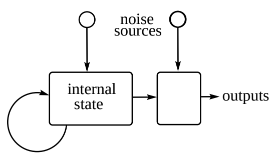
Lecture 16:

Markov and Hidden Markov Models

March 8, 2006

Probabilistic Models for Time Series

- Generative models for time-series:
  To get interesting variability need *noise*.
  To get correlations across time, need some system *state*.



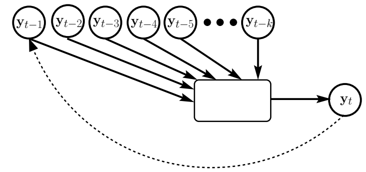
- Time: discrete
  States: discrete or continuous
  Outputs: discrete or continuous
- Today: discrete state
  similar to finite state automata; Moore/Mealy machines

Markov Models

- Use past as state. Next output depends on previous output(s):

$$\mathbf{y}_t = f[\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots]$$

order is number of previous outputs



- Add noise to make the system probabilistic:

$$p(\mathbf{y}_t | \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots, \mathbf{y}_{t-k})$$

- Markov models have two problems:
  - need big order to remember past "events"
  - output noise is confounded with state noise

Learning Markov Models

- The ML parameter estimates for a simple Markov model are easy:

$$p(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_T) = p(\mathbf{y}_1 \ldots \mathbf{y}_k) \prod_{t=k+1}^{T} p(\mathbf{y}_t | \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots, \mathbf{y}_{t-k})$$

$$\log p(\{\mathbf{y}\}) = \log p(\mathbf{y}_1 \ldots \mathbf{y}_k) + \sum_{t=k+1}^{T} \log p(\mathbf{y}_t | \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots, \mathbf{y}_{t-k})$$

- Each window of $k+1$ outputs is a training case for the model $p(\mathbf{y}_t | \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots, \mathbf{y}_{t-k})$.
- Example: for discrete outputs (symbols) and a 2nd-order markov model we can use the multinomial model:

$$p(y_t = m | y_{t-1} = a, y_{t-2} = b) = \alpha_{mab}$$

The maximum likelihood values for α are:

$$\alpha^*_{mab} = \frac{\mathrm{num}[t\ s.t.\ y_t = m, y_{t-1} = a, y_{t-2} = b]}{\mathrm{num}[t\ s.t.\ y_{t-1} = a, y_{t-2} = b]}$$

Hidden Markov Models (HMMs)

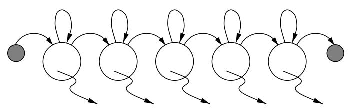
Add a latent (hidden) variable x_t to improve the model.

- HMM $\equiv$ " probabilistic function of a Markov chain":
  1. 1st-order Markov chain generates hidden state sequence (path):

     $$\mathsf{P}(x_{t+1} = j | x_t = i) = S_{ij} \qquad \mathsf{P}(x_1 = j) = \pi_j$$

  2. A set of output probability distributions $A_j(\cdot)$ (one per state) converts state path into sequence of observable symbols/vectors

     $$\mathsf{P}(\mathbf{y}_t = y | x_t = j) = A_j(y)$$



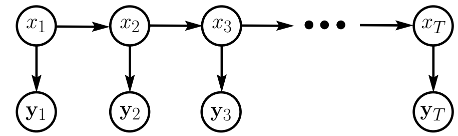
(state transition diagram)

- Even though hidden state seq. is 1st-order Markov, the output process is not Markov of *any* order [ex. 1111121111311121111131...]

Applications of HMMs

- Speech recognition.
- Language modeling.
- Information retrieval.
- Motion video analysis/tracking.
- Protein sequence and genetic sequence alignment and analysis.
- Financial time series prediction.
- ...

