}(z) & \text{if } z > a \\ \mathbf{h_q}(z) & \text{otherwise} \end{cases} \tag{2.17}$$

Note that also $\mathbf{h}$ is differentiable because both $\mathbf{h_p}$ and $\mathbf{h_q}$ are so and the domains (z-components) and images overlap. □

Building upon this, we now present a proof of the Miranda Theorem for Curves (Theorem 2.5):

Proof. Without loss of generality, consider a box $B = [-1, 1]^3$ satisfying $C_1^\mathbf{f}(B) \wedge JC_3^\mathbf{f}(B) \wedge MK_3^\mathbf{f}(B)$.

For such a box B, the faces B_z^+ and B_z^- lie in the planes $z = 1$ and $z = -1$.

By the standard Miranda theorem applied to pair of functions $(f(x, y, 1), g(x, y, 1))$, we can see that there is a point $\mathbf{p}_1 = (p_1, p_2, 1) \in B_z^+$ such that $\mathbf{f}(\mathbf{p}_1) = \mathbf{0}$. By applying the quantitative IFT, we can construct the graph of a function $g_1 : [1-\mu, 1+\mu] \to \mathbb{R}^2$ such that $\{(g_1(z), z) : z \in [1 - \mu, 1 + \mu]\}$ is a parameterization of $Zero(\mathbf{f})$ around $\mathbf{p}_1$. Then, we can choose a point $\mathbf{p}_2 = (g_1(1-\mu), 1-\mu) \in B$

on the curve. This point is in B and thus we can apply the theorem again to get another function

$$g_2 : [1-\mu, 1+\mu] \to \mathbb{R}^2$$

whose graph is a parameterization of $Zero(\mathbf{f})$ around the point $\mathbf{p}_2$. Continuing in this way, we can construct a curve through the points $\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_k$ (for some $k \geqslant 1$) until the curve reaches the face B_z^-.

Now, let Γ be such a curve.

As $MK_3(B)$ holds in B, it can trivially be seen that the curve Γ cannot be intersecting the pair of facets $B_1^{\pm}$ or the pair of faces $B_2^{\pm}$. $(\forall i \in \{1,2\}.\ \ f_i(B_1^-) < 0 < f_i(B_1^+) \implies 0 \notin f_i(B_1^-) \wedge 0 \notin f_i(B_1^-))$

This implies:

$$\forall p = (p_1, p_2, p_3) \in \Gamma.\ \ p_1, p_2, p_3 \in [-1, 1] \tag{2.18}$$

Moreover, as the curve Γ connects the faces B_z^+ and B_z^-, we have:

$$\forall z \in [-1,1].\ \ \exists x, y \in [-1,1].\ \ (x, y, z) \in \Gamma \tag{2.19}$$

We now claim that $Zero(\mathbf{f}) \cap B$ is contained in Γ, which proves our theorem. Suppose, for the sake of contradiction, that there is some point $\mathbf{p}' = (x', y', z')' \in (Zero(\mathbf{f}) \cap B) \setminus \Gamma$.

For such a $\mathbf{p}' = (x', y', z')$, we have $x', y', z' \in [-1, 1]$ by (2.18)

However, for the same z', we must also have: $\exists x, y \in R.\ \ (x, y, z') \in \Gamma$ by (2.19).
$((x, y) \neq (x', y')$ is known as $\mathbf{p}' \notin \Gamma)$

This implies that, for the box $B' = [[-1, 1], [-1, 1], z']$ we must have:

$$|Zero(f, g) \cap B'| \geqslant 2$$

As we have $\{(x, y, z), (x, y, z')\} \subseteq Zero(f, g) \cap B'$ and $(x, y) \neq (x', y')$

Moreover, as $B' \subseteq B$, we have $JC_3^{\mathbf{f}}(B')$ (as a result of $JC_3^{\mathbf{f}}(B)$).

However, by Lemma 2.1, we know $JC_3^{\mathbf{f}}(B') \implies |Zero(f, g) \cap B'| \leqslant 1$

Thus, we have $JC_3^{\mathbf{f}}(B') \wedge |Zero(f, g) \cap B'| \geqslant 2 \wedge (JC_3^{\mathbf{f}}(B') \implies |Zero(f, g) \cap B'| \leqslant 1)$, which is a contradiction.

Hence, our assumption that there exists $\mathbf{p}' = (x', y', z')' \in (Zero(\mathbf{f}) \cap B) \setminus \Gamma$ must be false, proving our claim that $Zero(\mathbf{f}) \cap B$ is contained in Γ.

This completes the proof of the Miranda Theorem for Curves. □

2.6 Preconditioning

Unfortunately, the Miranda conditions discussed in the preceding sections are not guaranteed to hold as we consider smaller and smaller boxes around an arbitrary non-singular space curve. As discussed in [13], a preconditioning operation is needed to provide such guarantees.

To describe an adequate preconditioning operation, let us denote by e_i for $i \in \{1, 2, 3\}$ the unit vectors $(1, 0, 0)^T$, $(0, 1, 0)^T$, and $(0, 0, 1)^T$ respectively. Moreover, let us denote by δ_{ij} the Kronecker delta, i.e. $\delta_{ij} = 1$ if and only if $i = j$. Otherwise $\delta_{ij} = 0$. We use ∂_i for the partial derivative along the i-th coordinate direction. Denote for a matrix $M \in \mathbb{R}^{m \times n}$ by M_{ij} the i-th row and j-th column entry of M.

Then, the standard preconditioning of the MK uses the following transformation:

$$\forall p \in \mathbb{R}^3 : \quad \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix}_{(p)} = \begin{pmatrix} \nabla f_1^T \\ \nabla f_2^T \\ \nabla f_3^T \end{pmatrix}^{-1}_{(m_B)} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}_{(p)} \tag{2.20}$$

Lemma 2.7 (Derivaties for the standard preconditioning)**.** *The standard preconditioning aligns the function space of $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$ such that*

$$\forall i \in \{1, 2, 3\} : \quad \nabla \tilde{f}_i = e_i \tag{2.21}$$

Moreover every root of $\tilde{\mathbf{f}}$ is also a root of $\mathbf{f}$.

Proof. The second equation holds to linearity of derivatives:

$$\begin{aligned}
\partial_i \tilde{f}_j(m) &= e_j^T \begin{pmatrix} \partial_i \tilde{f}_1 \\ \partial_i \tilde{f}_2 \\ \partial_i \tilde{f}_3 \end{pmatrix}_{(p)} \\
&= \underbrace{e_j^T \begin{pmatrix} \nabla f_1^T \\ \nabla f_2^T \\ \nabla f_3^T \end{pmatrix}^{-1}_{(m_B)}}_{e_i} \cdot \begin{pmatrix} \partial_i f_1 \\ \partial_i f_2 \\ \partial_i f_3 \end{pmatrix}_{(p)} \\
&= \delta_{ij}
\end{aligned}$$

Therefore, $\nabla \tilde{f}_i = e_i$ as claimed.

The transformation matrix is invertible and therefore $\forall p \in \mathbb{R}^3$:

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix}_{(p)} = \begin{pmatrix} \nabla f_1^T \\ \nabla f_2^T \\ \nabla f_3^T \end{pmatrix}^{-1}_{(m_B)} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}_{(p)} = 0 \quad \Leftrightarrow \quad \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}_{(p)} = 0$$

I.e. a root of $\tilde{\mathbf{f}}$ is also a root of $\mathbf{f}$. □

It has been proven that this preconditioning ensures the halting of the *MK* test in a small enough box around roots if $\mathbf{f}$ in [13]. Thus, this preconditioning can be used before the *MK* test in a subdivision algorithm searching for the roots of $\mathbf{f}$ (where boxes far from the roots are eventually discarded by $C_0^{\mathbf{f}}$) to guarantee eventual halting of the search.

Unfortunately the curve $\tilde{f}_1^{-1}(0) \cap \tilde{f}_2^{-1}(0)$ in transformed space does not directly tell us anything about the curve $f_1^{-1}(0) \cap f_2^{-1}(0)$ in primal space. As a result, preconditioned Miranda tests

cannot be directly combined with the Miranda Theorem for Curves (Theorem 2.5) for an isotopic approximation of the traced curve.

On the other hand, the information regarding the roots of $\mathbf{f}$ (given by information regarding the roots of $\tilde{\mathbf{f}}$) is valuable, as it can be seen by its use in our algorithm in the proceeding chapters.

3 Algorithm Design

Here, we present two algorithms which, combined, allow us to approximate space curves. These algorithms build upon the tests and theoretical results provided in Chapter 2.

3.1 Curve-Isolating Subdivision Algorithm

Let us first define a subdivision of a box B_0 as a set $\bar{\mathbf{B}}$ of disjoint boxes for which $\bigcup_{B \in \bar{\mathbf{B}}} B = B_0$ holds. Let us consider the elements of such a $\bar{\mathbf{B}}$ connected if and only if the boxes share a face.

Let us also define the concept of an octree subdivision of a box B_0 as a subdivision of B_0 achievable by starting with the set $\bar{\mathbf{B}} = \{B_0\}$ and splitting an arbitrary element of $\bar{\mathbf{B}}$ to eight boxes sharing the same ratios of dimensions as the original box iteratively for an arbitrary number of iterations.

Moreover, for an arbitrary box $B \subseteq \mathbb{R}^3$, let $Faces(B)$ be the set of all 2D faces of B.

Now, for arbitrary box or box face B and real number $c \geq 1$, define $B.scale(c)$ as the box/box face sharing the same mid-point and proportions as B but having all interval widths multiplied by c.

For $\mathbf{f} = (f_1, f_2)$ implicitly defining a space curve and satisfying the presented assumptions on an initial bounded box $B_0 \subseteq \square\mathbb{R}^3$:

For all $face \in Faces(B)$ for some $B \subseteq \square\mathbb{R}^3$, consider the standard preconditioning defined as follows:

$$\tilde{\mathbf{f}} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}_{(p)} = \begin{pmatrix} \nabla f_1^T \\ \nabla f_2^T \end{pmatrix}^{-1}_{(m_{face})} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_{(p)}$$

Then, consider the preconditioned test: $F\_PMK_c^{\mathbf{f}}(face) \equiv MT^{\tilde{\mathbf{f}}}(face.scale(c))$

(Where MT is the simple miranda test defined as in (2.13))

Such a test can be equivalently written as: $F\_PMK_c^{\mathbf{f}}(face) \equiv$

$$\left[(\tilde{f}_1((face_1^-).scale(c)) < 0 < \tilde{f}_1((face_1^+).scale(c))) \wedge (\tilde{f}_2((face_2^-).scale(c)) < 0 < \tilde{f}_2((face_2^+).scale(c)))\right] \tag{3.1}$$

Then, define preconditioned MK test for boxes defined as follows:

$$PMK_c^{\mathbf{f}}(B) \equiv \left[\bigvee_{face \in faces(B)} F\_PMK_c^{\mathbf{f}}(face.scale(c))\right] \tag{3.2}$$

Using the test $PMK_c^{\mathbf{f}}(B)$, alongside C_0, C_1, and JC defined in previous sections, we can define the following algorithm:

Algorithm 1 $Subdivide^{\mathbf{f}}(B_0, c, \epsilon)$

Input: $B_0 \subseteq \mathbb{R}^3, c \geq 1, \epsilon \in \mathbb{R}^+ \cup \{+\infty\}$

Output: A subset $\mathbf{B}$ of an octree subdivision of B_0

Ensure: $\forall B \in \mathbf{B}.\ (\neg C_0^{\mathbf{f}}(B)) \wedge (size(B) \leq \epsilon) \wedge C_1^{f_1}(B) \wedge C_1^{f_2}(B) \wedge JC^{\mathbf{f}}(B.scale(4)) \wedge PMK_c^{\mathbf{f}}(B)$

$\mathbf{B} \leftarrow \emptyset$

$Q \leftarrow \{B_0\}$

while $Q \neq \emptyset$ **do**

  $B \leftarrow Q.pop()$

  if $\neg C_0^{\mathbf{f}}(B)$ **then** ▷ If B not excluded

    if $(size(B) \leq \epsilon) \wedge C_1^{f_1}(B) \wedge C_1^{f_2}(B) \wedge JC^{\mathbf{f}}(B.scale(4)) \wedge PMK_c^{\mathbf{f}}(B)$ **then**

      $\mathbf{B}.push(B)$ ▷ Subdivision halted for box B

    else

      $Q.push(B.split())$ ▷ Further subdivision required

    end if

  end if

end while

Where Boolean expressions are evaluated using short-circuit evaluation. This ensures that each test in $(size(B) \leq \epsilon) \wedge C_1^{f_1}(B) \wedge C_1^{f_2}(B) \wedge JC^{\mathbf{f}}(B.scale(4)) \wedge PMK_c^{\mathbf{f}}(B)$ is only carried out if the preceding tests are true.

Note that as all $B \in \mathbf{B}$ satisfies $PMK_c^{\mathbf{f}}(B)$, we can see that there exists $face \in Faces(B.scale(c))$ such that the 2D MK test on $face.scale(c)$ succeeds.

By the Miranda theorem ([6]), this implies:

$$\forall B \in \mathbf{B}.\ \exists face \in Faces(B.scale(c)).\ \exists \mathbf{p} \in face.scale(c).\ \mathbf{f}(\mathbf{p}) = 0$$

which means that there is a point $\mathbf{p}$ close to the volume of box B which lies on the traced curve $S = B \cap \mathbf{f}^{-1}(\mathbf{0})$. This closeness is determined by c passed to the algorithm.

We also know that for the traced curve cannot be passing through any volume not covered by a $B \in \mathbf{B}$ as all boxes in such volumes must have been discarded by the C_0 (exclusion) predicate.

Moreover, as $B.scale(4)$ for all boxes $B \in \mathbf{B}$, we know by Lemma 2.4 that there are no closed loops of the traced curve in $B.scale(4)$ for all $B \in \mathbf{B}$.

