Signal Reconstruction from Zero-Crossings

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August 1998
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Abstract

We present a method for recovering (to within a constant factor) periodic, octave band-limited signals given the times of the zero-crossings. Recovery involves taking the singular-value decomposition of a size $N \times 2M$ matrix, where N is the number of zero-crossings within one period and M is product of the octave bandwidth and the period length. We also discuss approximate approaches which can be used to reconstruct aperiodic or very-long-period signals. Our algorithm achieves an inversion of Logan's theorem in the case where such is possible.

1 Logan's Theorem

Sampling theorems provide conditions under which continuous signals may be represented by countable sets of real numbers. In the usual setting, we agree on a regular grid of time points and provide samples of the signal amplitude at those times. The Nyquist-Shannon theorems tell us that a low-pass signal may be reconstructed exactly, so long as the times of samples are spaced at least as densely as half the period of the highest frequency. However, we could also agree upon a regular grid of *amplitude* levels and provide the *times* at which the signal crossed those levels. This has been dubbed *implicit sampling* by Bar-David [Bond and Cahn, 1958, Bar-David, 1974]; although there is no clear theory relating the frequency content of the signal and the number of levels required. *Logan's theorem* [Logan, Jr., 1977] addresses a special case of this general problem of signal reconstruction from level crossings. It states that if a signal is band-limited to a single octave then the times of the zero crossings are sufficient to reconstruct the signal – to within a constant factor of course.[1] For Nyquist-style sampling (uniform

[1]There are some additional important technical caveats. The signal must also have

in time), the reconstruction procedure is well known and involves simply convolving the given samples with a particular kernel (the *sinc* function) that performs the required interpolation. However, to our knowledge no practical inversion scheme exists for Logan's theorem.

All strictly band-limited signals must be infinite in time. Such signals are either periodic or aperiodic. If an infinite signal is aperiodic, an infinite number of zero-crossing times must be measured and the signal itself requires an infinite amount of information to specify. However, if an infinite signal is periodic, only a finite number of zero-crossing times (those within one period) need to be measured and the signal is entirely specified by one period. Because Logan's paper also shows that signals *with* free zeros *cannot* be uniquely reconstructed from their zero-crossings, it is practical (and indeed meaningful) to consider reconstruction only of *periodic signals with no free zeros.*

In what follows, we address exactly this problem. We present a method for recovering (to within a constant factor) periodic octave band-limited signals (with no free zeros) given the times of the zero-crossings within the period. Recovery requires taking the singular-value decomposition of a size $N \times 2M$ matrix, where N is the number of zero-crossings and M is product of the octave bandwidth and the period length. We also discuss approximate approaches which can be used to reconstruct aperiodic or very-long-period signals.

2 Bandlimited Periodic Signals

Any signal $s(t)$ that is periodic with period T can be written as an infinite Fourier series in the (co)sinusoids of real frequency $f_0 = 1/T$ and harmonics of frequencies hf_0 where $h = 2, 3, \ldots$. If the signal is also band limited, it can be written as a finite series containing only those harmonics which lie in the band of interest. In particular, a signal of period T that is band limited to the octave $\pm[f_{\min}, 2f_{\min})$ in real-frequency space can be written in the form:

$$s(t) = \sum_{k=1}^{M} [a_k \cos(2\pi f_k t) + b_k \sin(2\pi f_k t)] \tag{1}$$

no *free zeros*. Free zeros are those zeros which can be removed without destroying the bandpass property of the signal. Logan showed that the free zeros of a signal are those zeros which it shares with its Hilbert transform. It also must have no coincident zeros of even multiplicity. In other words, under the convention $sign(0) = 0$ the function $sign(s(t))$ must be zero only when $s(t)$ changes sign. ???

in other words, as a series of pure tones with frequencies $f_k, k = 1 \dots M$ that are given[2] by:

$$f_1 = \lceil f_{\min}/f_0 \rceil f_0 \,, \tag{2a}$$

$$f_M = \lfloor 2f_{\min}/f_0 \rfloor f_0 \,, \qquad \text{and} \tag{2b}$$

$$f_{k+1} - f_k = f_0 \,. \tag{2c}$$

This arrangement is shown in figure 1.