HMM Graphical Model



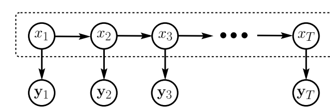
- Hidden states $\{x_t\}$, outputs $\{\mathbf{y}_t\}$
  Joint probability factorizes:

$$\mathsf{P}(\{x\}, \{\mathbf{y}\}) = \prod_{t=1}^{T} \mathsf{P}(x_t | x_{t-1}) \mathsf{P}(\mathbf{y}_t | x_t)$$
$$= \pi_{x_1} \prod_{t=1}^{T-1} S_{x_t, x_{t+1}} \prod_{t=1}^{T} A_{x_t}(\mathbf{y}_t)$$

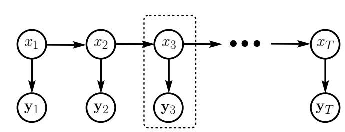
- NB: Data are *not* i.i.d.
  There is no easy way to use plates to show this model. (Why?)

Links to Other Models

- You can think of an HMM as:
  A Markov chain with stochastic measurements.



or
A mixture model with states coupled across time.



- The future is independent of the past given the present.
  However, conditioning on all the observations couples hidden states.

Probability of an Observed Sequence

- To evaluate the probability $\mathsf{P}(\{\mathbf{y}\})$, we want:

$$\mathsf{P}(\{\mathbf{y}\}) = \sum_{\{x\}} \mathsf{P}(\{x\},\{\mathbf{y}\})$$

$$\mathsf{P}(\text{observed sequence}) = \sum_{\text{all paths}} \mathsf{P}(\text{ observed outputs , state path })$$

- Looks hard! (#paths = #states^T). But joint probability factorizes:

$$\mathsf{P}(\{\mathbf{y}\}) = \sum_{x_1}\sum_{x_2}\cdots\sum_{x_T}\prod_{t=1}^{T} \mathsf{P}(x_t|x_{t-1})\mathsf{P}(\mathbf{y}_t|x_t)$$

$$= \sum_{x_1} \mathsf{P}(x_1)\mathsf{P}(\mathbf{y}_1|x_1)\sum_{x_2} \mathsf{P}(x_2|x_1)\mathsf{P}(\mathbf{y}_2|x_2)\cdots\sum_{x_T} \mathsf{P}(x_T|x_{T-1})\mathsf{P}(\mathbf{y}_T|x_T)$$

- By moving the summations inside, we can save a lot of work.

The forward (α) recursion

- We want to compute:

$$L = \mathsf{P}(\{\mathbf{y}\}) = \sum_{\{x\}} \mathsf{P}(\{x\},\{\mathbf{y}\})$$

- There is a clever "forward recursion" to compute the sum efficiently.

$$\alpha_j(t) = \mathsf{P}(\ \mathbf{y}_1^t\ ,\ x_t = j\ )$$
$$\alpha_j(1) = \pi_j A_j(\mathbf{y}_1)$$
$$\alpha_k(t+1) = \{\sum_j \alpha_j(t) S_{jk}\} A_k(\mathbf{y}_{t+1})$$

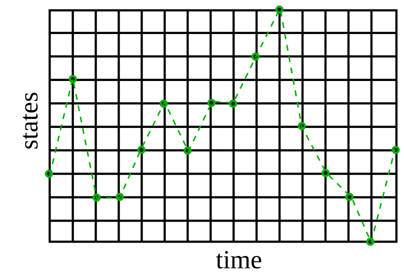
- Notation: $x_a^b \equiv \{x_a, \ldots, x_b\}$; $\mathbf{y}_a^b \equiv \{\mathbf{y}_a, \ldots, \mathbf{y}_b\}$
- This enables us to easily (cheaply) compute the desired likelihood L since we know we must end in some possible state:

$$L = \sum_k \alpha_k(T)$$

Bugs on a Grid

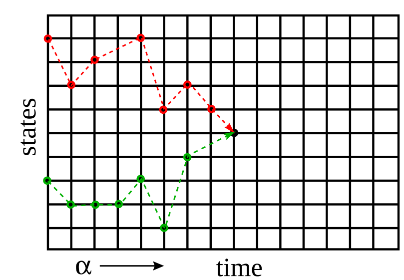
- Naive algorithm:

1. start bug in each state at t=1 holding value 0
2. move each bug forward in time: make copies & increment the value of each copy by transition prob. + output emission prob.
3. go to 2 until all bugs have reached time T
4. sum up values on all bugs



Bugs on a Grid - Trick

- Clever recursion:
  adds a step between 2 and 3 above which says: at each node, replace all the bugs with a single bug carrying the sum of their values



- This trick is called *dynamic programming*, and can be used whenever we have a summation, search, or maximization problem that can be set up as a grid in this way. The axes of the grid don't have to be "time" and "states".