Unfortunately, however, we have no proof of further properties regarding the traced curve's behavior in the outputted set of boxes $\mathbf{B}$ as the results of the Miranda Theorem for Curves (Theorem 2.5) do not directly imply any guarantees when a preconditioning step is used. (As explained in Section 2.6)

However, experimentally, $Subdivide^{\mathbf{f}}(B_0, c, \epsilon)$ has been observed to output a set of boxes $\mathbf{B}$ whose connected components $K_1, ..., K_m$ each cover exactly one of the connected components of the traced curve in B_0. (i. e. connected components of $\mathbf{B}$ isolate the connected components of the traced curve in B_0)

3.2 Reconstruction Algorithm

Let Q_{curve} be a set of boxes, outputted by $Subdivide^{\mathbf{f}}(B_0, c, \epsilon)$ for some $B_0 \subseteq \mathbb{R}^3$, $c > 1$, $\epsilon > 0$. Then, we present the following algorithm to find approximations for the connected components of the traced curve in B_0:

Algorithm 2 $Reconstruct(Q_{\text{curve}})$
Input: $Q_{\mathbf{curve}} \subseteq \square\mathbb{R}^3$
Output: An ϵ-approximation of the curve(s) in $\mathbf{f} = \mathbf{0}$

Create a directed graph G that contains:
    A vertex for each box in Q_{curve}
    A directed edge from B_1 to $B_2 \Leftrightarrow$ both boxes share a piece of an edge and the direction of the edge conforms with $JC(B_1)$ and $JC(B_2)$
Split the graph G into simple connected components $K_1, K_2, \ldots$

for each component K_i **do**
    if $\exists$ directed cycle in K_i with $length > 2$ **then**
        Output *short(est)* directed cycle in K_i with $length > 2$
    else
        Output a path realizing the graph diameter of the undirected version of K_i
    end if
end for

where the graph diameter for the undirected version K_i is determined with the following algorithm:

Algorithm 3 *Find_Diameter*(G)
Input: undirected graph $G = (V, E)$
Output: A path $v, ..., w$ realizing the diameter of G

1: Let $u \in V$
2: Use BFS to find a vertex v with maximum distance from u
3: Use BFS to find a vertex w with maximum distance from v
4: Return a shortest path from v to w

In the *Reconstruct* algorithm, A directed edge from B_1 to B_2 is considered to conform with $JC(B_1)$ and $JC(B_2)$ if and only if the vector pointing from the center of B_1 to the center of B_2 does not contradict the parameterization implied by the $JC(B_1)$ and $JC(B_2)$ conditions.

With this, the *Reconstruct* algorithm works by constructing directed graphs for each connected component K_i of the input set Q_{curve} where the direction of each edge represents a potential way the traced curve passes between the boxes in the connected component.

If such a graph for a connected component includes cycles with length greater than two, the *Reconstruct* algorithm returns the shortest of such cycles. While we currently do not posses theoretical guarantees for such an approximation, we have experimentally observed that such cycles have only been detected in the created graphs when the traced curve piece was a closed loop inside the related connected component (for which the smallest of such cycles presented a simple approximation).

If the graph for a connected component does not include any cycles with length greater than two, then we know that the traced curve must not have a cycle of length greater than

two in the connected component as the predicates used are conservative (and thus only ever over-approximate the possibilities for the real behavior of the curve). For such cases, the approximation of the traced curve piece by a path realizing the graph diameter has been experimentally a good approximation.

Here, it must be noted that the limit of two for cycle length is necessary as cycles of length two occur sporadically in the created graphs due to the way directed edges are created. While this has not caused the erroneous approximation of any traced curve components in our experiments, further work is needed in either providing guarantees or improving this method.

4 Implementation and Experiments

4.1 Implementation

The algorithms described in Chapter 3 have been implemented in Matlab alongside the needed data structures, predicates, and interval arithmetic.

In this process, the *Subdivide* algorithm has been implemented with the *parfor* function of Matlab for parallelizing the subdivision of boxes at each depth as follows:

```
...

%Depth limit for phase 1
depthlimit = 8;
%Depth limit for phase 2
numiterMKlimit = 6;

...

%% Subdivision Phase 1
Q = B0; %Input of the first phase of subdivision
QJac = []; %Output of first phase of subdivision

% Depth for phase 1
depth = 0;

% Create a subdivision of boxes, which all satisfy the predicates untill C1 tests
% and the Jacobian tests hold (where boxes satisfying C0 get excluded at each level)
while ~isempty(Q) && depth <= depthlimit
    Q_next = cell(length(Q),1);
    QJac_add = cell(length(Q),1);

    disp(['Phase 1: depth = ', num2str(depth), ' | length(Q) = ', num2str(length(Q))]);

    %Parallel Subdivision
    parfor i = 1:length(Q)
        B_par = Q(i);
        if ~local_predicate.C0(B_par,f,1) && ~local_predicate.C0(B_par,g,2)
            if B_par.radius<MAXEPS && local_predicate.C1(B_par,df,3) &&
↪  local_predicate.C1(B_par,dg,4) && ... %&& local_predicate.C1cross(B_par,df,dg,5)
```

```
                local_predicate.Jacobian(B_par,df,dg,5)
                QJac_add{i} = B_par;
            else
                children = B_par.split;
                Q_next{i} = children;
            end
        end
    end

    accepted = [QJac_add{:}];

    disp(['# accepted boxes = ', num2str(length(accepted))]);

    %Collection of results
    Q = [Q_next{:}];
    QJac = [QJac, QJac_add{:}];

    depth = depth+1;
end

if ~isempty(Q) && depth == depthlimit + 1
    disp("Phase 1 stopped due to depth limit");
end

disp(['Time for Phase 1: ',num2str(toc(tStart)),'s']);

disp("Phase 1 finalized.");
disp("Proceeding to phase 2...");

tPhase2 = tic;

%% Subdivision Phase 2
QMK = QJac; %Input of the second phase of subdivision
Qcurve = []; %Output of second phase of subdivision
%%
%Depth for phase 2
numiterMK = 1;

while ~isempty(QMK) && numiterMK <= numiterMKlimit
    %Iteration cell arrays
    QMK_next = cell(length(QMK),1);
    Qcurve_add = cell(length(QMK),1);

    disp(['Phase 2: numiterMK = ', num2str(numiterMK), ' | length(QMK) = ',
↪   num2str(length(QMK))]);

    %Parallel Subdivision
    parfor i = 1:length(QMK)
        B_par = QMK(i);
        if ~local_predicate.C0(B_par,f,1) && ~local_predicate.C0(B_par,g,2)
            if local_predicate.Jacobian(B_par,df,dg,5) &&
↪   local_predicate.MK_face(B_par,f,df,g,dg,7) %local_predicate.MK_face(B_par,f,df,g,dg,7)
                if any(B_par.testresults{7})
                    Qcurve_add{i} = B_par;
                end %Else: MK test has succeeded not in finding a root, but in excluding all in
↪   internal call to C0_faces
            else
                children = B_par.split;
                QMK_next{i} = children;
            end
        end
```

```
88      end
89
90      accepted = [Qcurve_add{:}];
91
92      disp(['# accepted boxes = ', num2str(length(accepted))]);
93
94      %Collection of results
95      QMK = [QMK_next{:}];
96      Qcurve = [Qcurve, Qcurve_add{:}];
97
98      numiterMK = numiterMK+1;
99  end
```

As it can be seen, the subdivide algorithm has been split into two steps here to better understand the depth of subdivision required for all boxes to satisfy the *JC* condition and the additional depth of subdivision required to satisfy the *MK* test.

Then, the output of this subdivision step *Qcurve* is used to create the required graph structures, and is used to find approximations for the traced curve pieces. (Appendix A.1 can be seen for the whole main code including these steps for Algorithm 2.)

The code presented in Appendix A can be seen for further details regarding these steps, as well as the implementations of the related functions and the main data structures.

4.2 Experiments and Results

In the following pages, the results for the experimental approximations of several space curves (each implicitly defined by two functions) can be seen.

When discussing the result, the first subdivision loop will be referred as Phase 1 and the second subdivision loop will be referred as Phase 2.

4.2.1 Experiment 1

Approximation of an implicit space curve defined by $\mathbf{f} = (f_1, f_2)$ given by:

$$f_1(x, y, z) = x^2 + y^2 - z$$
$$f_2(x, y, z) = x^2 + y^2 + z^2 - 1$$

where inputs for $Subdivide^{\mathbf{f}}(B_0, c, \epsilon)$ are as follows

$$B_0 = [[-1, 1], [-1, 1], [-1, 1]]$$
$$c = 1.3$$
$$\epsilon = \infty$$

presents the following behavior:

Table 4.1: Experiment 1 - Phase 1

Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	1	0
1	8	0
2	64	0
3	224	44
4	128	4
Time: 0.84525s		

Table 4.2: Experiment 1 - Phase 2

Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	48	36
1	96	0
Time: 1.1794s		

The outputted curve approximation can then be seen drawn on top of the boxes used in the approximation in the following figures:

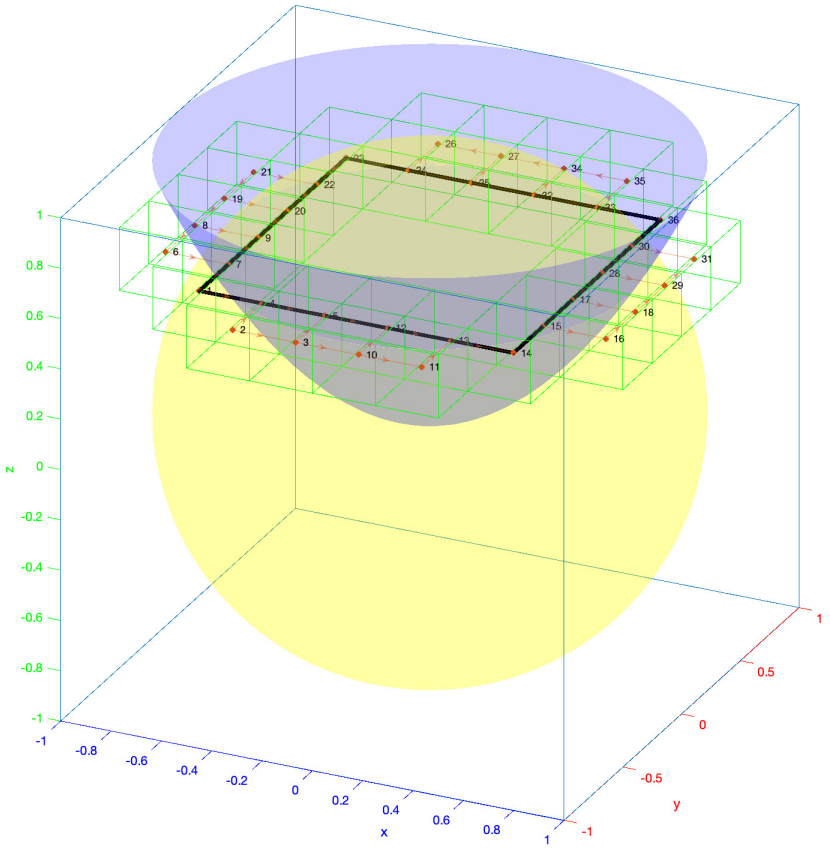
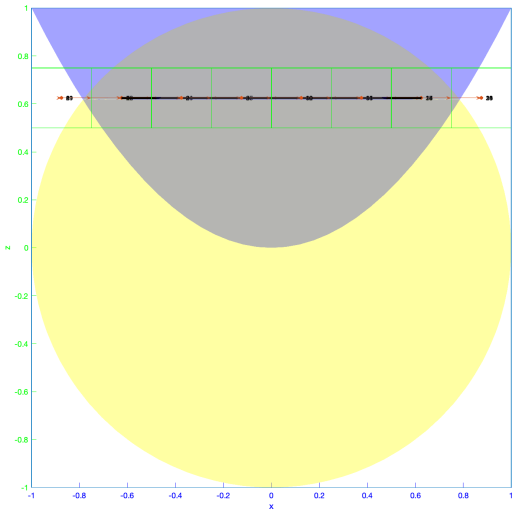
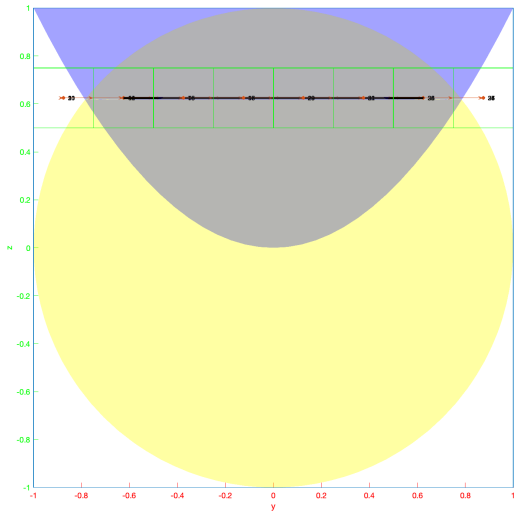


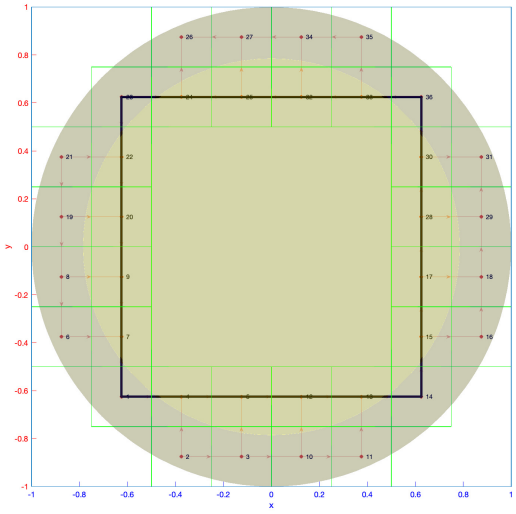
Figure 4.1: The output for experiment 1 seen in 3D



(a) Front view



(b) Side view



(c) Top view

Figure 4.2: The output for experiment 1 viewed from the front, the side, and top

4.2.2 Experiment 2

Approximation of an implicit space curve defined by $\mathbf{f} = (f_1, f_2)$ given by:

$$f_1(x, y, z) = x^2 + y^2 - z$$
$$f_2(x, y, z) = x^2 + y^2 + z^2 - 1$$

where inputs for $Subdivide^{\mathbf{f}}(B_0, c, \epsilon)$ are as follows:

$$B_0 = [[-1, 1], [-1, 1], [-1, 1]]$$
$$c = 1.3$$
$$\epsilon = 0.05$$

presents the following behavior and results:

Table 4.3: Experiment 2 - Phase 1

Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	1	0
1	8	0
2	64	0
3	224	0
4	480	0
5	928	0
6	1664	424
Time: 3.5449s		

Table 4.4: Experiment 2 - Phase 2

Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	424	408
1	128	0
Time: 13.8679s		

The outputted curve approximation can then be seen drawn on top of the boxes used in the approximation in the following figures:

