Lecture 20:

Hugin Inference Algorithm

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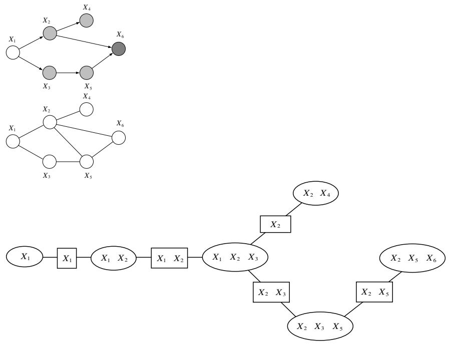
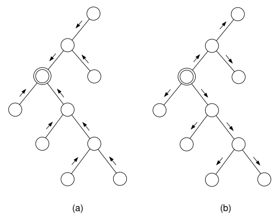
The Main Setup

- Three main steps:
  1. Pre-processing (compiling) the graphical model to prepare for inference: building the clique junction tree.
  2. Conditioning on the evidence.
  3. Marginalizing out the non-query nodes efficiently.



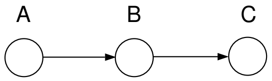


Adjusting Clique Potentials

- The goal of the junction tree algorithm is to adjust the clique potentials ψ_c given some evidence $\bar{\mathbf{x}}_E$ so that they are unnormalized marginals:

$$\psi_C(\mathbf{x}_C) = p(\mathbf{x}_C, \bar{\mathbf{x}}_E)$$

- Normally clique potentials *do not* correspond to marginals; some of them must be conditionals.
- Example: $\psi_{AB} = p(\mathbf{x}_A, \mathbf{x}_B) \quad ; \quad \psi_{BC} = p(\mathbf{x}_C|\mathbf{x}_B)$



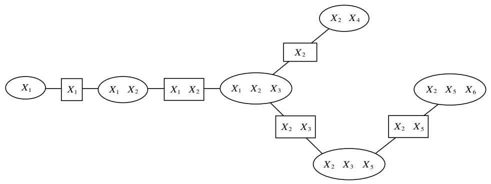
- We can always adjust the potentials to make them marginals (e.g. above we could multiply by $p(\mathbf{x}_B)$) but then the product of potentials will not be proportional to the joint probability.
- What can we do? Extend the representation!

Separator Potentials

- On each edge of the clique tree, we place a potential ϕ_S over the variables in the intersection of the two adjacent cliques it joins.
- These intersections are called *separator sets* and are themselves cliques (fully connected in the underlying graph) although of course they are no longer maximal.
- Now our representation of the joint probability is defined as:

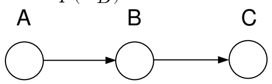
$$p(\mathbf{X}) = \frac{\prod_C \psi_C(\mathbf{x}_C)}{\prod_S \phi_S(\mathbf{x}_S)}$$

where the normalizer is absorbed into a special separator $\phi_{\oslash}$.



Extended Representation

- Now we can have clique potentials proportional to marginals *and* a representation of the joint distribution at the same time.
- e.g. $p(\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C) = \frac{p(\mathbf{x}_A,\mathbf{x}_B)p(\mathbf{x}_B,\mathbf{x}_C)}{p(\mathbf{x}_B)}$



- Furthermore, this extended representation still obeys the Hammersly-Clifford theorem, i.e. it still represents exactly the same family of distributions with the correct conditional independencies.
- Initialization? Set all separator potentials to be unity.
- What about division by zero? It will turn out that a separators is only zero if both clique potentials it is connected to are also zero. In this case we define the ratio to be zero.

