Lecture 7:

Fully Observed Trees

February 1, 2006

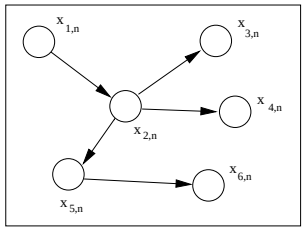
Directed Tree Graphical Models 1

- Directed trees are DAGMs in which each variable x_i has exactly one other variable as its parent x_{π_i} except the "root" x_{root} which has no parents. Thus, the probability of a variable taking on a certain value depends only on the value of its parent:

$$p(\mathbf{x}) = p(x_{\text{root}}) \prod_{i \neq \text{root}} p(x_i|x_{\pi_i})$$

- Trees are the next step up from assuming independence. Instead of considering variables in isolation, consider them in pairs.

NB: each node (except root) has exactly one parent, but nodes may have more than one child.



Undirected Tree Graphical Models 2

- Undirected trees are connected, acyclic graphs with exactly (D-1) edges if there are D nodes (variables).
- For undirected trees, the cliques are all pairs of connected nodes.

$$p(\mathbf{x}) = \frac{1}{Z} \prod_i \psi_i(x_i, x_{\pi_i})$$

where we can make $Z = 1$ with the choice $\psi_i = p(x_i|x_{\pi_i})$ except for one clique involving the root: $\psi_j = p(x_r)p(x_j|x_{\pi_j})$

- Trees have no "explaining-away" (converging arrows). Therefore, d-separation and regular separation are equivalent.
- Directed and undirected trees are equivalent and the choice of root is arbitrary (for fully observed models).
- Another characterization of trees: there is exactly one path between any pair of nodes (without doubling back).

Likelihood function 3

- Notation:
  $\mathbf{y}_i \equiv$ a node x_i and its single parent x_{π_i}.
  $\mathbf{V}_i \equiv$ set of joint configurations of node i and its parent x_{π_i}
  ($\mathbf{y}_{\text{root}} \equiv x_{\text{root}}$ and $\mathbf{V}_{\text{root}} \equiv \mathbf{v}_{\text{root}}$)
- Directed model likelihood:

$$\begin{aligned}
\ell(\theta; \mathcal{D}) &= \sum_n \log p(\mathbf{x}^n) = \sum_n \left[\log p_r(x_r^n) + \sum_{i \neq r} \log p(x_i{}^n|x_{\pi_i}{}^n) \right] \\
&= \sum_n \sum_i \sum_{\mathbf{v} \in \mathbf{V}_i} [\mathbf{y}_i^n = \mathbf{v}] \log p_i(\mathbf{v}) \qquad \text{indicator trick} \\
&= \sum_i \sum_{\mathbf{v} \in \mathbf{V}_i} N_i(\mathbf{v}) \log p_i(\mathbf{v})
\end{aligned}$$

where $N_i(\mathbf{v}) = \sum_n [\mathbf{y}_i^n = \mathbf{v}]$ and $p_i(\mathbf{v}_i) = p(x_i|x_{\pi_i})$.

- Undirected model likelihood:

$$\begin{aligned}\ell(\theta;\mathcal{D}) &= \sum_n \log \prod_i \psi_i(\mathbf{y}_i^n) \\ &= \sum_n \sum_i \sum_{\mathbf{v}\in\mathbf{V}_i} [\mathbf{y}_i^n = \mathbf{v}] \log \psi_i(\mathbf{v}) \\ &= \sum_i \sum_{\mathbf{v}\in\mathbf{V}_i} N_i(\mathbf{v}) \log \psi_i(\mathbf{v})\end{aligned}$$

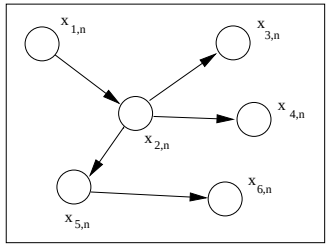
  where $N_i(\mathbf{y}) = \sum_n [\mathbf{y}_i^n = \mathbf{y}]$ and $\psi_i(\mathbf{y}_i) = p(x_i|x_{\pi_i})$.
  (Except for one clique involving the root: $\psi_j = p(x_r)p(x_j|x_{\pi_j})$)
- Directed and undirected likelihoods are the same!
- Trees are in the exponential family with $\mathbf{y}_i$ as sufficient statistics.