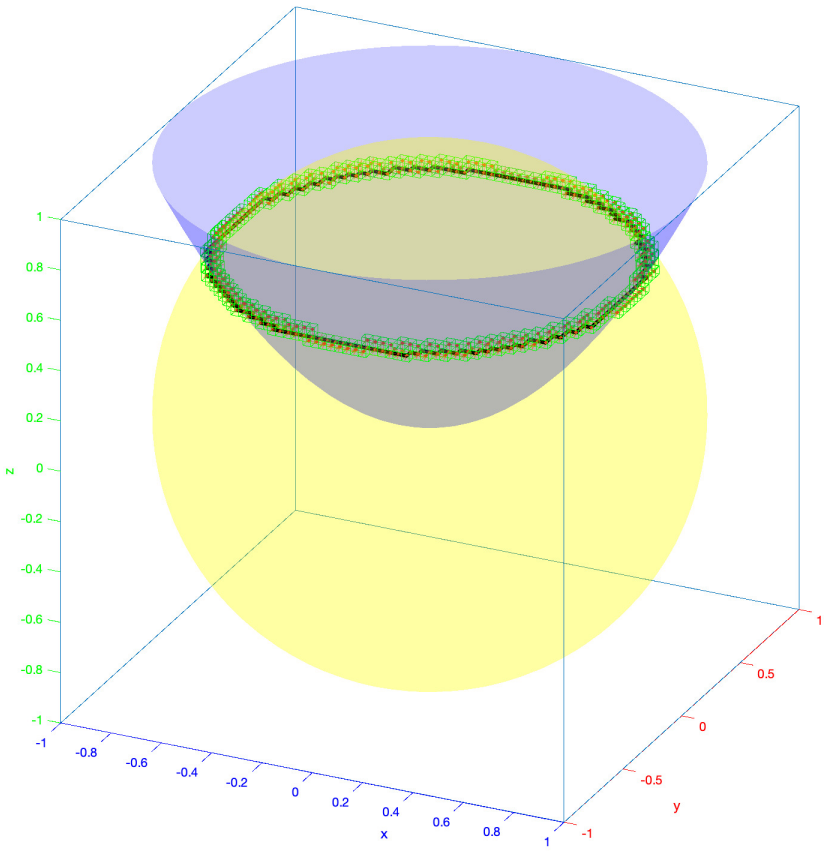
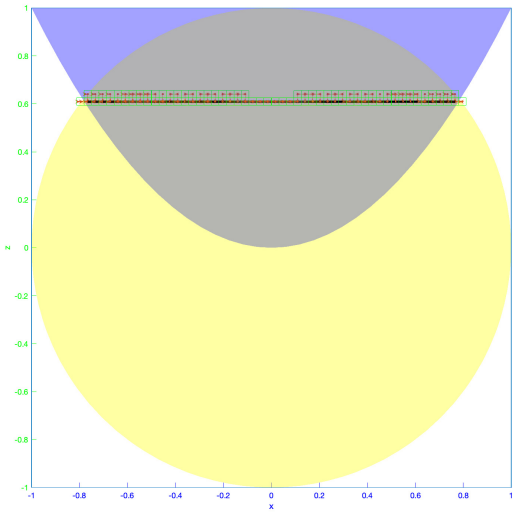
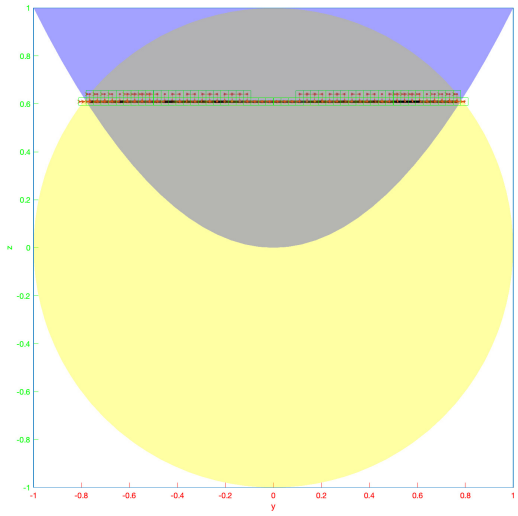


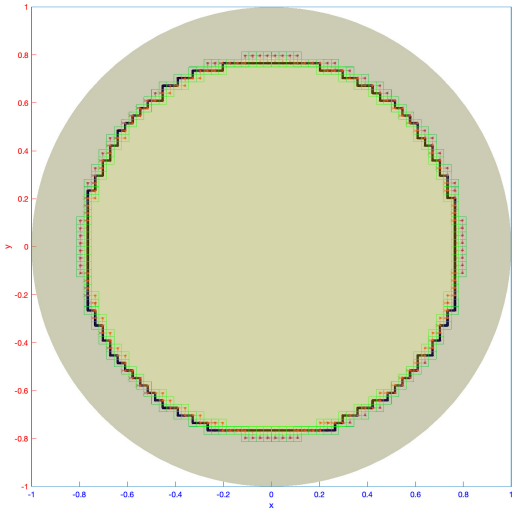
Figure 4.3: The output for experiment 2 seen in 3D



(a) Front view



(b) Side view



(c) Top view

Figure 4.4: The output for experiment 2 viewed from the front, the side, and top

4.2.3 Experiment 3

Approximation of an implicit space curve defined by $\mathbf{f} = (f_1, f_2)$ given by:

$$f_1(x, y, z) = x^4 + 2x^2y^2 + y^4 - 2(x^2 + y^2) + 1 - z$$

$$f_2(x, y, z) = 0.5 - z$$

where inputs for $Subdivide^{\mathbf{f}}(B_0, c, \epsilon)$ are as follows:

$$B_0 = [[-1.2, 1.2], [-1.2, 1.2], [-1.2, 1.2]]$$

$$c = 1.3$$

$$\epsilon = \infty$$

presents the following behavior and results:

Table 4.5: Experiment 3 - Phase 1

Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	1	0
1	8	0
2	32	0
3	128	0
4	480	0
5	1920	108
6	3456	660
7	96	0
Time: 10.9647s		

Table 4.6: Experiment 3 - Phase 2

Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	768	364
1	1824	0
2	736	0
Time: 31.1397s		

The outputted curve approximation can then be seen drawn on top of the boxes used in the approximation in the following figures:

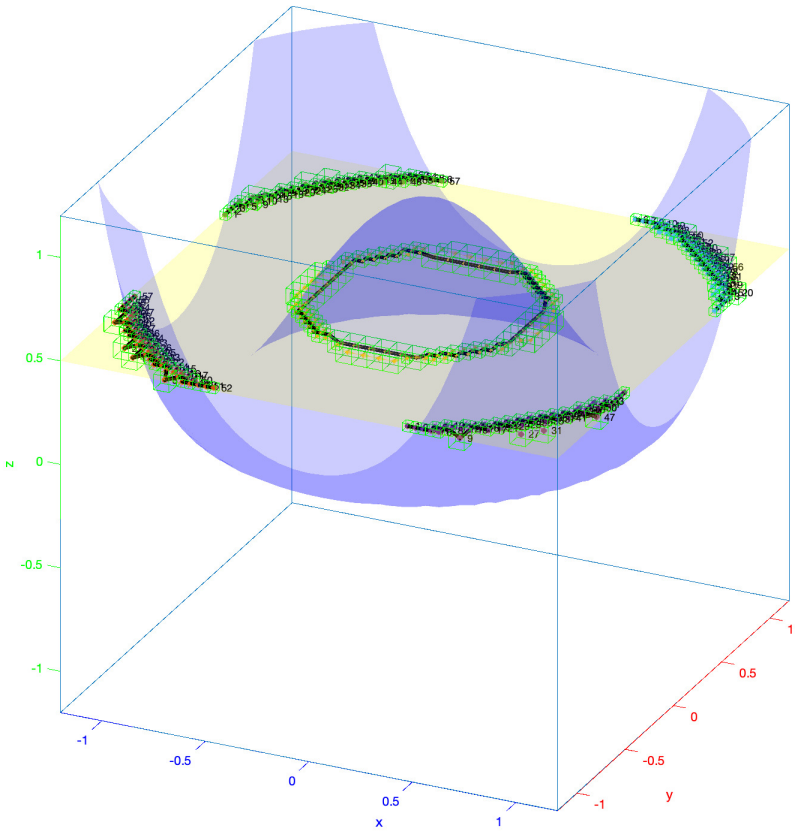
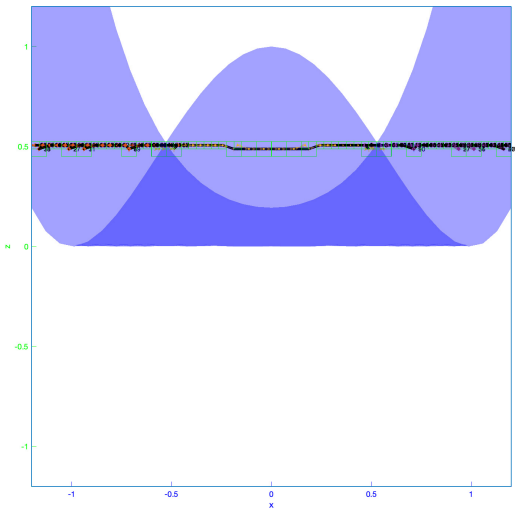
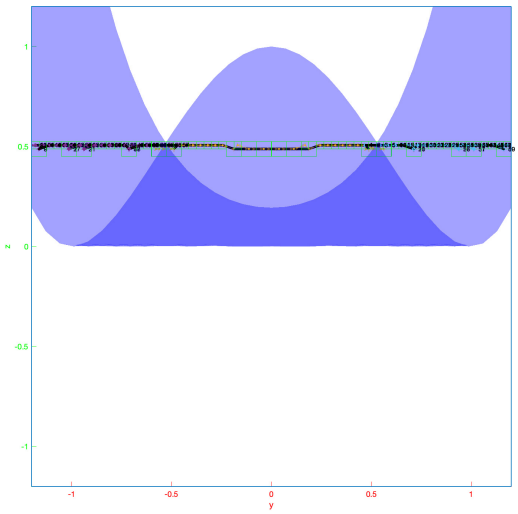


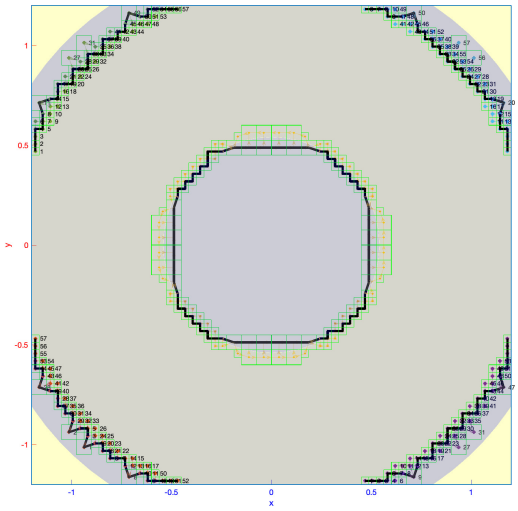
Figure 4.5: The output for experiment 3 seen in 3D



(a) Front view



(b) Side view



(c) Top view

Figure 4.6: The output for experiment 3 viewed from the front, the side, and top

4.2.4 Experiment 4

Approximation of an implicit space curve defined by $\mathbf{f} = (f_1, f_2)$ given by:

$$f_1(x, y, z) = x^2 + y^2 - z^2 - 2$$
$$f_2(x, y, z) = x^2 - y^2 + z^2 - 1$$

where inputs for $Subdivide^{\mathbf{f}}(B_0, c, \epsilon)$ are as follows:

$$B_0 = [[-3, 3], [-3, 3], [-3, 3]]$$
$$c = 1.3$$
$$\epsilon = \infty$$

presents the following behavior and results:

Table 4.7: Experiment 4 - Phase 1

Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	1	0
1	8	0
2	64	0
3	384	16
4	896	200
5	384	40
Time: 3.1143s		

Table 4.8: Experiment 4 - Phase 2

Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	256	224
1	256	0
2	256	0
Time: 23.6716s		

The outputted curve approximation can then be seen drawn on top of the boxes used in the approximation in the following figures:

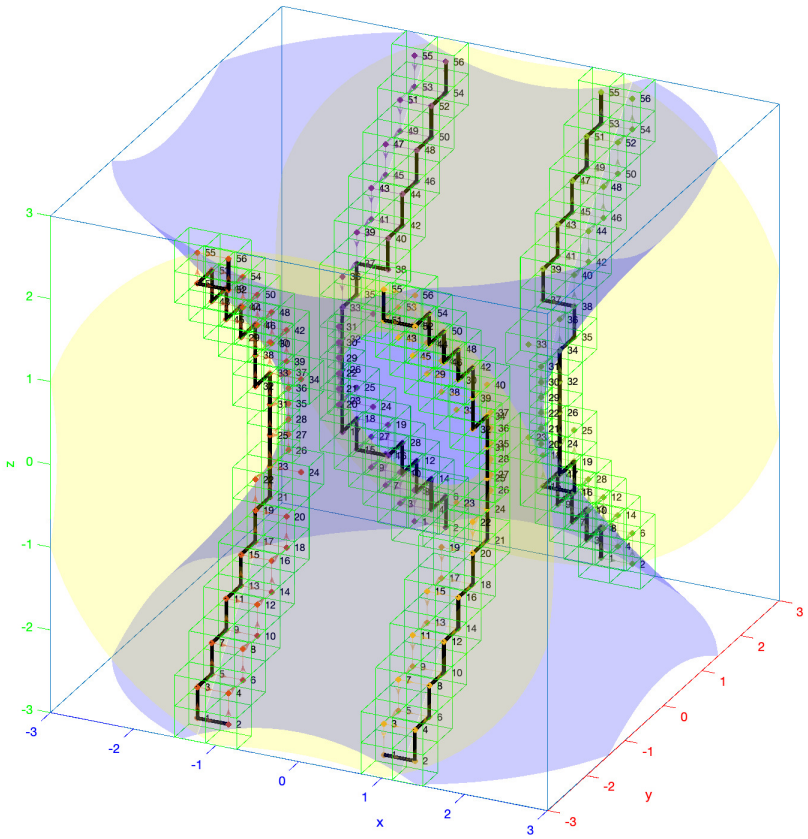
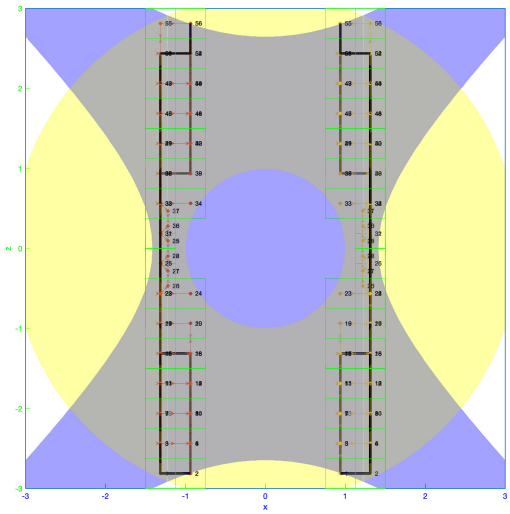
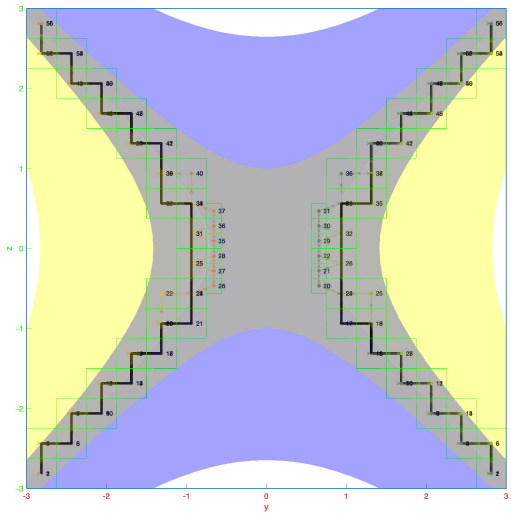