Update Effects

- After performing the updates on the previous slide, we are guaranteed that ψ_V^{**} and ψ_W^{**} are consistent with respect to S:

$$\sum_{V\backslash S} \psi_V^{**} = \sum_{V\backslash S} \frac{\phi_S^{**}}{\phi_S^*}\psi_V^* = \frac{\phi_S^{**}}{\phi_S^*}\sum_{V\backslash S}\psi_V^* = \frac{\phi_S^{**}}{\phi_S^*}\phi_S^* = \phi_S^{**} = \sum_{W\backslash S}\psi_W^{**}$$

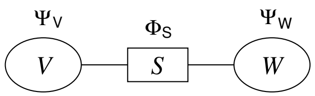
- But the updates leave the joint distribution $p(\mathbf{x}_W, \mathbf{x}_V)$ unchanged:

$$\frac{\psi_V^*\psi_W^*}{\phi_S^*} = \frac{\psi_V\psi_W\phi_S^*}{\phi_S\phi_S^*} = \frac{\psi_V\psi_W}{\phi_S}$$
$$\frac{\psi_V^{**}\psi_W^{**}}{\phi_S^{**}} = \frac{\psi_V^*\psi_W^*\phi_S^{**}}{\phi_S^*\phi_S^{**}} = \frac{\psi_V^*\psi_W^*}{\phi_S^*} = \frac{\psi_V\psi_W}{\phi_S}$$

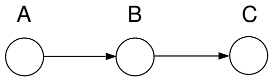
Local Consistency

- Since cliques overlap, some variables appear in more than one clique. If we sum out non-intersection variables, any pair of cliques must give same marginals for nodes they have in common.
- Let us focus on *local consistency*: how to make two adjacent clique agree on marginals over separator variables.
- Consider the following updates:

$$\phi_S^* = \sum_{V\backslash S}\psi_V \qquad \psi_W^* = \frac{\phi_S^*}{\phi_S}\psi_W \qquad \psi_V^* = \psi_V$$
$$\phi_S^{**} = \sum_{W\backslash S}\psi_W^* \qquad \psi_W^{**} = \psi_W^* \qquad \psi_V^{**} = \frac{\phi_S^{**}}{\phi_S^*}\psi_V^*$$



Example



With no evidence.
- initially: $\psi_{ab} = p(a,b) \quad \psi_{bc} = p(c|b) \quad \phi_b = 1$
- updates:

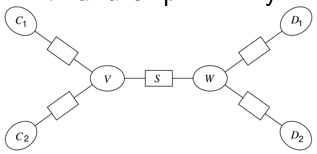
$$\phi_b^* = \sum_a p(a,b) = p(b)$$
$$\psi_{bc}^* = \frac{p(b)}{1}p(c|b) = p(b,c)$$

With evidence $a = 1$.
- initially: $\bar{\psi}_{ab} = p(\bar{a}=1,b) \quad \psi_{bc} = p(c|b) \quad \phi_b = 1$
- updates:

$$\phi_b^* = p(\bar{a}=1,b)$$
$$\psi_{bc}^* = p(\bar{a}=1,b)p(c|b) = p(\bar{a}=1,b,c)$$

Clique Tree Propagation

- What happens when we have a tree of cliques instead of just a pair? How can we achieve *global consistency* so that all cliques containing a variable x_i agree on its marginal $p(x_i)$?
- Two things:
  1. Arrange the cliques into a *junction tree* so that *local consistency implies global consistency*.
     We don't need to consider all pairs of cliques that share variables.
  2. Order updates to ensure that updates between V and W do not ruin consistency between V and U previously achieved.



When can one node safely pass a message to another?

Junction Tree Correctness

- The key property of the junction tree is that *local consistency implies global consistency*.
- In other words, conside a variable x_i which appears in two cliques. In a junction tree, it will also appear in every clique on the path between those two and nowhere else.
- If the cliques along that path are *pariwise consistent* with respect to x_i then they will also be *jointly consistent* with respect to x_i.
- Thus, running the pairwise message passing on a junction tree of cliques will achieve local *and* global consistency. We can get the same answer for x_i by consulting any clique node containing x_i.
- Futhermore, this answer will be exactly the correct marginal.

Message-Passing-Protocol

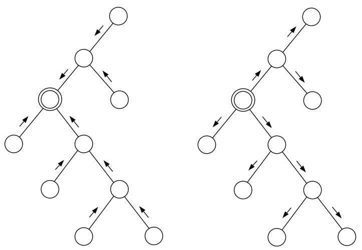
A clique can send a message to a neighbour only when it has received messages from all its other neighbours.

