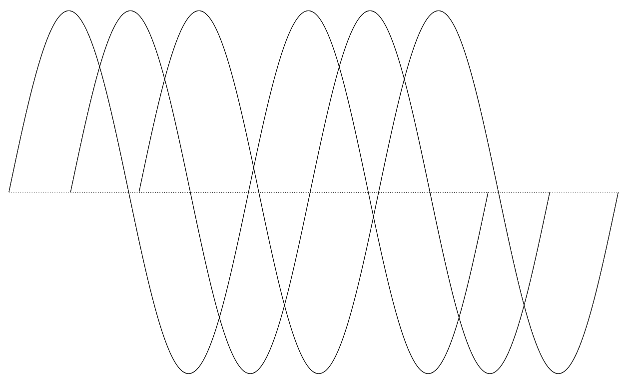
Adding Up Oscillations

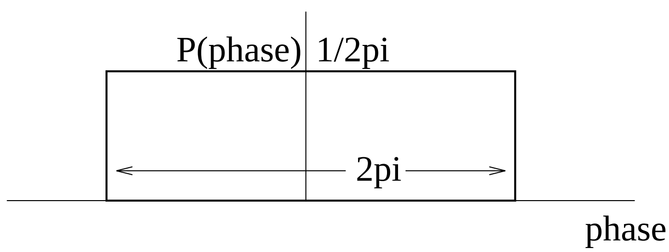
- Imagine taking the average of many sinusoids which all have the same frequency and amplitude but which have different phases



- Q: What should we expect the average to look like ?
- A: It depends on two things:
  - How many of them we add up.
  - How their phases are distributed.
- Let us ask how *big* we should expect the average to be – that is how do the amplitude and power of the *average of the average* scale ?

Two Extreme Cases

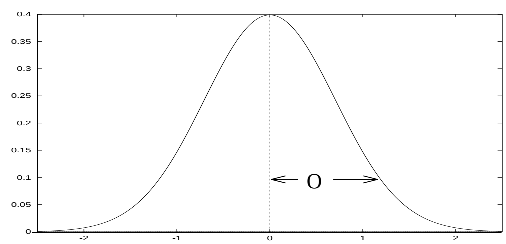
- Assume we are adding up oscillations of unit power.
- The obvious case: all the oscillations are perfectly synchronized.
- In this case, the answer is easy: the amplitude and power of the average are both equal to 1.
- The other obvious case: all the phases are random – no synchrony at all.



- In this case:
  The amplitude of the average scales as:
  $$\frac{1}{\sqrt{N}}$$
  and the power of the average scales as:
  $$\frac{1}{N}$$

A More General Case

- What if there is some synchrony but not perfect synchrony ?
- Model this case with phases that are gaussian distributed about some mean phase. This is "synchrony with slop":



- In this case:
  The amplitude of the average scales as:

$$\sqrt{\frac{1}{N} + \left(1 - \frac{1}{N}\right) e^{-\sigma^2}}$$

  and the power of the average scales as:

$$\frac{1}{N} + \left(1 - \frac{1}{N}\right) e^{-\sigma^2}$$

- Adding up sine waves of equal amplitude and frequency but different phases is like adding up vectors from the unit circle.
- If we add up vectors of unit norm then the norm-squared of the resultant vector is the *power* of the resulting waveform and the norm of the resultant vector is the *amplitude*.
- Consider power:

$$\langle \overline{x}^2 + \overline{y}^2 \rangle$$

$$\langle \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2 + \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2 \rangle$$

$$\frac{1}{N}\langle x_i^2 + y_i^2 \rangle + \left(1 - \frac{1}{N}\right)\langle x_i \rangle \langle x_j \rangle + \left(1 - \frac{1}{N}\right)\langle y_i \rangle \langle y_j \rangle$$

$$\frac{1}{N} + \left(1 - \frac{1}{N}\right)\left(\langle x_i \rangle \langle x_j \rangle + \langle y_i \rangle \langle y_j \rangle\right)$$

where $\langle x_i^2 + y_i^2 \rangle = 1$ since all the vectors we add up have unit norm.

More Mathematics

- What about $\langle x_i \rangle$ and $\langle y_i \rangle$?
  Move to the complex plane and consider:
  $$\langle x_i \rangle = Im[\langle e^{j\theta} \rangle] \qquad \langle y_i \rangle = Re[\langle e^{j\theta} \rangle]$$
- For phases perfectly synchronized at 0, $\langle e^{j\theta} \rangle = 1$ and so $P = 1$.
- For phases θ uniformly distributed over the circle, $\langle e^{j\theta} \rangle = 0$ and so $P = \frac{1}{N}$.
- For phases gaussian distributed about 0,
  do the integral:
  $$\int_{-\infty}^{+\infty} \frac{e^{-\theta^2/\sigma^2}}{\sqrt{2\pi\sigma}} e^{j\theta} d\theta$$
  to get $\langle x_i \rangle = 0$ and $\langle y_i \rangle = e^{-\sigma^2}$.