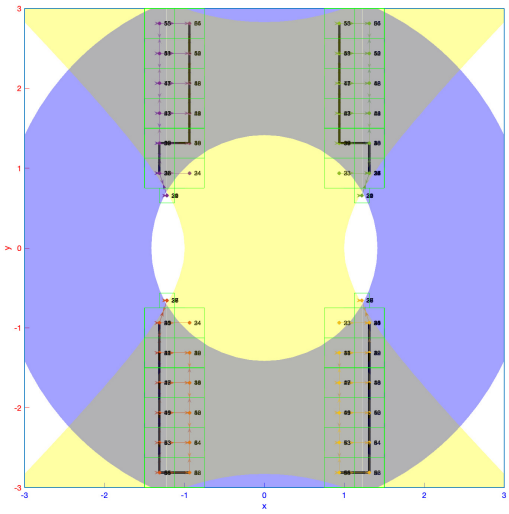
Figure 4.7: The output for experiment 4 seen in 3D



(a) Front view



(b) Side view



(c) Top view

Figure 4.8: The output for experiment 4 viewed from the front, the side, and top

4.2.5 Experiment 5

For this experiment, we look at the approximation of an implicit space curve defined by $\mathbf{f} = (f_1, f_2)$ given by:

$$f_1(x, y, z) = \|\langle qp1, p1 - [x; y; z]\rangle\|/\|qp1\| - \|\langle qp2, p2 - [x; y; z]\rangle\|/\|qp2\|$$
$$f_2(x, y, z) = \|\langle qp1, p1 - [x; y; z]\rangle\|/\|qp1\| - \|\langle qp3, p3 - [x; y; z]\rangle\|/\|qp3\|$$

where $p1, p2, p3$ represent coordinates of three points and $qp1, qp2, qp3$ represent 3D vectors, given as follows:

$$p1 = [0; 0; 4]$$
$$p2 = [0; 0; -4]$$
$$p3 = [1; 1; 0]$$
$$qp1 = [4; 3; 0]$$
$$qp2 = [4; -3; 0]$$
$$qp3 = [4; 1; 4]$$

With this, the given f_1 function implicitly defines the surface of equal distance to the line defined by point $p1$ and vector $qp1$ and the line defined by point $p2$ and vector $qp2$. Similarly, f_2 implicitly defines the surface of equal distance to the line defined by point $p1$ and vector $qp1$ and the line defined by point $p3$ and vector $qp3$.

Thus, the 1D manifold implicitly defined by f_1 and f_2 is the Voronoi diagram of the lines defined by the given points and the corresponding vectors.

For the experiment, inputs for $Subdivide^{\mathrm{f}}(B_0, c, \epsilon)$ are given as follows:

$$B_0 = [[-10, 10], [-10, 10], [-10, 10]]$$

$$c = 1.3$$

$$\epsilon = \infty$$

and we observe the following behavior and results:

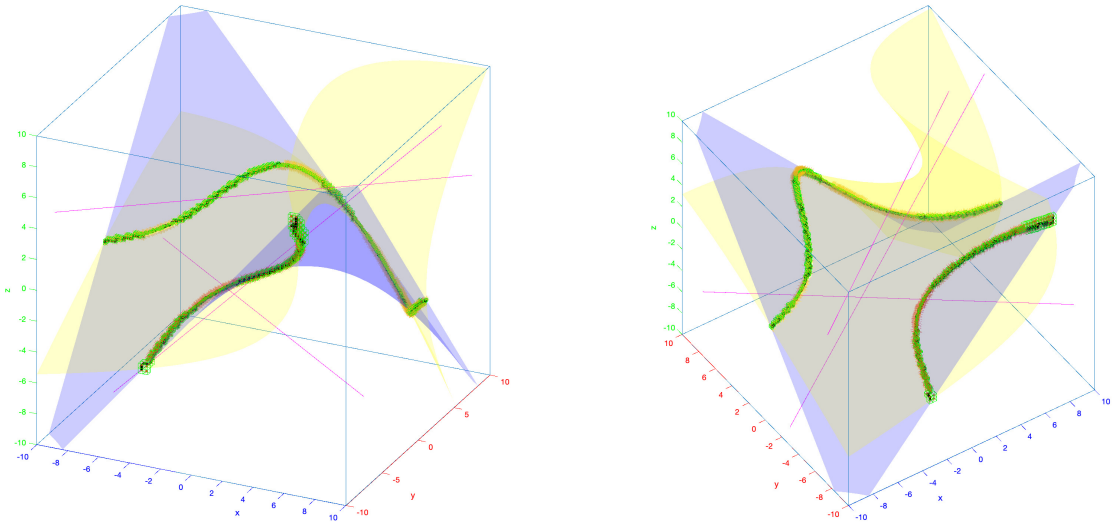
Table 4.9: Experiment 5 - Phase 1

Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	1	0
1	8	0
2	64	0
3	368	16
4	1368	200
5	4504	40
6	10424	186
7	20376	2945
8	19056	4949
Time: 305.3464s		

Table 4.10: Experiment 5 - Phase 2

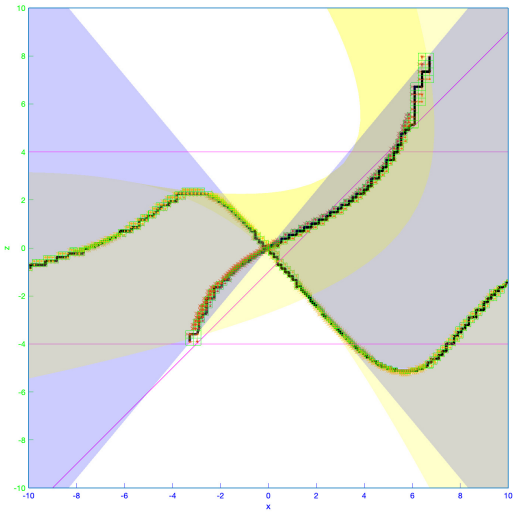
Depth	Number of Evaluated Boxes	Number of Accepted Boxes
0	8080	1551
1	44312	31
2	59488	0
3	45816	0
4	11504	0
5	296	0
Time: 4451.3464s		

The outputted curve approximation can then be seen drawn on top of the boxes used in the approximation in the following figures:

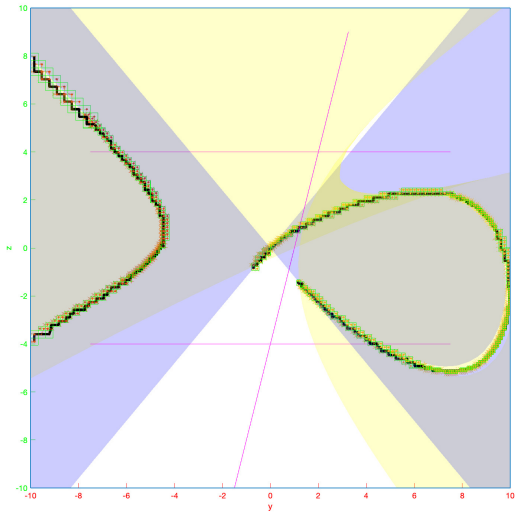


(a) View 1 **(b)** View 2

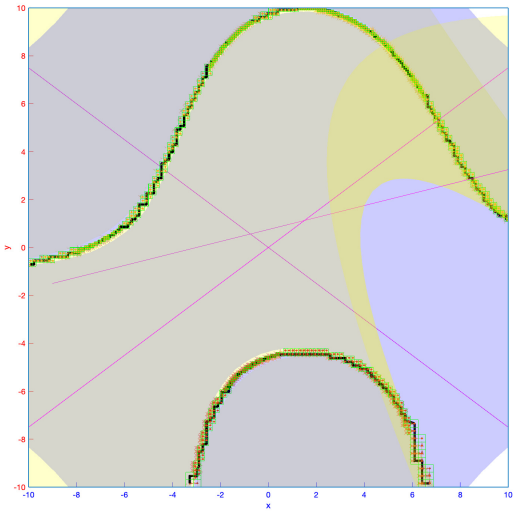
Figure 4.9: The output for experiment 5 seen in 3D, from two different angles



(a) Front view



(b) Side view



(c) Top view

Figure 4.10: The output for experiment 5 viewed from the front, the side, and top

5 Conclusion

The isotopic approximation of implicitly-defined co-dimension 2 manifolds in n dimensional Euclidean space is an open problem in computational geometry. Building upon previous research on the approximation of co-dimension 1 manifolds, we have approached this problem using interval arithmetic and adaptive subdivision.

We have discussed predicates over interval boxes informative of the behavior of implicit space curves and presented proofs for guarantees provided by the combinations of such predicates.

We further discussed a preconditioning operation, which is currently used to guarantee the eventual detection of non-singular roots with the MK test for boxes containing the roots ([13]) at the cost of the apparent loss of some guarantees provided by our predicates.

Building upon this theory, we designed and implemented a new algorithm for the approximation of space curves and demonstrated the experimental results observed with this algorithm.

While the presented algorithm has not been shown to carry all the theoretical guarantees required for a guarantee for providing isotopic approximations, it has been demonstrated to provide accurate experimental results for the experiments studied in this thesis.

A Appendix

A.1 Main code for the implemented algorithms

```
%Main code for parallelized subdivision

%% Initialization
tic;
syms f_syms g_syms d_syms x y z
assume(x,'real');
assume(y,'real');
assume(z,'real');

% %Example 1
% f_syms(x,y,z) = x.^2+y.^2-z;
% g_syms(x,y,z) = x.^2+y.^2+z.^2-1;
% B0 = freecurvebox([-ones(3,1),ones(3,1)]);
% MAXEPS = inf;

% %Example 2 - Same as Example 1, but with a limit imposed on box sizes by MAXEPS
% f_syms(x,y,z) = x.^2+y.^2-z;
% g_syms(x,y,z) = x.^2+y.^2+z.^2-1;
% B0 = freecurvebox([-ones(3,1),ones(3,1)]);
% MAXEPS = 0.05;

% %Example 3
% f_syms(x,y,z) = x.^4+2*x.^2*y.^2+y.^4 -2*(x.^2+y.^2) +1-z;
% g_syms(x,y,z) = 0.5-z;
% % B0 = freecurvebox([zeros(3,1),ones(3,1)]);
% B0 = freecurvebox([-ones(3,1),ones(3,1)]).scale(1.2);
% % B0 = freecurvebox([-1,-0.8;-1,-0.8;0.4,0.6]);
% % B0 = freecurvebox([-ones(2,1),ones(2,1);[0.1,2.1]]);
% MAXEPS = inf;

% % %Example 4
% f_syms(x,y,z) = x.^2+y.^2-z.^2 - 2;
% g_syms(x,y,z) = x.^2-y.^2+z.^2 - 1;
% % B0 = freecurvebox([zeros(3,1),ones(3,1)]);
% B0 = freecurvebox([-ones(3,1),ones(3,1)]).scale(3);
% % B0 = freecurvebox([-1,-0.8;-1,-0.8;0.4,0.6]);
% % B0 = freecurvebox([-ones(2,1),ones(2,1);[0.1,2.1]]);
% MAXEPS = inf;