- This protocol maintains consistency.[1]
- Protocol is also realizable: designate one node of junction tree as root. Pass messages inward to root and then back out to leaves.

```
Formally:
CollectEvidence(root);
DistributeEvidence(root);

CollectEvidence(node):
  foreach child(node) {
    Update(node,CollectEvidence(child)) };
  return node;

DistributeEvidence(node):
  foreach child(node) {
    Update(child,node); DistributeEvidence(child)) };
```



(a) (b)

[1] Consider a message $W \to V$. If V has already sent its message to W, then it must have received all its other messages. The current message $W \to V$ will achieve consistency and no more messages will be exchanged. If V has not sent to W yet, it will wait until it has received all other messages and then send a final message to achieve consistency.

The Hugin Algorithm

- **Compilation.**
  Moralization: For directed graphs, join parents and drop directions of links.
  Triangulation: Many possible algorithms. Hard step.
  Identification of maximal cliques: easy in triangulated graphs.
  Construction of Junction Tree: maximal spanning tree over cliques using intersection size as weights.
- **Introduction of evidence.**
  At every node where we have observed some data, take appropriate slice of potential.
- **Initialization.**
  Each potential of original graph (possibly sliced) is multiplied onto exactly one clique of junction tree. Separators are initialized to unity.
- **Propagation of probabilities.**
  Pass messages according to MPP: designate root of clique tree and call CollectEvidence and DistributeEvidence from root. For a message from $V \to W$:

$$\phi_S^* = \sum_{V \backslash S} \psi_V \qquad \psi_W^* = \frac{\phi_S^*}{\phi_S}\psi_W$$

At termination, clique potentials and separator potentials are proportional to marginal probabilities of cliques/separator sets given evidence. Further marginalization can be performed for singletons or subsets.

Shafer-Shenoy Algorithm

- It is possible to develop an alternate (but equivalent) algorithm in which we don't explicitly store the separator potentials ϕ.
- Instead, we work out what the multiplicative update to a clique potential ψ would have been (by expressing separator potentials in terms of clique potentials), and perform that update explicitly.
- The updates are expressed as "messages" from clique i to j:

$$\mu_{ij} = \sum_{C_i \backslash C_j} \psi_{C_i} \prod_{k \neq i} \mu_{ki}$$

- Once a clique has received messages from all its neighbours, we can compute its marginal as the product of messages and evidence:

$$p(C_i) \propto \psi_{C_i} \prod_{k \neq i} \mu_{ki}$$

Together, these two equations are the Shafer-Shenoy Algorithm.

Correctness of Shafer-Shenoy Messages

- Two cases:
-If the first update on the link vw was from v to w, then we want $\mu_{vw} = \phi^*/\phi$. If all the other messages coming into i are correct, then using $\phi^* = \sum_{V \backslash S} \psi_V$ and the fact that the initial separator potential is unity, we can see that μ_{vw} will be correct.
–Otherwise, we want $\mu_{vw} = \phi^{**}/\phi^*$. Using the fact that $\phi_S^{**} = \sum_{W \backslash S} \frac{\phi_S^*}{\phi_S} \psi_W$ we can see again that the message will be correct.

Link between Shafer-Shenoy & Hugin

- Shafer-Shenoy looks a lot like belief propagation (sum-product).
- But we can relate it to the Hugin algorithm as follows:
Consider cliques V and W.
What is μ_{vw} were the "update factor" in the direction from $v \to w$?
- At the end of Shafer-Shenoy, the marginal would be the product of the original potential, multiplied by all the update factors – just like in the Hugin algorithm.
- So we just need to check that μ_{vw} defined as in SS:

$$\mu_{vw} = \sum_{C_v \backslash C_w} \psi_{C_v} \prod_{k \neq v} \mu_{kv}$$

are really equal to the update factors from Hugin.

Remaining Issues (see book)

- Generalized Viterbi: replace sum with max in ψ, phi updates
- Computational Complexity
- Proofs:
  - MPP on junction tree gives correct marginals.
  - Many async. updates without MPP still achieve consistency.
  - Extended representation obeys Hammersley-Clifford.
  - Separator potentials are zero only when both cliques are.
  - Decomposable $\Leftrightarrow$ Triagulated $\Leftrightarrow$ $\exists$Junction tree.
  - Elimination performs triagulation.
  - Maximal spanning tree with $w = |S|$ is junction tree.