Inference of Hidden States 12

- What if we we want to estimate the hidden states given observations? To start with, let us estimate a single hidden state:

$$
\begin{aligned}
p(x_t|\{\mathbf{y}\}) = \gamma(x_t) &= \frac{p(\{\mathbf{y}\}|x_t)p(x_t)}{p(\{\mathbf{y}\})} \\
&= \frac{p(\mathbf{y}_1^t|x_t)p(\mathbf{y}_{t+1}^T|x_t)p(x_t)}{p(\mathbf{y}_1^T)} \\
&= \frac{p(\mathbf{y}_1^t, x_t)p(\mathbf{y}_{t+1}^T|x_t)}{p(\mathbf{y}_1^T)} \\
p(x_t|\{\mathbf{y}\}) = \gamma(x_t) &= \frac{\alpha(x_t)\beta(x_t)}{p(\mathbf{y}_1^T)}
\end{aligned}
$$

$$
\text{where} \qquad
\begin{aligned}
\alpha_j(t) &= p(\, \mathbf{y}_1^t \,,\, x_t = j \,) \\
\beta_j(t) &= p(\mathbf{y}_{t+1}^T \mid x_t = j \,) \\
\gamma_i(t) &= p(x_t = i \mid \mathbf{y}_1^T)
\end{aligned}
$$

Forward-Backward Algorithm 13

- We compute these quantites efficiently using another recursion. Use total prob. of all paths going through state i at time t to compute the *conditional* prob. of being in state i at time t:

$$
\begin{aligned}
\gamma_i(t) &= p(x_t = i \mid \mathbf{y}_1^T) \\
&= \alpha_i(t)\beta_i(t)/L
\end{aligned}
$$

  where we defined:

$$\beta_j(t) = p(\mathbf{y}_{t+1}^T \mid x_t = j \,)$$

- There is also a simple recursion for $\beta_j(t)$:

$$
\begin{aligned}
\beta_j(t) &= \sum_i S_{ji}\beta_i(t+1)A_i(\mathbf{y}_{t+1}) \\
\beta_j(T) &= 1
\end{aligned}
$$

- $\alpha_i(t)$ gives total *inflow* of prob. to node (t, i)
  $\beta_i(t)$ gives total *outflow* of prob.

Forward-Backward Algorithm 14

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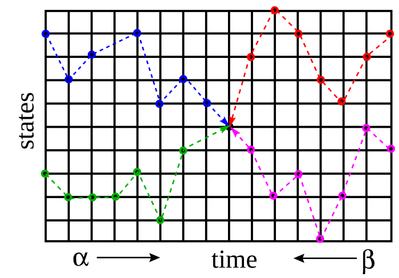


- Bugs again: we just let the bugs run forward from time 0 to t and backward from time T to t.
- In fact, we can just do one forward pass to compute all the $\alpha_i(t)$ and one backward pass to compute all the $\beta_i(t)$ and then compute any $\gamma_i(t)$ we want. Total cost is $O(K^2T)$.

Likelihood from Forward-Backward Algorithm 15

- Since $\sum_{x_t} \gamma(x_t) = 1$, we can compute the likelihood at any time using the results of the $\alpha - \beta$ recursions:

$$L = p(\{\mathbf{y}\}) = \sum_{x_t} \alpha(x_t)\beta(x_t)$$

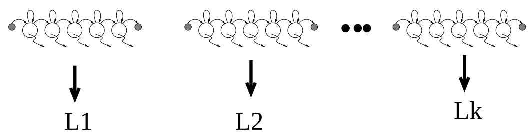
- In the forward calculation we proposed originally, we did this at the final timestep $t = T$:

$$L = \sum_{x_T} \alpha(x_T)$$

  because $\beta_T = 1$.