```

```
%Example 5: trisector
% https://mathworld.wolfram.com/Point-LineDistance3-Dimensional.html
%implement the sqrt function for intervals
p = [0, 0,1;...
     0, 0,1;...
     4,-4,0];
direction = [4, 4,4;...
             3,-3,1;...
             0, 0,4];
q = p+direction;
% d_syms(p,q,x,y,z) = norm(cross(q-p,p-[x;y;z]))/norm(q-p);
% f_syms(x,y,z) = norm(cross(q(:,1)-p(:,1),p(:,1)-[x;y;z]))/norm(q(:,1)-p(:,1))-norm(cross(q(:,2)-p(:,2),p(:,2)-[x;y;z]))/norm(q(:,2)-p(:,2));
% g_syms(x,y,z) = norm(cross(q(:,1)-p(:,1),p(:,1)-[x;y;z]))/norm(q(:,1)-p(:,1))-norm(cross(q(:,3)-p(:,3),p(:,3)-[x;y;z]))/norm(q(:,3)-p(:,3));

qp1 = q(:,1)-p(:,1);
p1 = p(:,1);
qp2 = q(:,2)-p(:,2);
p2 = p(:,2);
qp3 = q(:,3)-p(:,3);
p3 = p(:,3);
f_syms(x,y,z) = norm(cross(qp1,p1-[x;y;z]))/norm(qp1)-norm(cross(qp2,p2-[x;y;z]))/norm(qp2);
g_syms(x,y,z) = norm(cross(qp1,p1-[x;y;z]))/norm(qp1)-norm(cross(qp3,p3-[x;y;z]))/norm(qp3);

B0 = freecurvebox( 10*[-ones(3,1),ones(3,1)]);
MAXEPS = inf;


%##################################################
%0000000000000000000000000000000000000000000000000
%##################################################


df_syms = gradient(f_syms,[x,y,z]);
dg_syms = gradient(g_syms,[x,y,z]);
f_mat = matlabFunction(f_syms);
g_mat = matlabFunction(g_syms);
df_mat = matlabFunction(df_syms);
dg_mat = matlabFunction(dg_syms);
[f,g,df,dg] = funmanipulation.boxfunction(f_mat,g_mat,df_mat,dg_mat);

[fig,ax] = createfigure(B0);
plotspace = reshape(B0.scale(4).boxdimensions',1,[]);
alphavalue = 0.2;
hf = fimplicit3(ax,f,plotspace,'EdgeColor','none','FaceAlpha',alphavalue,'FaceColor','b');
hg = fimplicit3(ax,g,plotspace,'EdgeColor','none','FaceAlpha',alphavalue,'FaceColor','y');
B0.plotbox(ax);
drawnow;

%% for trisector example
M = cat(3,p+100*(q-p),q+100*(p-q));
hline1 = plot3(ax,squeeze(M(1,:,:))',squeeze(M(2,:,:))',squeeze(M(3,:,:))','m');
%% end trisector part

filename = which('parallel_local_tracecurve');
[filepath,name,ext] = fileparts(filename);

%% Settings
%Rules for inherited test results
```

```
B0.testresults = cell(1,8);
↪  %1:C0(f),2:C0(g),3:C1(f),4:C1(g),5:Jacobian,6:MK,7:MK_face,8:C0_face
B0.inherittestindices = 1:4;

%Depth limit for phase 1
depthlimit = 8;
%Depth limit for phase 2
numiterMKlimit = 6;

tStart = tic;

%% Subdivision Phase 1
Q = B0; %Input of the first phase of subdivision
QJac = []; %Output of first phase of subdivision

% Depth for phase 1
depth = 0;

% Create a subdivision of boxes, which all satisfy the predicates untill C1 tests
% and the Jacobian tests hold (where boxes satisfying C0 get excluded at each level)
while ~isempty(Q) && depth <= depthlimit
    Q_next = cell(length(Q),1);
    QJac_add = cell(length(Q),1);

    disp(['Phase 1: depth = ', num2str(depth), ' | length(Q) = ', num2str(length(Q))]);

    %Parallel Subdivision
    parfor i = 1:length(Q)
        B_par = Q(i);
        if ~local_predicate.C0(B_par,f,1) && ~local_predicate.C0(B_par,g,2)
            if B_par.radius<MAXEPS && local_predicate.C1(B_par,df,3) &&
↪  local_predicate.C1(B_par,dg,4) && ... %&& local_predicate.C1cross(B_par,df,dg,5)
               local_predicate.Jacobian(B_par,df,dg,5)
                QJac_add{i} = B_par;
            else
                children = B_par.split;
                Q_next{i} = children;
            end
        end
    end

    accepted = [QJac_add{:}];

    disp(['# accepted boxes = ', num2str(length(accepted))]);

    %Collection of results
    Q = [Q_next{:}];
    QJac = [QJac, QJac_add{:}];

    depth = depth+1;
end

if ~isempty(Q) && depth == depthlimit + 1
    disp("Phase 1 stopped due to depth limit");
end

disp(['Time for Phase 1: ',num2str(toc(tStart)),'s']);

disp("Phase 1 finalized.");
disp("Proceeding to phase 2...");

```

```
tPhase2 = tic;

%% Subdivision Phase 2
QMK = QJac; %Input of the second phase of subdivision
Qcurve = []; %Output of second phase of subdivision
%%
%Depth for phase 2
numiterMK = 1;

while ~isempty(QMK) && numiterMK <= numiterMKlimit
    %Iteration cell arrays
    QMK_next = cell(length(QMK),1);
    Qcurve_add = cell(length(QMK),1);

    disp(['Phase 2: numiterMK = ', num2str(numiterMK), ' | length(QMK) = ',
    ↪  num2str(length(QMK))]);

    %Parallel Subdivision
    parfor i = 1:length(QMK)
        B_par = QMK(i);
        if ~local_predicate.C0(B_par,f,1) && ~local_predicate.C0(B_par,g,2)
            if local_predicate.Jacobian(B_par,df,dg,5) &&
    ↪  local_predicate.MK_face(B_par,f,df,g,dg,7) %local_predicate.MK_face(B_par,f,df,g,dg,7)
                if any(B_par.testresults{7})
                    Qcurve_add{i} = B_par;
                end %Else: MK test has succeeded not in finding a root, but in excluding all in
    ↪  internal call to C0_faces
            else
                children = B_par.split;
                QMK_next{i} = children;
            end
        end
    end

    accepted = [Qcurve_add{:}];

    disp(['# accepted boxes = ', num2str(length(accepted))]);

    %Collection of results
    QMK = [QMK_next{:}];
    Qcurve = [Qcurve, Qcurve_add{:}];

    numiterMK = numiterMK+1;
end

if ~isempty(QMK) && numiterMK == numiterMKlimit + 1
    disp("Phase 2 stopped due to depth limit");
end

disp(['Time for Phase 2: ',num2str(toc(tPhase2)),'s']);

disp("Phase 2 finalized.");

%% Outputs
disp(['Total time for subvision: ',num2str(toc(tStart)),'s']);

disp(['Number of boxes in Q: ',num2str(length(Q))]);

%boxes which were not classified and fail the Jacobian
for i = 1:length(Q)
    Q(i).plotbox(ax,'r');
```

```
end

disp(['Number of boxes in QMK: ',num2str(length(QMK))]);

%boxes which were not classified but pass the Jacobian
for i = 1:length(QMK)
    QMK(i).plotbox(ax,'y');
end

disp(['Number of accepted boxes: ',num2str(length(Qcurve))]);

%boxes which pass the MK test and satisfy all the C1/Jacobian requirements
for i = 1:length(Qcurve)
    Qcurve(i).plotbox(ax,'g');
end

%% ####################




%% Curve Construction
tic;

leavessub = leaves(B0);
n = length(leavessub);
nodesnearcurvelogical = false(n,1);
centers = zeros(3,n);

parfor i = 1:n
    B = leavessub(i);
    leavessub(i).boxid = i;
    centers(:,i) = leavessub(i).center;
    if any(B.testresults{7}) %all boxes in the final subdivision which pass the MK test also
    ↪  pass Jacobian
        nodesnearcurvelogical(i) = true;
    end
end
nodes = 1:n;
nodesnearcurve = nodes(nodesnearcurvelogical); %list of nodes according to initial IDs

Afull = logical(sparse(n,n));

for i = 1:n %Cannor directly use parfor here
    neighbors = leavessub(i).neighbors;
    for j = 1:length(neighbors) %probably not worth doing with parfor
        Afull(i,neighbors(j).boxid) = true;
    end
end

Afull = Afull | Afull';
Gfull = graph(Afull);

Gnearcurve = subgraph(Gfull,nodesnearcurve);

bins = conncomp(Gnearcurve);
numcomps = numel(unique(bins));
curvepieces = cell(1,numcomps);

disp(["Numcomps: ", num2str(numcomps)])
```

```
275
276 for comp = 1:numcomps
277     nodescomponentlogical = bins == comp;
278     nodescomponent = nodesnearcurve(nodescomponentlogical);
279     Gcomponent = subgraph(Gnearcurve,nodescomponentlogical);
280
281     ndirected = length(nodescomponent);
282
283     disp(["Comp ", num2str(comp), ":"])
284     disp(["Component size:", ndirected])
285
286     if ndirected == 1
287         disp("Skipped comp of size 1.")
288         disp("-----ooooo-----")
289         continue
290     end
291
292     Adirected = logical(sparse(ndirected,ndirected));
293     edges = table2array(Gcomponent.Edges);
294     for i = 1:size(edges,1)
295         edge = edges(i,:);
296         B1 = leavessub(nodescomponent(edge(1)));
297         B2 = leavessub(nodescomponent(edge(2)));
298         center1 = B1.center;
299         center2 = B2.center;
300         direction = (B1.radius+B2.radius)/sqrt(3) - (center2-center1) < 10*eps;
301         direction = direction*(-1)^(center1(direction)<center2(direction));%B1 has smaller
    ↪ coordinates than B2 in direction
302         if all((direction')*B1.testresults{5} >= 0) && all((direction')*B2.testresults{5} >=0)
303             Adirected(edge(1),edge(2)) = true;
304         elseif all((direction')*B1.testresults{5} <= 0) && all((direction')*B2.testresults{5}
    ↪ <=0)
305             Adirected(edge(2),edge(1)) = true;
306         end
307     end
308     Gdirected = digraph(Adirected);
309     hGdirected = plot(ax,Gdirected,'XData',centers(1,nodescomponent),'YData',centers(2,nodesc⌋
    ↪ omponent),'ZData',centers(3,nodescomponent));
310     hGdirected.ArrowSize = 5; %7.5;
311     hGdirected.LineWidth = 0.5; %0.75;
312
313     %%Find cycle or path
314     %Use BFS to create a tree with shortest paths
315     %Then run DFS to check for paths that have all boxes nearby (neighbors of neighbors)
316     %Repeat for all neighbors of the starting vertex and select the shortest path
317     [path,foundcycle] = findcycle(Gdirected);
318     if ~foundcycle
319         disp("Cycle not found!")
320         path = findpath(Gdirected);
321     else
322         disp("Cycle found.")
323     end
324     disp(["Path length:", length(path)])
325     disp("-----ooooo-----")
326     curvepieces{comp} = path;
327
328     %Plot curve
329     Coordinates = zeros(3,length(path));
330     for i = 1:length(path)
331         Coordinates(:,i) = leavessub(nodescomponent(path(i))).center;
332     end
```

```
333         plot3(ax,Coordinates(1,:),Coordinates(2,:),Coordinates(3,:),'-k','Linewidth',3);
334     end
335     disp(['time for graph algorithm: ',num2str(toc),'s']);
```

A.2 Code for the cycle finding algorithm used in reconstruction

```
function [cycle,foundcycle] = findcycle(G)
    if nargin < 1
        s = [1 1 2 3 3 4 4 6 6 7 8 7 5];
        t = [2 3 4 4 5 5 6 1 8 1 3 2 8];
        G = digraph(s,t);
        plot(G);
        hold on;
    end
    %We use node 1 or any of its neighbors as start vertex

    edges = table2array(G.Edges);
    n = size(G.Nodes,1);
    neighbors = (edges(:,1) == 1) | (edges(:,2) == 1); %any(edges == 1,2);
    neighbors = unique(edges(neighbors,:));
    foundcycle = false;
    cycle = [];
    i = 1;
    while ~foundcycle && i <= length(neighbors)
        s = neighbors(i);
        [~,D] = shortestpathtree(G,'all',s);
        maxdist = max(D(~isinf(D)));

        farthestnodesbool = D >= maxdist-2 & ~isinf(D) & D>1;
        farthestnodes = 1:n;
        farthestnodes = farthestnodes(farthestnodesbool);
        j = 1;
        while maxdist < inf && ~foundcycle && j <= length(farthestnodes)
            if findedge(G,farthestnodes(j),s) || findedge(G,s,farthestnodes(j))
                foundcycle = true;
                t = farthestnodes(j);
                TR = shortestpathtree(G,t,s,'OutputForm','cell');
                cycle = [TR{1},t];
            end
            j = j+1;
        end
        i = i+1;
    end
end
```

A.3 Code for the path finding algorithm used in reconstruction

```
function path = findpath(G)
    if nargin < 1
        s = [1 3 3 4 6 6 7 8 7 5];
        t = [2 4 5 5 1 8 1 3 2 8];
        G = digraph(s,t);
        plot(G);
        hold on;
    end
    edges = table2array(G.Edges);
    Gundirected = graph(edges(:,1),edges(:,2));
%     H = flipedge(G);
    v = 1;
    [~,D] = shortestpathtree(Gundirected,'all',v);
    s = find(D==max(D),1);
    [~,D] = shortestpathtree(Gundirected,'all',s);
    t = find(D==max(D),1);
    TR = shortestpathtree(Gundirected,s,t,'OutputForm','cell');
    path = TR{1};

    %Warning: this is not 100% correct, we would also need to additionally check that the first
↪   and last boxes are on the domain boundary
end
```