- Trees are just a special case of fully observed graphical models.
- For discrete data x_i with values v_i, each node stores a conditional probability table (CPT) over its values given its parent's value. The ML parameter estimates are just the empirical histograms of each node's values given its parent:

$$p^*(x_i = v_i|x_{\pi_i} = v_j) = \frac{N(x_i = v_i, x_{\pi_i} = v_j)}{\sum_{\mathbf{v}_i} N(x_i = v_i, x_{\pi_i} = v_j)} = \frac{N_i(\mathbf{y}_i)}{N_{\pi_i}(v_j)}$$

  except for the root which uses marginal counts $N_r(v_r)/N$.
- For continuous data, the most common model is a two-dimensional Gaussian at each node. The ML parameters are just to set the mean of $p_i(\mathbf{y}_i)$ to be the sample mean of $[x_i; x_{\pi_i}]$ and the covariance matrix to the sample covariance.
- In practice we should use some kind of smoothing/regularization.

- What about the tree structure (links)?
  How do we know which nodes to make parents of which?



- Bold idea: how can we also *learn* the optimal structure?
  In principle, we could search all combinatorial structures, for each compute the ML parameters, and take the best one.
- But is there a better way? Yes. It turns out that structure learning in tree models can be converted to a good old computer science problem: maximum weight spanning tree.

- Let us rewrite the likelihood function:

$$\begin{aligned}\ell(\theta;\mathcal{D}) &= \sum_{\mathbf{x}\in\mathbf{V}_{\text{all}}} N(\mathbf{x}) \log p(\mathbf{x}) \\ &= \sum_{\mathbf{x}} N(\mathbf{x}) \left(\log p(\mathbf{x}_r) + \sum_{i\neq r} \log p(x_i|x_{\pi_i}) \right)\end{aligned}$$

- ML parameters, are equal to the observed frequency counts $q(\cdot)$:

$$\begin{aligned}\frac{\ell^*}{N} &= \sum_{\mathbf{x}\in\mathbf{V}_{\text{all}}} q(\mathbf{x}) \left(\log q(\mathbf{x}_r) + \sum_{i\neq r} \log q(x_i|x_{\pi_i}) \right) \\ &= \sum_{\mathbf{x}} q(\mathbf{x}) \left(\log q(\mathbf{x}_r) + \sum_{i\neq r} \log \frac{q(x_i, x_{\pi_i})}{q(x_{\pi_i})} \right) \\ &= \sum_{\mathbf{x}} q(\mathbf{x}) \sum_{i\neq r} \log \frac{q(x_i, x_{\pi_i})}{q(x_i)q(x_{\pi_i})} + \sum_{\mathbf{x}} q(\mathbf{x}) \sum_i \log q(\mathbf{x}_i)\end{aligned}$$

- NB: second term does not depend on structure.

- Each term in sum $i \neq r$ corresponds to an edge from i to its parent.

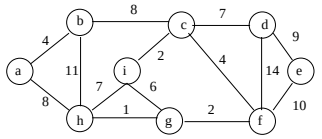
$$\begin{aligned}\frac{\ell^*}{N} &= \sum_{\mathbf{x}} q(\mathbf{x}) \sum_{i \neq r} \log \frac{q(x_i, x_{\pi_i})}{q(x_i) q(x_{\pi_i})} + C \\ &= \sum_{i \neq r} \sum_{x_i, x_{\pi_i}} q(x_i, x_{\pi_i}) \log \frac{q(x_i, x_{\pi_i})}{q(x_i) q(x_{\pi_i})} + C \\ &= \sum_{i \neq r} \sum_{\mathbf{y}_i} q(\mathbf{y}_i) \log \frac{q(\mathbf{y}_i)}{q(x_i) q(x_{\pi_i})} + C \\ &= \sum_{i \neq r} W(i; \pi_i) + C\end{aligned}$$

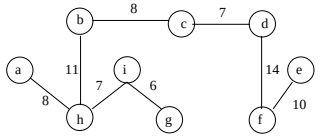
where the edge weights W are defined by *mutual information*:

$$W(i;j) = \sum_{x_i, x_j} q(x_i, x_j) \log \frac{q(x_i, x_j)}{q(x_i) q(x_j)}$$

- So overall likelihood is sum of weights on edges that we use. We need the maximum weight spanning tree.