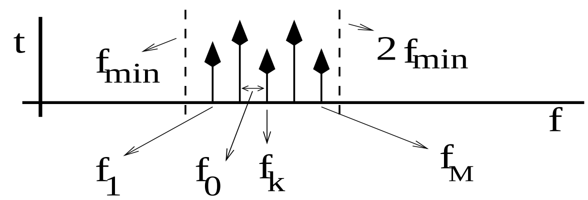

Figure 1: Fourier series for a band-limited periodic signal. The signal has period $T = 1/f_0$, is band-limited to $[f_{\min}, 2f_{\min})$ and the Fourier series has M terms.

Our reconstruction method is straightforward: we force the above finite series (1) for $s(t)$ to vanish at the measured zero-crossing times and solve for the resulting coefficients a_k and b_k. Each zero crossing gives an equation that is linear in the coefficients; the simultaneous solution of these linear equations yields the admissible set of coefficients. It turns out, not surprisingly, that the coefficients can be determined only to within a single degree of freedom, which represents the unknown scaling of $s(t)$. The octave band-limit restriction ensures that there are enough equations to determine the set of possible coefficients. (??? need to say a bit more here about that, see comment in Logan's paper top p.490)

3 Solution by Null Space Identification

Given T, $f_{\min}$, and N zero-crossing times $\mathbf{Z}_t = \{t_1, \dots, t_j, \dots, t_N\}$, the algorithm proceeds as follows:

1) Compute $f_0 = 1/T$ and f_k, $k = 1 \dots M$ as in (2).

2) Compute the matrix $X_{jk} = 2\pi t_j f_k$.

[2] The notation $\lceil x \rceil$ denotes the smallest integer greater than *or equal to* x ("ceil") and $\lfloor x \rfloor$ is the greatest integer less than x ("floor"). Notice that if x is an integer $\lceil x \rceil = x$ while $\lfloor x \rfloor = x - 1$.

3) Find the null space of the $N \times 2M$ matrix $[\cos(X) \sin(X)]$. (In the absence of noise, this matrix will be rank deficient by exactly one dimension, in other words it will have rank $\min(N, 2M) - 1$ and its null space will be one-dimensional.) Let the first M components of this null space vector be $\hat{a}$ and the last M components be $\hat{b}$.
(Note: the null space is the set of all coefficients for which the above Fourier series (1) exactly vanishes at the times $\mathbf{Z}_t$.)

4) The reconstructed signal (up to an unknown constant factor) is

$$\hat{s}(t) = \sum_{k=1}^{M} \left[\hat{a}_k \cos(2\pi f_k t) + \hat{b}_k \sin(2\pi f_k t) \right] \tag{3}$$

Finding the null space is the only nontrivial computation. Numerically, it is achieved using the singular-value decomposition[3] (SVD), taking the eigenvector corresponding to the smallest singular-value:

3.1) Compute the SVD of the $N \times 2M$ matrix $[\cos(X) \sin(X)]$ to give U, V, σ (where the columns of U are unit vectors which form an orthonormal basis for $\Re^N$, the columns of V are unit vectors which form an orthonormal basis for $\Re^{2M}$, and σ is a vector of the $\min(N, 2M)$ singular-values).

3.2) The null space which defines $\hat{a}$ and $\hat{b}$ is the column of V with smallest singular-value. (Ideally the smallest singular-value would be zero, but due to noise it will be a small but finite value.)

Figure 2 shows the result of applying this algorithm to reconstruct a simple synthetic signal.[4]

4 Practical Recovery Strategies

If we are accurately given the zero-crossing times, the period, and the octave to which its power is limited then the reconstruction of a signal is just a

[3]It is difficult to provide a single reference for the SVD, but a common early citation is [Eckart and Young, 1939]. The original numerical routines by Golub and Reinsch were published in Chapter I.10 of [Wilkinson and Reinsch, 1971]. A good general reference is [Golub and Loan, 1989] and citations therein especially the intriguing [Beltrami, 1873].

