CSC412/2506 – Assignment #1

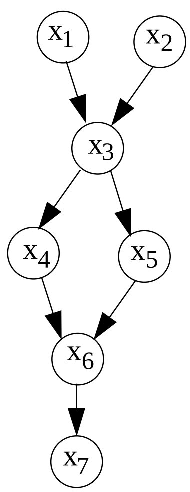
Due: Jan23, 2pm at the **START** of class
Worth: 10%
Late assignments not accepted.

1 Graphical Model Distributions

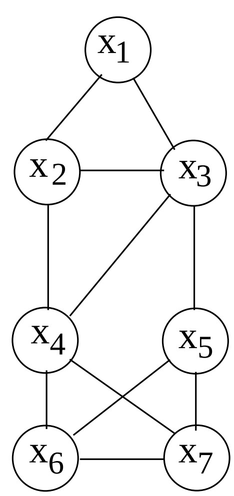
The figure below shows a directed graphical model (D) and an undirected graphical model (U) each representing a joint distribution over six random variables.

- For each of the following statements and each graphical model, prove whether the statement *must* be true of the distribution represented by the model, *could be true* but we don't know, or *cannot be true*.

  1. $\mathbf{x}_4$ is conditionally independent of $\mathbf{x}_5$ given $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$
  2. $\mathbf{x}_3$ is conditionally *dependent* on $\mathbf{x}_7$ given $\mathbf{x}_4$ and $\mathbf{x}_5$
  3. $\mathbf{x}_1$ is marginally independent of $\mathbf{x}_2$
  4. $\mathbf{x}_4$ is conditionally independent of $\mathbf{x}_5$ given $\mathbf{x}_3, \mathbf{x}_6$



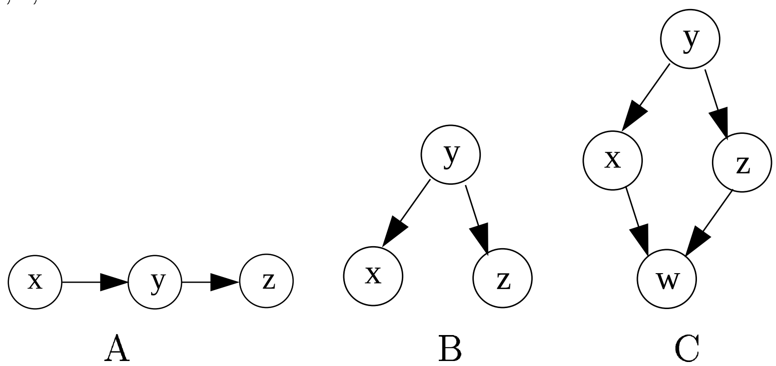
D



U

2 Directed Models

For each of the directed graphs A,B,C below:



- Give all possible *topological orderings* of the variables.
- List all conditional independencies implied by the directed model.
- If there is an undirected model which captures exactly the same conditional independencies, draw it.
  If not, remove the fewest possible edges from the directed model until it has an undirected model capturing the equivalent independencies, and show the new directed model and the corresponding undirected model.
- [Hard] Prove or disprove the following statement: If both the original directed graph and the undirected graph obtained by ignoring the direction of the edges are acyclic then there is an undirected model which captures exactly the same conditional independence assumptions as the directed model.

3 Complete Graphs

- Draw a directed graphical model on 5 variables which (a) can capture *any* joint distribution and (b) is acyclic.
- Can any edges be added to or removed from your graph and still preserve both the properties (a) and (b) above? If so, show the addition or removal, if not say why not.
- Draw an undirected graphical model on 4 variables which can capture *any* joint distribution.
  List all the maximal cliques.
- Can any edges be added to or removed from your graph and still preserve the above property?
  If so, show the addition or removal. If not, say why not.
- [Bonus] Prove or disprove the following statement: all directed acyclic graphical models on k variables which can capture any joint distribution have exactly $k^2/2 - k/2$ edges.
- [Extra Bonus] Prove or disprove the following statement: all undirected graphical models on k variables which can capture any joint distribution have exactly $k^2/2 - k/2$ edges.

4 Incompatible Conditionals

In this question you'll try to convince yourself that the representation

$$p(\mathbf{X}) = \prod_i \mathsf{P}(\mathbf{x}_i | \mathbf{x}_{\text{neighbours}(i)})$$

does not allow arbitrary conditional probabilities $\mathsf{P}(\mathbf{x}_i | \mathbf{x}_{\text{neighbours}(i)})$.

- Consider two binary variables $\mathbf{x}$ and $\mathbf{y}$ with the conditional distributions:

$$p(x = 1 | y = 0) = 3/4 \qquad p(x = 1 | y = 1) = 2/3$$
$$p(y = 1 | x = 0) = 2/3 \qquad p(y = 1 | x = 1) = 4/7$$

- Show that the function $f(x, y) = p(x|y)p(y|x)$ is not a valid joint distribution $p(x, y)$.
- Write down a valid joint distribution $p(x, y)$ that has the given conditionals.
- What condition must the joint $p(x, y)$ satisfy in order to be able to be written as a product of the conditionals $p(x|y)$ and $p(y|x)$?
  Show your reasoning.
- Write down two conditional distributions, like the ones above, on binary variables x and y, so that $p(x|y)p(y|x)$ *is* a valid joint. (Don't set any probabilities to zero or one.)