Using HMMs for Recognition

- Use many HMMs for recognition by:
  1. training one HMM for each class (requires *labelled* training data)
  2. evaluating probability of an unknown sequence under each HMM
  3. classifying unknown sequence: HMM with highest likelihood

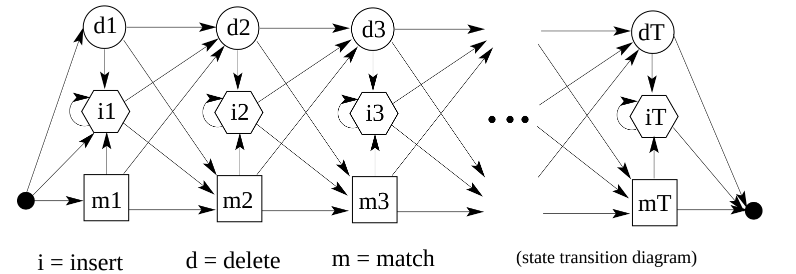


- This requires the solution of two problems:
  1. Given model, evaluate prob. of a sequence.
     (We can do this exactly & efficiently.)
  2. Give some training sequences, estimate model parameters.
     (We can find the local maximum of parameter space nearest our starting point using Baum-Welch (EM).)

Viterbi Decoding

- The numbers $\gamma_j(t)$ above gave the probability distribution over all states at any time.
- By choosing the state $\gamma_*(t)$ with the largest probability at each time, we can make a "best" state path. This is the path with the *maximum expected number of correct states*.
- But it *is not* the single path with the highest likelihood of generating the data. In fact it may be a path of prob. zero!
- To find the single best path, we do *Viterbi decoding* which is just Bellman's dynamic programming algorithm applied to this problem.
- The recursions look the same, except with $\max$ instead of $\sum$.
- Bugs once more: same trick except at each step kill all bugs but the one with the highest value at the node.

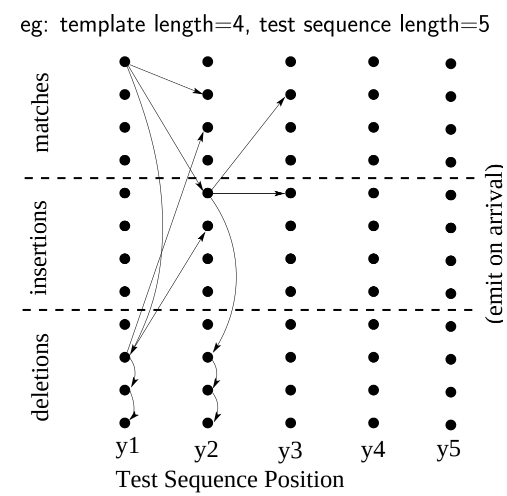
Profile (String-Edit) HMMs



- A "profile HMM" or "string-edit" HMM is used for probabilistically matching an observed input string to a stored template pattern with possible insertions and deletions.
- Three kinds of states: match, insert, delete.
  m_n – use position n in the template to match an observed symbol
  i_n – insert extra symbol(s) observations after template position n
  d_n – delete (skip) template position n

DP for Profile HMMs

- How do we fill in the costs for a DP grid using a string-edit HMM?
- Almost the same as normal except:
  - Now the grid is 3 times its normal height.
  - It is possible to move down without moving right if you move into a deletion state.