[4]The signal used in the example is $s(t) = cos(2\pi 11x) + (1/2)sin(2\pi 11x) + (1/33)cos(2\pi 12x) + (1/4)sin(2\pi 12x) + cos(2\pi 13x) + (1/8)sin(2\pi 13x) + cos(2\pi 14x) + (1/7)sin(2\pi 14x) + (1/22)cos(2\pi 15x) + (1/3)sin(2\pi 15x) + cos(2\pi 16x) + (1/12)sin(2\pi 16x) + cos(2\pi 17x) + (1/40)sin(2\pi 17x) + cos(2\pi 18x) + (1/2)sin(2\pi 18x) + (1/3)cos(2\pi 19x) + (1/2)sin(2\pi 19x)$.

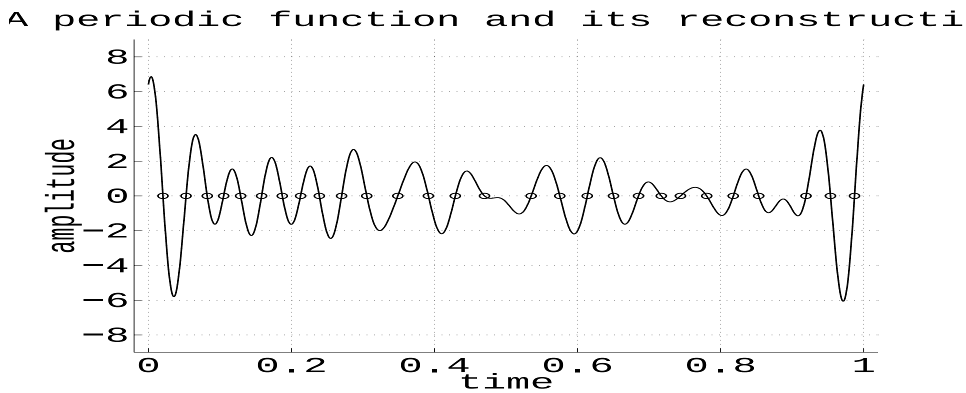

Figure 2: Reconstruction of the one-second period of a signal band limited to [10,20) Hz from its zero crossings. The reconstruction and original signal are plotted on top of each other; no difference can be seen. The arbitrary scaling of the reconstruction has been chosen to match the original signal. The inputs to the reconstruction algorithm are the zero-crossing times, the period (1 sec) and the octave (10 Hz).

direct application of the SVD technique described above. Furthermore, the spectrum of the singular-values contains important diagnostic information about the quality of the reconstruction. In particular, the ratio of the two smallest singular-values indicates the consistency of the input information. If this ratio is large (i.e. many orders of magnitude), then the null space is very nearly one-dimensional and the zero-crossing times, period, and octave are all consistent with each other.

However, in practical recovery situations some information may be noisy or missing entirely. For noisy information, the ratio of the two smallest singular-values may be near unity, indicating that the input information is inconsistent. Such inconsistency may be caused by uncertainty in the measurements of zero-crossing times, by imperfect knowledge of the period or octave, or by violation of the periodic, band-limited, or no free zero assumptions. Fortunately, however, we can attempt to clean up or fill in such noisy or missing information by applying the constraint that the solution must lie in a one-dimensional null space, i.e. that the smallest singular-value must be much less than the second smallest.

Consider, for example, a scenario in which the period of the signal is unknown. We know the zero-crossing times, measured during one period; thus we know that the period is greater than the last zero-crossing time.