A.4 Code for the implemented tests

```
classdef local_predicate < handle

    %Assuming the following indexing for inherited test results:
    %1:C0(f),2:C0(g),3:C1(f),4:C1(g),5:JC,6:MK,7:MK_face,8:C0_face

    properties (Constant)
        JC_scale = 4;
        MK_scale = 2;
        %For the test used on 6 sides:
        MK_box_scale = 1.3; %Scales each box before determining the faces, must be above 1 for current implementation
        MK_face_scale = 1.3; %Scales each face of the box, must be above 1 for current implementation
        MK_jac_ind = 6;
        C0_depth = 0; %0 means just evalaute on the main box, and gives the best performence by far (for C0_depth)
        C0_face_depth = 0; %0 means just evalaute on the main box, and seems to give the best performence (for  C0_face_depth)
    end

    methods
        function this = local_predicate()
            local_tracecurve;
        end
    end
    methods(Static)
        % ##### C0 #####
        function bool = C0(B,f,ind)
            if nargin > 2
                if isempty(B.testresults{ind})
                    bool = local_predicate.C0core(B,f);
                    B.testresults{ind} = bool;
                else
                    bool = B.testresults{ind};
                end
            else
                bool = local_predicate.C0core(B,f);
            end
        end
        function bool = C0core(B,f)
            %Returns true if the surface f=0 surely does not pass through B
            %bool = 0 \notin f(B)
            bool = funexcludes0(f,B,local_predicate.C0_depth);
%~interval.zeros(1).subset(f(B));
        end
        % ####################



        % ##### C1 #####
        function bool = C1(B,df,ind)
            if nargin > 2
                if isempty(B.testresults{ind})
                    bool = local_predicate.C1core(B,df);
                    B.testresults{ind} = bool;
```

```
            else
                bool = B.testresults{ind};
            end
        else
            bool = local_predicate.C1core(B,df);
        end
    end
    function bool = C1core(B,df)
        %C1(B,df, IND)
        %Returns true if the surface f=0 satisfies the C1 condition
        %bool = 0 \notin < df(B) , df(B) >
            Bint = B.interval;
            bool = ~interval.zeros(3,1).subset(df(Bint)'*df(Bint));
    end
    function bool = C1cross(B,df,dg,ind)
        if nargin > 2
            if isempty(B.testresults{ind})
                bool = local_predicate.C1crosscore(B,df,dg);
                B.testresults{ind} = bool;
            else
                bool = B.testresults{ind};
            end
        else
            bool = local_predicate.C1crosscore(B,df);
        end
    end
    function bool = C1crosscore(B,df,dg)
        %C1cross(B,df,dg,ind)
        %Returns true if the curve defined by f=0 and g=0 satisfies the C1 condition
        %bool = 0 \notin < df(B) x dg(B) , df(B) x dg(B)>
            Bint = B.interval;
            cross_fg = cross(df(Bint),dg(Bint));
            bool = ~interval.zeros(3,1).subset(cross_fg'*cross_fg);
    end
    % ####################



    % ##### JC #####
    function [bool,v] = Jacobian(B,df,dg,ind)
        if nargin > 3
            if isempty(B.testresults{ind})
                [bool,v] = local_predicate.Jacobiancore(B,df,dg);
                B.testresults{ind} = v;
            else
                v = B.testresults{ind};
                bool = any(v);
            end
        else
            [bool,v] = local_predicate.Jacobiancore(B,df,dg);
        end
        B.passesJacobian = B.passesJacobian || bool; %for function markcurve
    end
    function [bool,v] = Jacobiancore(B,df,dg)
        B = B.scale(local_predicate.JC_scale);
        v = zeros(3,1);
        dB = [df(B),dg(B)];
        for i = 1:3
            testinterval = -(-1)^i*det(dB(setdiff([1,2,3],i),:));
            if 0 < testinterval
                v(i) = 1;
```

```
                elseif testinterval < 0
                    v(i) = -1;
                else
                    v(i) = 0; %Jacobian did not succeed
                end
            end
            bool = any(v);
        end
        % ####################

        % ##### MK #####
        function bool = MK(B,f,df,g,dg,ind,ax)
            if nargin > 5
                if isempty(B.testresults{ind})
                    if nargin > 6
                        bool = local_predicate.MKcore(B,f,df,g,dg,ax);
                    else
                        bool = local_predicate.MKcore(B,f,df,g,dg);
                    end
                    B.testresults{ind} = bool;
                else
                    bool = B.testresults{ind};
                end
            else
                bool = local_predicate.MKcore(B,f,df,g,dg);
            end
        end
        function bool = MKcore(B,f,df,g,dg,ax)
            %Returns true if there are two pairs of opposite faces of B
            %with f having opposite sign on one pair, and g on the other.
            %Includes preconditioning around the box center
            B = B.scale(local_predicate.MK_scale);
            m = B.center;
            dfm = df(m); dgm = dg(m);

                Jm = [dfm,dgm,cross(dfm,dgm)]';
            invJm = Jm^(-1);
            h = @(varargin) cross(dfm,dgm)'*(funmanipulation.convert2vec(varargin{:})-m);
    ↪  %varargin can be x,y,z components or box or interval vector
            fprime = @(varargin) invJm(1,:) * [f(varargin{:});g(varargin{:});h(varargin{:})];
            gprime = @(varargin) invJm(2,:) * [f(varargin{:});g(varargin{:});h(varargin{:})];
            hprime = @(varargin) invJm(3,:) * [f(varargin{:});g(varargin{:});h(varargin{:})];


            faces = B.facets;
            if ~funsmaller0(fprime,faces(1,1)) || ~funsmaller0(@(B) -fprime(B),faces(1,2)) ||
    ↪  ...
               ~funsmaller0(gprime,faces(2,1)) || ~funsmaller0(@(B) -gprime(B),faces(2,2)) ||
    ↪  ...
               ~funsmaller0(hprime,faces(3,1)) || ~funsmaller0(@(B) -hprime(B),faces(3,2))
                bool = false;
            else
                bool = true;
            end


            if nargin>5
                Bint = B.interval;
                plothandle = B.plotbox(ax,'b');
               if ~exist('axorig','var') || isempty(axorig) || ~isgraphics(axorig) %#ok<NODEF>
                    subplot(1,3,1,ax);
```

```
                    axorig = subplot(1,3,2);
                    r = groot;
                    Monitors = r.MonitorPositions;
                    [~,M] = max(Monitors(:,3));
                    fig = ax.Parent;
                    sizex = Monitors(M,3);
                    originalPos = fig.Position;
                    newPos = originalPos;

                    set(fig, 'units', 'pixels', 'position',newPos);
                    prepareaxes(axorig);
                end
                if ~exist('axtrans','var') || isempty(axtrans) || ~isgraphics(axtrans)
    ↪ %#ok<NODEF>
                    axtrans = subplot(1,3,3);
                    prepareaxes(axtrans);
                end

                plothoriginal = local_predicate.plotfunctions(axorig,f,g,h,Bint);
                plothtransformed =
    ↪ local_predicate.plotfunctions(axtrans,fprime,gprime,hprime,Bint);
                ploth = [plothoriginal,plothtransformed];
            end

            if nargin>5
                if bool
                    titletext = 'success';
                else
                    titletext = [...
                        '$ f^\sim(B_x^-)$ =
    ↪ ',num2str(fprime(faces(1,1)).bounds),']',newline,...
                        '$-f^\sim(B_x^+)$ =
    ↪ ',num2str(uminus(fprime(faces(1,2))).bounds),']',newline,...
                        '$ g^\sim(B_y^-)$ =
    ↪ ',num2str(gprime(faces(2,1)).bounds),']',newline,...
                        '$-g^\sim(B_y^+)$ =
    ↪ ',num2str(uminus(gprime(faces(2,2))).bounds),']',newline,...
                        '$ h^\sim(B_y^-)$ =
    ↪ ',num2str(hprime(faces(3,1)).bounds),']',newline,...
                        '$-h^\sim(B_y^+)$ =
    ↪ ',num2str(uminus(hprime(faces(3,2))).bounds),']',newline];
                end
                title(axtrans,titletext,'interpreter','latex','FontSize',16);

                for i = 1:length(ploth)
                    delete(ploth{i});
                end
                delete(plothandle);
            end
        end
        % ####################

        % ##### Plot #####
        function ploth = plotfunctions(ax,fun1,fun2,fun3,B)
        %plot the functions within the box for testing reasons
            if nargin == 5
                plotinterval = [B(1).bounds,B(2).bounds,B(3).bounds];
                axis(ax,plotinterval);
            else
                plotinterval = axis(ax);
            end
```

```
            %Use the function arrayfun for the next 3 lines possibly
            fun1_element = @(x,y,z) elementwisefunction(fun1,x,y,z);
            fun2_element = @(x,y,z) elementwisefunction(fun2,x,y,z);
            fun3_element = @(x,y,z) elementwisefunction(fun3,x,y,z);
            ploth = cell(1,3);
            ploth{1} = fimplicit3(ax,fun1_element,plotinterval,'b');
            ploth{2} = fimplicit3(ax,fun2_element,plotinterval,'r');
            ploth{3} = fimplicit3(ax,fun3_element,plotinterval,'g');
            alpha(ploth{1},0.5);
            alpha(ploth{2},0.5);
            alpha(ploth{3},0.5);
        end
        function ploth = plotfunctions2(ax,fun1,fun2,B)
        %plot the functions within the box for testing reasons
            if nargin == 5
                plotinterval = [B(1).bounds,B(2).bounds,B(3).bounds];
                axis(ax,plotinterval);
            else
                plotinterval = axis(ax);
            end
            %Use the function arrayfun for the next 3 lines possibly
            fun1_element = @(x,y,z) elementwisefunction(fun1,x,y,z);
            fun2_element = @(x,y,z) elementwisefunction(fun2,x,y,z);
            ploth = cell(1,2);
            ploth{1} = fimplicit3(ax,fun1_element,plotinterval,'b');
            ploth{2} = fimplicit3(ax,fun2_element,plotinterval,'r');
            alpha(ploth{1},0.5);
            alpha(ploth{2},0.5);
        end
        % ####################

        % ##### MK_face #####
        function [bool,v] = MK_face(B,f,df,g,dg,ind,ax)
            if nargin > 5
                if isempty(B.testresults{ind})
                    if nargin > 6
                        [bool,v] = local_predicate.MK_face_core(B,f,df,g,dg,ax);
                    else
                        [bool,v] = local_predicate.MK_face_core(B,f,df,g,dg);
                    end
                    B.testresults{ind} = v;
                else
                    v = B.testresults{ind};
                    bool = any(v);
                end
            else
                [bool,v] = local_predicate.MK_face_core(B,f,df,g,dg);
            end
        end
        function [bool,v] = MK_face_core(B,f,df,g,dg)
            %Matrix showing which surfaces intersect the curve (populated below):
            res = zeros(3,2);
            %(Does not consider intersections in non-parameterizable direction's faces)

            spandir = zeros(3,1);

            %C0 on the faces of the initial box
            [C0_face_small, ~] = local_predicate.C0_face_core(B.scale(1),f,g);
            if C0_face_small %MK test terminates true but no faces are found, will not be
    ↪  considered during reconstruction later
                bool = true;
```

```
283                 v = reshape(res.',1,[]);
284                 return
285             end
286 
287             Bscaled = B.scale(local_predicate.MK_box_scale);
288 
289             %C0 on the faces of the scaled box
290             [C0_face, exclusions] = local_predicate.C0_face_core(Bscaled,f,g);
291             if C0_face %MK test terminates true but no faces are found, will not be considered
    ↪  during reconstruction later
292                 bool = true;
293                 v = reshape(res.',1,[]);
294                 return
295             end
296 
297             [jacbool,jac] = local_predicate.Jacobian(B,df,dg, local_predicate.MK_jac_ind);
298             if ~jacbool %Jacobian test fails, need to split
299                 bool = false;
300                 v = reshape(res.',1,[]);
301                 return
302             end
303 
304             faces = Bscaled.facets;
305 
306             for i = 1:3
307                 if jac(i) ~= 0 %Parameterizable in this direction
308                     for j = 1:2
309                         if exclusions(2*(i-1)+j) ~= 1 %do not bother if face would be excluded
310                             face = faces(i,j);
311                             m = face.center;
312                             dfm = df(m); dgm = dg(m);
313 
314                             %Remove the i'th dimension from dfm/dfg to get 2d versions
315                             dfm(i,:) = [];
316                             dgm(i,:) = [];
317 
318                             Jm = [dfm,dgm]';
319                             invJm = Jm^(-1);
320 
321                             fprime = @(varargin) invJm(1,:) * [f(varargin{:});g(varargin{:})];
322                             gprime = @(varargin) invJm(2,:) * [f(varargin{:});g(varargin{:})];
323 
324                             edges = face.scale(local_predicate.MK_face_scale).facets;
325 
326                             %facebool
327                             if ~funsmaller0(fprime,edges(1,1)) || ~funsmaller0(@(B)
    ↪  -fprime(B),edges(1,2)) || ...
328                                ~funsmaller0(gprime,edges(2,1)) || ~funsmaller0(@(B)
    ↪  -gprime(B),edges(2,2))
329                                 facebool = false;
330                             else
331                                 facebool = true;
332                             end
333 
334                             res(i,j) = facebool;
335                         end
336                     end
337                     dirbool = res(i,1) && res(i,2);
338                     spandir(i) = dirbool;
339                 end
340             end
```

```
341
342             %Flatten result for output
343             v = reshape(res.',1,[]);
344             bool = any(v);
345
346         end
347         %####################
348
349         % ##### C0_face #####
350         function [bool,v] = C0_face(B,f,g,ind) %Returns true iff all surfaces can be excluded
351             if nargin > 3
352                 if isempty(B.testresults{ind})
353                     [bool,v] = local_predicate.C0_face_core(B,f,g);
354                     B.testresults{ind} = v;
355                 else
356                     v = B.testresults{ind};
357                     bool = all(v);
358                 end
359             else
360                 [bool,v] = local_predicate.C0_face_core(B,f,g);
361             end
362         end
363         function [bool,v] = C0_face_core(B,f,g)
364             %Matrix showing which surfaces can be excluded
365             res = zeros(3,2);
366
367             faces = B.facets;
368
369             for i = 1:3
370                 for j = 1:2
371                     face = faces(i,j);
372
373                     res(i,j) = funexcludes0(f,face,local_predicate.C0_face_depth) ||
    ↪  funexcludes0(g,face,local_predicate.C0_face_depth); %Can replace by calls to C0_core
374                 end
375             end
376
377             %Flatten result for output
378             v = reshape(res.',1,[]);
379             bool = all(v);
380         end
381         %####################
382     end
383 end
```