- To find the maximum weight spanning tree A on a graph with nodes U and weighted edges E:
  1. $A \leftarrow$ empty
  2. Sort edges E by nonincreasing weight: $e_1, e_2, \ldots, e_K$.
  3. for $k = 1$ to K $\{A \mathrel{+}= e_k$ unless doing so creates a cycle$\}$





We can now completely solve the tree learning problem:

1. Compute the marginal counts $q(x_i)$ for each node and pairwise counts $q(x_i, x_j)$ for all pairs of nodes.
2. Set the weights to the mutual informations:

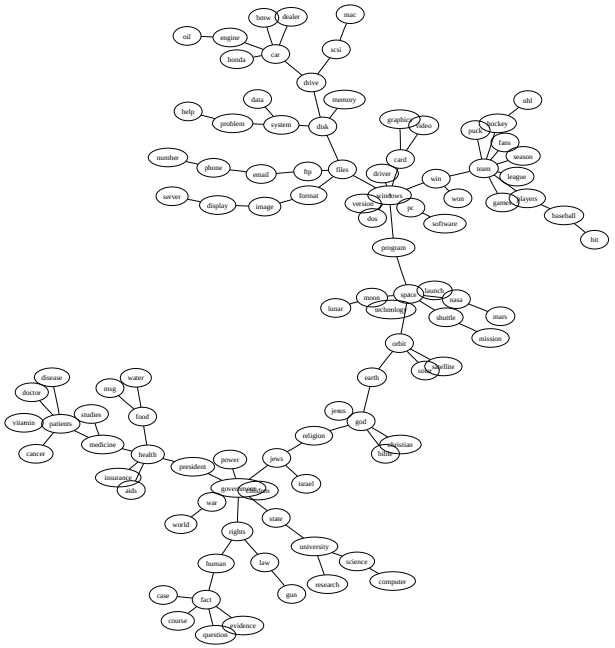
$$W(i;j) = \sum_{x_i, x_j} q(x_i, x_j) \log \frac{q(x_i, x_j)}{q(x_i) q(x_j)}$$

3. Find the maximum weight spanning tree A=MWST(W).
4. Using the undirected tree A chosen by MWST, pick a root arbitrarily and orient the edges away from the root. Set the conditional functions to the observed frequencies:

$$p(x_i | x_{\pi_i}) = \frac{q(x_i, x_{\pi_i})}{\sum_{x_i} q(x_i, x_{\pi_i})} = \frac{q(x_i, x_{\pi_i})}{q(x_{\pi_i})}$$

- *Any* directed tree consistent with the undirected tree found by the algorithm above will assign the same likelihood to any dataset.
- Amazingly, as far as likelihood goes, the root is arbitrary. We can just pick one node and orient the edges away from it. Or we can work with undirected models.
- For continuous nodes (e.g. Gaussian), the situation is similar, except that computing the mutual information requires an integral.
- Mutual information is the *Kullback-Leibler* divergence (cross-entropy) between a distribution and the product of its marginals. Measures how far from independent the joint distribution is.

$$W(i;j) = \mathrm{I}[x_i; x_j] = \mathrm{KL}[q(x_i, xj) \| q(x_i) q(x_j)]$$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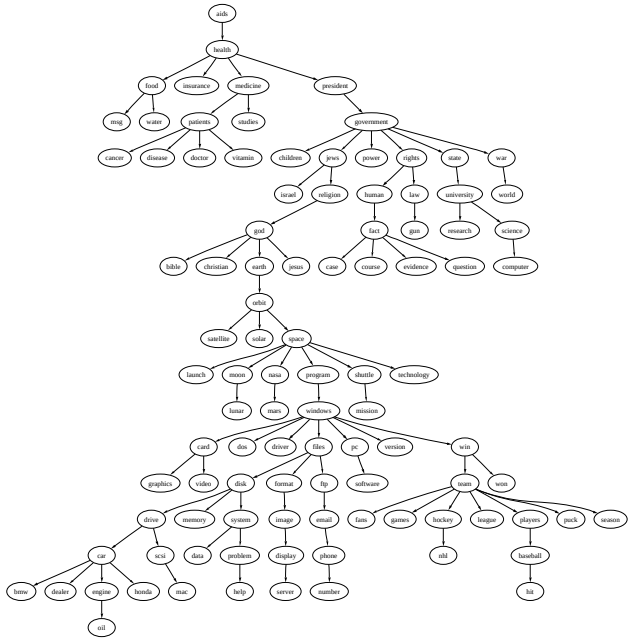
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