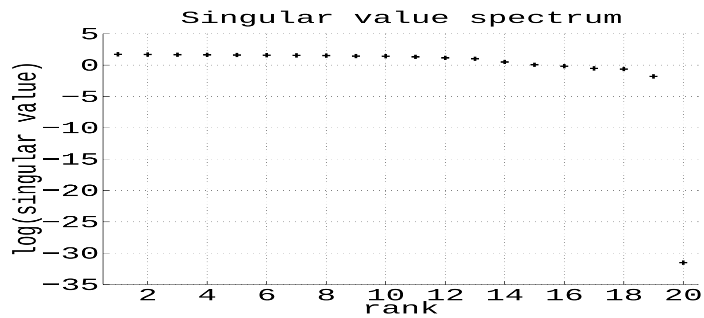

Figure 3: Singular-value spectrum for the reconstruction in figure 2. The log of the singular-values is plotted against their rank; the smallest is clearly far below the others indicating that the null space is indeed one-dimensional

We also know the period is not much more than the last zero-crossing plus a few times the mean inter-zero interval. (??? theorem about this) Within this range, we can search for the true period by computing the matrix X above for each candidate period and examining the ratio of the two smallest singular-values in the SVD of $[\cos(X)\sin(X)]$. We select the period that maximizes this ratio and use that period to reconstruct the signal. Figure 4 shows this ratio computed in a range around the last zero crossing for the same synthetic signal as in figure 2.

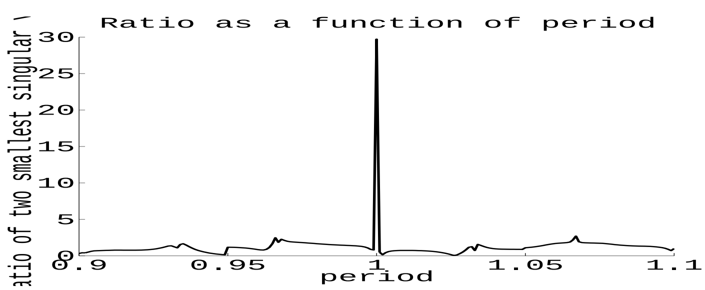

Figure 4: Searching for the true period. The natural log of the ratio of the two smallest singular-values is plotted against the period. The true period is unity; a peak in the ratio is visible around this value.

A similar one-dimensional search could be performed for an unknown octave. A good search range for the octave are those which contain the mean period – equal to twice the mean inter-zero interval. In principal, it might be possible to extend this approach to deal with noisy measurements of the zero-crossing times. Given a reliable estimate of the period and octave, a noisy zero-crossing could be corrected by moving it earlier and later in time to maximize the ratio of the two smallest singular values.

5 Long Period and Aperiodic Signals

[this section is still under heavy construction]

For band-limited signals which are either aperiodic or are periodic with extremely long periods, the approach outlined above is not practical. Despite Logan's warning that "the overall recovery procedure is obviously hopeless except in the case of periodic functions" (Logan, p.507), we would nonetheless like to devise a strategy to reconstruct such signals from only their zero-crossings. One possibility is to break long signals into many short sections and approximate each section by a periodic signal. A potential concern with this approach is that the original signals have power spectra which are very dense within the band of interest. The approximating periodic signals, on the other hand, have power spectra which are sparse combs. The essential question is whether a *single* dense spectrum can be approximated by a *sequence* of sparse comb spectra.

The problem can be posed as a communication game. Imagine a sender who has access to the original band-limited signal. He transmits to the receiver a sequence of real numbers, each one representing the time difference between successive zero-crossings. The job of the receiver is to reconstruct the signal using only these numbers and her prior knowledge of the octave in which the signal's power is band-limited.

The receiver is faced with two fundamental issues. First, she needs to decide when to cut the stream of times and attempt to reconstruct a section of the signal using the zero-crossings she has obtained so far. The longer she waits, the more computationally demanding her task will be. But the finer will be the spacing of the harmonics she can use in her reconstruction, and so the better she will be able to approximate the dense spectrum of the true signal. Second, once she has decided on a batch of zero-crossings with which to attempt a reconstruction, she must decide on the fictitious period of the (periodic) signal that will approximate the true signal section. Having made these two decisions, she can then apply the algorithm above and attempt to recover a portion of the original signal.

Acknowledgements

The authors thank Gayle Wittenberg for useful discussions in the early stages of this work. Wilhelm Schlag provided useful information about the maximum and minimum times between zero crossings of real trigonometric series.

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