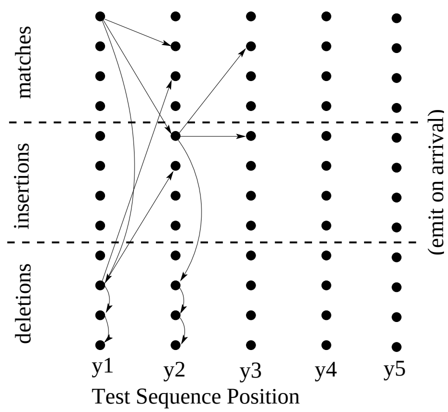


String-Edit HMM Grid Costs

$C_{x \to x'} = -\log T_{x,x'} - \log A_{x'}(\mathbf{y}_t)$ if x' is match or insert

$C_{x \to x'} = -\log T_{x,x'}$ if x' is a delete state

State $x \in \{m_n, i_n, d_n\}$ has nonzero transition probabilities only to states $x' \in \{m_{n+1}, i_n, d_{n+1}\}$.



Forward-Backward for Profile HMMs

- The equations for the delete states in profile HMMs need to be modified slightly, since they don't emit any symbols.
- For delete states k, the forward equations become:

$$\alpha_k(t) = \sum_j \alpha_j(t) S_{jk}$$

  which should be evaluated after the insert and match state updates.
- For all states, the backward equations become:

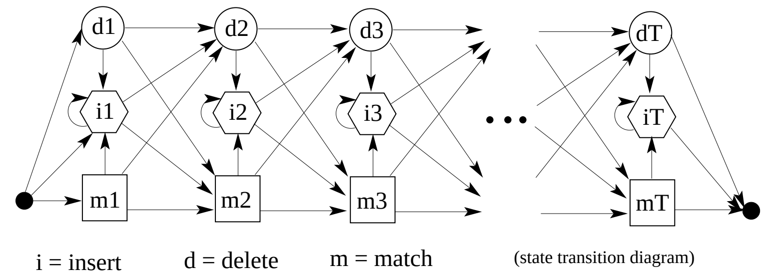
$$\beta_k(t) = \sum_{i \in \text{match,ins}} S_{ki} \beta_i(t+1) A_i(\mathbf{y}_{t+1}) + \sum_{j \in \text{del}} S_{kj} \beta_j(t)$$

  which should be evaluated first for delete states k; then for the rest.
- The gamma equations remain the same:

$$\gamma_i(t) = p(x_t = i \mid \mathbf{y}_1^T) = \alpha_i(t)\beta_i(t)/L$$

- Notice that each summation above contains only three terms, regardless of the number of states!

Profile HMMs have Linear Costs



- number of states = 3(length_template)
- Only insert and match states can generate output symbols.
- Once you visit or skip a match state you can never return to it.
- At most 3 destination states from any state, so S_{ij} very sparse.
- Storage/Time cost *linear* in #states, not quadratic.
- State variables and observations no longer in sync. (e.g. y1:m1 ; d2 ; y2:i2 ; y3:i2 ; y4:m3 ; . . .)

Initializing Forward-Backward for Profile HMMs

- The initialization equations for Profile HMMs also need to be fixed up, to reflect the fact that the model can only begin in states m_1, i_1, d_1 and can only finish in states m_N, i_N, d_N.
- In particular, $\pi_j = 0$ if j is not one of m_1, i_1, d_1.
- When initializing $\alpha_k(1)$, delete states k have zeros, and all other states have the product of the transition probabilities through only delete states up to them, plus the final emission probability.
- When initializing $\beta_k(T)$, the same kind of adjustment must be made.

- Markov ('13) and later Shannon ('48,'51) studied *Markov chains*.
- Baum et. al (BP'66, BE'67, BS'68, BPSW'70, B'72) developed much of the theory of "probabilistic functions of Markov chains".
- Viterbi ('67) (now Qualcomm) came up with an efficient optimal decoder for state inference.
- Applications to speech were pioneered independently by:
  - Baker ('75) at CMU (now Dragon)
  - Jelinek's group ('75) at IBM (now Hopkins)
  - communications research division of IDA (Ferguson '74 unpublished)
- Dempster, Laird & Rubin ('77) recognized a general form of the Baum-Welch algorithm and called it the *EM* algorithm.
- A landmark open symposium in Princeton ('80) hosted by IDA reviewed work till then.