A.5 Code defining intervals and interval arithmetic

```
classdef interval < handle & matlab.mixin.Copyable
    %Interval arithmetic operations
        %functions

        %Overloading operartors for the interval class
        %a+b
        %a-b
        %-a
        %+a
        %a.*b
        %a*b
        %a./b
        %a.\b
        %a/b
        %a\b
        %a.^b
        %a^b
        %a < b
        %a > b
        %a <=b
        %a >= b
        %a ~= b
        %a == b
            %%%Not included
            %%% a&b
            %%% a|b
            %%% ~a
            %%% a:d:b
            %%% a:b
        %a'
        %a.'
            %%% [a,b]
            %%% [a;b]
            %%% a(s1,s2,...,sn)
            %%% a(s1,s2,...,sn) = b
            %%% b(a)
        %sqrt
        %nthroot
        %abs
        %sign
        %norm %L_2 norm of interval vectors
        %norm_p %L_p norm
        %disp
        %int2str
        %extractbounds: used when working with matrices of intervals
        %sum
        %min
        %max
        %cross(a,b): computes the cross product of two interval vectors of size 3
        %det: computes the determinant of a 2x2 interval matrix
        %a.cap(b):      computes the intersection of intervals a and b
        %a.cup(b):      computes the union of intervals a and b

        %a \subset b:   a.subset(b)
        %a \superset b: a.superset(b)
```

```
        %x \in a:       a.element(x)

       %scale(factor): if [a,b] is bounded then this doubles the length of the interval, while keeping the midpoint
        %               if [a,b] is unbounded, then this halves the length of the uncovered number line

        %linspace(n): creates a mesh with each interval getting uniformly split into n values
    properties
        bounds %[a,b]
            %if a<=b: I = [a,b]
            %if b<a:  I = [-inf,a] \cup [b,inf]
            %The empty interval is represented by [inf,-inf]
            %Imagine the ends of the number line being connected at infiniity
    end

    methods
        function this = interval(lowerbounds,upperbounds)
            %Accepts two matrices of same size as bounds to create a matrix of intervals
            if nargin > 0
                if nargin == 1
                    upperbounds = lowerbounds;
                end
                v = size(lowerbounds);
                if all(size(lowerbounds) == size(upperbounds))
                    numv = prod(v);
                    if numv > 1
                        for ind = numv:-1:1
                            this(ind) = interval(lowerbounds(ind),upperbounds(ind));
                        end
                        if length(v) == 1
                            v = [v,1];
                        end
                        this = reshape(this,v);
                    else
                        this.bounds = [lowerbounds,upperbounds];
                    end
                else
                    warning('wrong interval definition');
                end
            end
        end
        %Overloading operators with MATLAB:
        %https://ch.mathworks.com/help/matlab/matlab_oop/implementing-operators-for-your-class.html
        function result = plus(a,b)
            %a+b
            fct = @interval.plus_element;
            result = interval.elementwiseoperator(fct,a,b);
        end
        function result = minus(a,b)
            %a-b
            result = a+(-b);
        end
        function result = uminus(a)
            %-a
            fct = @interval.uminus_element;
            result = interval.elementwiseoperator(fct,a);
        end
        function result = uplus(a)
            %+a
```

```
            result = a;
        end
        function result = times(a,b)
            %a.*b
            fct = @interval.times_element;
            result = interval.elementwiseoperator(fct,a,b);
        end
        function result = mtimes(a,b)
            %a*b
            [i,j] = size(a);
            [j2,k] = size(b);
            result = interval(zeros([i,k]));
            if j == j2
                if ~isa(a,'interval')
                    a = interval(a);
                end
                if ~isa(b,'interval')
                    b = interval(b);
                end
                for iind = 1:i
                    for kind = 1:k
                        currentsum = interval(0,0);
                        for jind = 1:j
                            currentsum =
↪ currentsum+interval.times_element(a(iind,jind).bounds,b(jind,kind).bounds);
                        end
                        result(iind,kind) = currentsum;
                    end
                end
            else
                error('The matrix dimensions are not matching');
            end
        end
        function result = rdivide(a,b)
            %a./b
            if isa(b, 'interval')
                result = a.*inverse(b);
            else
                result = a.*(1/b);
            end
        end
        function result = inverse(a)
            %1./a
            fct = @interval.inverse_element;
            result = interval.elementwiseoperator(fct,a);
        end
            %a.\b
            %a/b
            %a\b
        function result = power(a,b)
            %a.^b
            fct = @interval.power_element;
            result = interval.elementwiseoperator(fct,a,b);
        end
            %a^b
        function result = lt(a,b)
            %a < b
            fct = @interval.lt_element;
            result = interval.elementwiseoperator(fct,a,b);
        end
        function result = gt(a,b)
```

```
        %a > b
        result = b<a;
    end
    function result = le(a,b)
        %a <=b
        fct = @interval.le_element;
        result = interval.elementwiseoperator(fct,a,b);
    end
    function result = ge(a,b)
        %a >= b
        result = b <= a;
    end
    function result = ne(a,b)
        %a ~= b
        result = ~(a==b);
    end
    function result = eq(a,b)
        %a == b
        fct = @interval.eq_element;
        result = interval.elementwiseoperator(fct,a,b);
    end

    function result = inverse_elementwise(this)
        fct = @interval.inverse_element;
        result = interval.elementwiseoperator(fct,this);
    end

    function result = sqrt(a)
        %sqrt(a)
        fct = @interval.power_element;
        result = interval.elementwiseoperator(fct,a,1/2);
    end
    function result = nthroot(a,n)
        %nthroot(a,n)
        fct = @interval.power_element;
        result = interval.elementwiseoperator(fct,a,n^(-1));
    end

    function result = abs(a)
        %abs(a)
        fct = @interval.abs_element;
        result = interval.elementwiseoperator(fct,a);
    end

    function result = norm(a)
        %norm(a)
        result = norm_p(a,2);
    end
    function result = norm_p(a,p)
        %nthroot(a,b)
        if length(size(a)) == 2 && length(size(p)) == 2 && all(size(p) == [1,1])
            sizes = size(a);
            if sizes(1) == 1 %row vector
                result = sum(a.^p,2)%^(1/p);
            elseif sizes(2) == 1 %column vector
                result = sum(a.^p,1)%^(1/p);
            else %matrix
                error('matrix norm not yet implemented')
            end
        elseif length(size(a)) ~= 2
            error('invalid input a for norm_p(a,p)')
```

```
            else
                error('norm_p(a,b) function requires a single scalar p')
            end
        end

        function disp(this)
            disp('Interval');
            dispmat = interval.int2mat(this);
            disp(dispmat);
        end

        function str = int2str(this)
            str = mat2str(interval.int2mat(this));
        end

        function result = summinmax(this,directions,to_be_evaluated)
            if nargin<2
                directions = 1;
            end
            s = size(this);
            d = length(s);
            snew = s;
            snew(directions) = ones(1,length(directions));
            result = interval(zeros(snew));
            specifieddirections = false(1,d);
            specifieddirections(directions) = true(1,length(directions));

            vLim = s;
            vLim1 = s;
            vLim1(directions) = [];

            v1    = ones(1, length(vLim1));
            ready = false;
            while ~ready
                Index1 = arrayindexing.sub2indV(vLim1, v1);

                vLim2 = s(directions);
                v2    = ones(1, length(vLim2));
                ready = false;
                while ~ready
                    v = zeros(1,d);
                    v(~specifieddirections) = v1;
                    v(specifieddirections) = v2;
                    Index = arrayindexing.sub2indV(vLim, v);
                    eval(to_be_evaluated);
                    % Update the index vector:
                    [v2,ready] = arrayindexing.updateindexvec(v2,vLim2);
                end
                % Update the index vector:
                [v1,ready] = arrayindexing.updateindexvec(v1,vLim1);
            end
        end
        function result = sum(this,directions)
            to_be_evaluated = 'result(Index1) = result(Index1)+this(Index);';
            result = summinmax(this,directions,to_be_evaluated);
        end
        function result = min(this,varargin)
            to_be_evaluated = 'result(Index1) = result(Index1)+this(Index);';
            result = summinmax(this,directions,to_be_evaluated);
        end
        %max
```

```
296 %         function result = max(this,varargin)
297 %         end
298
299         function result = cross(a,b)
300             %cross(a,b)
301             if numel(a) == 3 && numel(b) == 3 %length(size(a)) == 2 && length(size(b)) == 2 &&
    ↪  all(size(a) == [1,3]) && all(size(b) == [1,3])
302                 result = [(a(2)*b(3)) - (a(3)*b(2)), (a(3)*b(1)) - (a(1)*b(3)), (a(1)*b(2)) -
    ↪  (a(2)*b(1))];
303             else
304                 error('cross function is only implemented for two interval vectors of size 3')
305             end
306         end
307
308         %det
309         function result = det(this)
310             if length(size(this)) == 2 && all(size(this) == [2,2])
311                 result = this(1)*this(4)-this(2)*this(3);
312             else
313                 error('det function has to be implemented for non-2x2 matrices')
314             end
315         end
316         %cap
317         %cup
318         function result = cup(this,b)
319             fct = @interval.cup_element;
320             result = interval.elementwiseoperator(fct,this,b);
321         end
322
323         function result = sign(this)
324             fct = @interval.sign_element;
325             result = interval.elementwiseoperator(fct,this);
326         end
327
328         function result = extractbounds(this)
329             s = size(this);
330             n = prod(s);
331             result = zeros(n,2);
332             for i = 1:n
333                 result(i,:) = this(i).bounds;
334             end
335             if s(end) == 1
336                 s(end) = [];
337             end
338             result = reshape(result,[s,2]);
339         end
340
341         function result = subset(a,b)
342             fct = @interval.subset_element;
343             result = all(interval.elementwiseoperator(fct,a,b));
344         end
345         function result = superset(this,bi)
346             %this superset of b
347             result = bi.subset(this);
348         end
349
350         function result = scale(this,factor)
351             result = interval(zeros(size(this)));
352             s = 1+(factor-1)/2;
353             for i=1:numel(this)
354                 result(i) = interval(s*this(i).bounds(1)+(1-s)*this(i).bounds(2),(1-s)*this(i
    ↪  ).bounds(1)+s*this(i).bounds(2));
```

```
            end
        end
        function result = linspace(this,n)
            result = zeros([numel(this),n]);
            for i=1:numel(this)
                result(i,:) = linspace(this(i).bounds(1),this(i).bounds(2),n);
            end
            s = size(this);
            if length(s) == 2 && s(2) == 1
                s(2) = [];
            end
            result = reshape(result,[s,n]);
        end

        %%%%%%%%%%%%% functions which need revision
%
%         function result = mrdivide(a,b)
%             %a/b
%             %%%%%%%%%%% Be careful if 0 is contained in numerator and denominator
%             result = a*b.inverseinterval;
%         end
% %         function result = mpower(a,b)
% %             %a^b
% % %             res1 =
% % %             result = a.interval+b.interval;
% %         end
    end


    methods(Static)
        function result = elementwiseoperator(fct,a,b)
            %Computes elementwise: fct(a,b) or fct(a) if b is not defined
            %Possible outputtupes are: 'interval' or 'logical'
            %allows for arrays a and b to have only 1 element while the other doesn't, similar
    ↪  to 1+[2,3] = [3,4]

            s = size(a);
            if isa(a,'double')
                a = interval(a);
            end
            if nargin >= 3
                if numel(a) == 1
                    s = size(b);
                end
                if isa(b,'double')
                    b = interval(b);
                end
            end
            snum = prod(s); %#ok<NASGU>
            for ind = prod(s):-1:1
                if nargin <= 2
                    input = {a(ind).bounds};
                else
                    if numel(a) == 1
                        input = {a.bounds,b(ind).bounds};
                    elseif numel(b) == 1
                        input = {a(ind).bounds,b.bounds};
                    else
                        input = {a(ind).bounds,b(ind).bounds};
                    end
```

```
                end
                result(ind) = fct(input{:});
            end
            result = reshape(result,s);
        end

        function result = plus_element(a,b)
            %a+b
            if (a(2)<a(1) && b(2)<b(1)) || (a(1)+b(1) <= a(2)+b(2) && (a(2)<a(1) || b(2) <
    ↪  b(1)))
                bounds = [-inf,inf];
            else
                bounds = a+b;
            end
            result = interval(bounds(1),bounds(2));
        end
        function result = uminus_element(a)
            %-a
            result = interval(-a(2),-a(1));
        end
        function result = times_element(a,b)
            %a*b
            values = a'*b;
            if a(1) <= a(2) && b(1) <= b(2)
                bounds = [min(values(:)),max(values(:))];
            else
                %make a contain infinity
                if a(1) <= a(2)
                    c = a;
                    a = b;
                    b = c;
                end
                if interval.subset_element([0,0],b) || (b(2) < b(1) &&
    ↪  interval.subset_element([0,0],a)) %0 in a or b
                    bounds = [-inf,inf];
                elseif b(2) < b(1) && ~interval.subset_element([0,0],a) &&
    ↪  ~interval.subset_element([0,0],b) %0 notin a or b
                    bounds = [min(values(values>0)),max(values(values<0))];
                else
                    if interval.subset_element([0,0],b)
                        c = a;
                        a = b;
                        b = c;
                    end
                    if ~interval.subset_element([0,0],a)
                        if 0 < b(1) %b positive
                            bounds = [a(1)*b(1),a(2)*b(1)];
                        else %b negative
                            bounds = [a(2)*b(2),a(1)*b(2)];
                        end
                    else
                        if 0<a(2) %0 < complement(a)
                            if 0 < b(1) %b positive
                                bounds = [a(1)*b(1),a(2)*b(2)];
                                if bounds(2)>bounds(1)
                                    bounds = [-inf,inf];
                                end
                            else %b negative
                                bounds = [a(2)*b(1),a(1)*b(2)];
                                if bounds(2)>bounds(1)
                                    bounds = [-inf,inf];
```

```
                            end
                        end
                    else %complement(a) < 0
                        if 0 < b(1) %b positive
                            bounds = [a(1)*b(2),a(2)*b(1)];
                            if bounds(2)>bounds(1)
                                bounds = [-inf,inf];
                            end
                        else %b negative
                            bounds = [a(1)*b(1),a(2)*b(2)];
                            if bounds(2)>bounds(1)
                                bounds = [-inf,inf];
                            end
                        end
                    end
                end
            end
        end
        result = interval(bounds(1),bounds(2));
    end
```

```
    function result = inverse_element(a)
        %Computes the inverse interval a^-1
        result = interval(1/a(2),1/a(1));
    end

    function result = abs_element(a)
        %|a|
        if a(1) <= a(2) %[a(1), a(2)]
            if a(1) >= 0
                result = interval(a(1), a(2));
            elseif a(2) <= 0
                result = interval(-a(2), -a(1));
            else %a(1) < 0 < a(2)
                result = interval(0, max(-a(1),a(2)));
            end
        else %[-inf, a(2)] \cup [a(1), inf]
            if a(2) >= 0 || a(1) <= 0
                result = interval(0, inf);
            else
                result = interval(min(abs(a(1)), abs(a(2))), inf);
            end
        end
    end

    function result = power_element(a,b)
        %Computes a^b assuming b is a real number
        %and a is non-negative when b is not an integer
        if a(2) >= a(1)
            if b(1) == b(2) %b can be treated as a single real number
                b = b(1);
                if rem(b(1),1) == 0 %b is an integer
                    if b >= 0
                        if rem(b,2) == 0 && a(1) <= 0 && 0 <= a(2)
                            result = interval(0,max(a(1)^b,a(2)^b));
                        else
                            res1 = a(1)^b;
```

```
                                res2 = a(2)^b;
                                result = interval(min(res1,res2),max(res1,res2));
                            end
                        else % b < 0
                            if rem(b,2) == 0 && a(1) <= 0 && 0 <= a(2)
                                result = interval(min(a(1)^b,a(2)^b), inf);
                            elseif a(1) <= 0 && 0 <= a(2) %rem(b,2) == 1
                                result = interval(-inf, inf);
                            else %a does not include 0
                                res1 = a(1)^b;
                                res2 = a(2)^b;
                                result = interval(min(res1,res2),max(res1,res2));
                            end
                        end
                    elseif a >= 0 %b is not an integer, but a is non-negative
                        res1 = a(1)^b;
                        res2 = a(2)^b;
                        result = interval(min(res1,res2),max(res1,res2));
                    else
                        error("interval method a.^b not implemented for non-integer b and
    ↪  negative a")
                    end
                else
                    error("interval method a.^b not implemented for an interval b")
                end
            else %a(1) > a(2) (a = [-\inf,a(2)] \cup [a(1),\inf])
                result = cup(interval(-inf,a(2)).^2, interval(a(1),inf).^2);
            end
        end

        function result = lt_element(a,b)
            %a < b
            result = (a(1)<=a(2)) && (b(1)<=b(2)) && a(2)<b(1);
        end
        function result = le_element(a,b)
            %a <=b
            result = (a(1)<=a(2)) && (b(1)<=b(2)) && a(2)<=b(1);
        end
        function result = eq_element(a,b)
            %a == b
            result = all(abs(a - b) == 3*eps);
        end

        function result = cup_element(a,b)
            if a(2) >= a(1) && b(2) >= b(1)
                % [a(1), a(2)] \cup [b(1), b(2)]
                result = interval(min(a(1),b(1)), max(a(2),b(2)));
            elseif a(2) >= a(1) && b(2) < b(1)
                % [a(1), a(2)] \cup ([-inf, b(2)] \cup [b(1), inf])
                if (a(1) <= b(2) && a(2) >= b(1)) || (a(1) >= b(2) && a(2) <= b(1))
                    result = interval(-inf, inf);
                elseif a(1) <= b(2) &&  a(2) < b(1)
                    result = interval(b(1), a(2));
                else % a(1) > b(2) &&  a(2) >= b(1)
                    result = interval(a(1), b(2));
                end
            elseif a(2) < a(1) && b(2) >= b(1)
                % ([-inf, a(2)] \cup [a(1), inf]) \cup [b(1), b(2)]
                %symmetric to the previous case
                c = a;
                a = b;
```

```
            b = c;
            if (a(1) <= b(2) && a(2) >= b(1)) || (a(1) >= b(2) && a(2) <= b(1))
                result = interval(-inf, inf);
            elseif a(1) <= b(2) &&  a(2) < b(1)
                result = interval(b(1), a(2));
            else % a(1) > b(2) &&  a(2) >= b(1)
                result = interval(a(1), b(2));
            end
        else % a(2) < a(1) && b(2) < b(1)
            % ([-inf, a(2)] \cup [a(1), inf]) \cup ([-inf, b(2)] \cup [b(1), inf])
            if max(a(2), b(2)) >= min(a(1), b(1))
                result = interval(-inf, inf);
            else
                result = interval(min(a(1),b(1)), max(a(2), b(2)));
            end
        end
        %to be checked
    end

    function result = sign_element(a)
        result = interval(min(sign(a(1)), sign(a(2))), max(sign(a(1)), sign(a(2))));
    end









    function result = subset_element(a,b)
        %a subset of b
        if a(1)<=a(2) && b(1) <= b(2)
            result = b(1) <= a(1) && a(2) <= b(2);
        elseif a(1)<=a(2) && ~(b(1) <= b(2))
            result = a(2) <= b(1) || b(2) <= a(1);
        elseif ~(a(1)<=a(2)) && b(1) <= b(2)
            result = false;
        elseif ~(a(1)<=a(2)) && ~(b(1) <= b(2))
            result = a(1)<=b(1) && b(2)<=a(2);
        end
    end









    function Index = sub2indV(Vlim,X)
        k     = [1, cumprod(Vlim)];
        Index = sum(k(1:length(X)) .* (X - 1)) + 1;
    end
    function v = ind2subV(Vlim, ind)
        ind = ind-1;
        v = zeros(1,0);
        for i = 1:length(Vlim)
            v(i) = 1+mod(ind,Vlim(i));
            ind = (ind-v(i)+1)/Vlim(i);
        end
    end
    function mat = int2mat(I)
```

```
            %Gather the interval bounds in a matrix
            sizeI = size(I);
            if sizeI(end) == 1
                sizeI(end) = [];
            end
            mat = zeros([sizeI,2]);
            num = numel(I);
            for i = 1:num
                if ~isempty(I(i).bounds)
                    mat([i,i+num]) = I(i).bounds;
                else
                    mat = ['empty ',num2str(sizeI),' array of intervals'];
                end
            end
        end

        function int = zeros(varargin)
            sizes = cell2mat(varargin);
            int = interval(zeros(sizes));
        end
        function int = ones(varargin)
            sizes = cell2mat(varargin);
            int = interval(ones(sizes));
        end
        function int = unit(varargin)
            sizes = cell2mat(varargin);
            if length(sizes) == 1
                A = zeros([sizes,1]);
                B = ones([sizes,1]);
            else
                A = zeros(sizes);
                B = ones(sizes);
            end
            int = interval(A,B);
        end


        function test()
            interval.test1;
        end
        function test1()
            a = interval(0,1);
            b = interval(3,4);
            disp(a+b);
            disp(a-b);
            disp(a.*b);
            disp(a./b);
        end
    end
end
```

A.6 Code defining generic boxes and functions on boxes

```
classdef box < handle & matlab.mixin.Copyable
    %box class for subdivision algorithms

    properties
        boxdimensions %D*2 vector with min and max coordinate of
            %the dimensions in each row[xmin,xmax; ymin,ymax,...]

        length0 %D*1 logical vector keeping track of the directions of 0 length
            %i.e. representing a lower dimensional box / face in D-dimensional space

        depth = 0;
    end
    properties (NonCopyable)
        parent %parent box
        children %2*2*...*2 array containing 2^d many children boxes
            %children(1,1,...,1) corresponds to the child with minimal
            %coordinates in each direction
            %d is the number of directions in which the box does not have 0 length

        plotboxhandle %handle to the plotted box boundary
    end

    methods
        function this = box(boxdimensions,length0)
            %box(boxdimensions)
            if nargin > 0
                this.boxdimensions = boxdimensions;
                if nargin > 1
                    this.length0 = length0;
                else
                    this.length0 = boxdimensions(:,1) == boxdimensions(:,2);
                end
            end
        end
        function disp(this)
            if length(this) == 1
                disp(this.boxdimensions);
            else
                disp([num2str(size(this)),' matrix of boxes']);
            end
        end
        function boxes = split(this)
            if ~isempty(this.children) || isempty(this.boxdimensions)
                boxes = [];
%                 warning("no box split performed");
                return
            end
            D = size(this.boxdimensions,1);
            d = D-sum(this.length0);

            childrenarry(2^d) = copy(this);
            if D == 1 || d == 1
                childrenarry = reshape(childrenarry,[2*ones(1,d),1]);
            else
                childrenarry = reshape(childrenarry,2*ones(1,d));
```

```
            end

            vLim = 2*ones(1,d);
            v     = ones(1, d);
            ready = false;
            while ~ready
                Index = arrayindexing.sub2indV(vLim, v);
                %If v(i) = 1, then the child will get the smaller coordinate in dimension i
                vcomplete = ones(1,D);
                vcomplete(~this.length0) = v;
                childrenarry(Index).boxdimensions = [this.boxdimensions(:,1),sum(this.boxdimensions,2)/2]+((vcomplete-1)'.*(this.boxdimensions(:,2)-this.boxdimensions(:,1))/2)*[1,1];
                childrenarry(Index).length0 = this.length0;
                childrenarry(Index).depth = this.depth+1;
                childrenarry(Index).parent = this;

                % Update the index vector:
                [v,ready] = arrayindexing.updateindexvec(v,vLim);
            end

            this.children = childrenarry;
            this.plotboxhandle = [];
            boxes = childrenarry(:)';
        end

        function c = center(this)
            c = sum(this.boxdimensions,2)/2;
        end
        function r = radius(this)
            r = norm(this.center-this.boxdimensions(:,1));
        end
        function C = corners(this)
            D = size(this.boxdimensions,1);
            d = D-sum(this.length0);
            C = zeros(D,2^d);

            for i = 1:2^d
                v = arrayindexing.ind2subV(2*ones(1,d),i);
                corner = zeros(D,1);
                corner(this.length0) = this.boxdimensions(this.length0,1);
                vind = 0;
                for j = 1:D
                    if ~this.length0(j)
                        vind = vind+1;
                        corner(j) = (v(vind)==1)*this.boxdimensions(j,1)+(v(vind)==2)*this.boxdimensions(j,2);
                    end
                end
                C(:,i) = corner;
            end
        end
        function f = facets(this)
            D = size(this.boxdimensions,1);
            d = D-sum(this.length0);
            dind = d;
            for i = D:-1:1
                if ~this.length0(i)
                    for j = 2:-1:1
                        B = copy(this);
                        B.boxdimensions(i,3-j) = B.boxdimensions(i,j);
                        B.length0 = this.length0;
```

```
                        B.length0(i) = true;
                        f(dind,j) = B;
                    end
                    dind = dind-1;
                end
            end
        end

        function Bscaled = scale(this,factor)
            classtype = class(this);
            scaledboxdimensions =
    ↪  interval(this.boxdimensions(:,1),this.boxdimensions(:,2)).scale(factor);
            extractedbounds = scaledboxdimensions.extractbounds; %#ok<NASGU>
            Bscaled = eval([classtype,'(scaledboxdimensions.extractbounds);']);
            Bscaled.length0 = this.length0;
        end

        %Returns a box without the removed dimension
        function Bnew = removedim(this,dim)
            classtype = class(this);
            dims = this.boxdimensions;
            len0 = this.length0;
            dims(dim,:) = [];
            len0(dim,:) = [];
            Bnew = eval([classtype,'(dims);']);
            Bnew.length0 = len0;
        end

        %Returns a box with an inserted dimension at the specified location
        %Inserts to the first/last position if the requested position is
        %out of bounds
        function Bnew = insertdim(this,dim,rowdims)
            classtype = class(this);
            dims = this.boxdimensions;
            len0 = this.length0;
            if dim < 1
                dim = 1;
            elseif dim > length(this.boxdimensions) + 1
                dim = length(this.boxdimensions) + 1;
            end
            dims = [dims(1:dim-1,:); rowdims; dims(dim:end,:)];
            len0 = [len0(1:dim-1,:); rowdims(1) == rowdims(2); len0(dim:end,:)];
            Bnew = eval([classtype,'(dims);']);
            Bnew.length0 = len0;
        end

        function Q = leaves(this)
            if isempty(this.children)
                Q = this;
            else
                Q = [];
                for i = 1:numel(this.children)
                    Q = [Q,leaves(this.children(i))]; %#ok<*AGROW>
                end
            end
        end
        function int = interval(this)
            int = interval(this.boxdimensions(:,1),this.boxdimensions(:,2));
        end

        function str = box2str(this)
```

```
        str = int2str(this.interval);
    end

    function Bcap = cap(this,B)
        %to be implemented
        Bcap = this;
    end
    function Bcup = cup(this,B)
        %to be implemented
        Bcup = this;
    end

    function varargout = coord(this)
        %Splits the box into its coordinate interval components
        %For 3D boxes:
        %[x,y,z] = B.splitcoordinates
        D = size(this.boxdimensions,1);
        varargout = cell(D,1);
        intvec = this.interval;
        output = mat2cell(intvec(:),ones(D,1))';
        [varargout{:}] = deal(output{:});
    end
    function plotsubdivision(this,ax,varargin)
        if isempty(this.children)
            this.plotbox(ax,varargin{:});
        else
            for i = 1:numel(this.children)
                this.children(i).plotsubdivision(ax,varargin{:});
            end
        end
    end
    function h = plotbox(this,ax,varargin)
        D = size(this.boxdimensions,1);
        d = D-sum(this.length0);
        switch d
            case 1
                %1-dimensional box possibly in higher dimensions
                corners = this.corners;
                switch D
                    case {1,2}
                        h = scatter(corners(1,:),corners(2,:),varargin);
                    case 3
                        h = scatter3(corners(1,:),corners(2,:),corners(3,:),varargin);
                end
            case 2
                %plot the box's edges
                corners = this.corners;
                corners = corners(:,[1,2,4,3,1]);
                switch D
                    case 2
                        h = plot(ax,corners(1,:),corners(2,:),varargin{:});
                    case 3
                        h = plot3(ax,corners(1,:),corners(2,:),corners(3,:),varargin{:});
                end
            case 3
                %plot the box's edges
                corners = this.corners;
                %collect the vertex indices making up the edges
                eind = [1,2;...
                        2,4;...
                        4,3;...
```

```
                                3,1;...
                                5,6;...
                                6,8;...
                                8,7;...
                                7,5;...
                                1,5;...
                                2,6;...
                                3,7;...
                                4,8];
                        XYZ = zeros(3,36);
                        for i = 1:12
                            XYZ(:,[3*i-2,3*i-1,3*i]) = [corners(:,eind(i,:)),[NaN;NaN;NaN]];
                        end
                        h = plot3(ax,XYZ(1,:),XYZ(2,:),XYZ(3,:),varargin{:});
                end
                this.plotboxhandle = h;
            end
            function zoom(this,ax)
                plotinterval = this.boxdimensions;
                plotinterval(this.length0,:) = this.boxdimensions(this.length0,:)+0.5*[-ones(sum(this.length0),1),ones(sum(this.length0),1)];
                plotinterval = plotinterval';
                plotinterval = reshape(plotinterval(:),1,[]);
                axis(ax,plotinterval);
            %                     r1 = boundingbox(1,2)-boundingbox(1,1);
            %                     r2 = boundingbox(2,2)-boundingbox(2,1);
            %                     pbaspect([r1, r2, max(r1,r2)])
                daspect(ax,[1 1 1]);
            end
%           function fillbox(this,axeshandle,color)
%               switch size(this.boxdimensions,1)
%                   case 2
%                       X = [this.boxdimensions(1,1);this.boxdimensions(1,2);this.boxdimensions(1,2);this.boxdimensions(1,1);this.boxdimensions(1,1)];
%                       Y = [this.boxdimensions(2,1);this.boxdimensions(2,1);this.boxdimensions(2,2);this.boxdimensions(2,2);this.boxdimensions(2,1)];
%                       this.filling = fill(axeshandle,X,Y,color);
%                   case 3
%               end
%           end

            function delete(this)
                if ~isempty(this.children)
                    while ~isempty(this.children)
                        this.children(1).delete;
                        this.children(1) = [];
                    end
                end
                if ~isempty(this.parent)
                    this.parent = [];
                end
            end


            %automatically transform into an interval for interval computations
            function result = plus(a,b)
                %a+b
                [a,b] = this.convert2interval(a,b);
                result = a+b;
            end
            function result = minus(a,b)
```

```
            %a-b
            [a,b] = this.convert2interval(a,b);
            result = a-b;
        end
        function result = uminus(a)
            %-a
            [a] = this.convert2interval(a);
            result = -a;
        end
        function result = uplus(a)
            %+a
            [a] = this.convert2interval(a);
            result = a;
        end
        function result = times(a,b)
            %a.*b
            [a,b] = this.convert2interval(a,b);
            result = a.*b;
        end
        function result = mtimes(a,b)
            %a*b
            [a,b] = this.convert2interval(a,b);
            result = a.*b;
        end
%         function result = rdivide(a,b)
%         end

    end
    methods(Static)
        function varargout = convert2interval(varargin)
            n = length(varargin);
            varargout = cell(1,n);
            for i = 1:n
                if isnumeric(varargin{1}) || isa(varargin{1},"interval")
                    varargout{i} = varargin{1};
                else
                    varargout{i} = varargin{1}.interval;
                end
            end
        end
        function [B,fig,ax] = test
            B = box([-1,1;-1,1;0,0],[false,false,true]);
            [fig,ax] = box.testsplit(B);
        end
        function [fig,ax] = testsplit(B)
            [fig,ax] = createfigure(B);
            B.split;
            B.children(1).split;
            B.children(1).children(2).split;
            B.plotsubdivision(ax,'k');
        end
    end
end
```

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