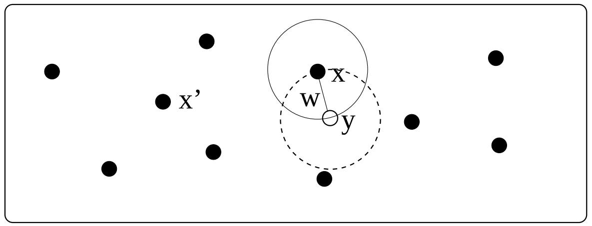
Lecture 18:

Shannon's Theorm Proof & Product Codes

November 13, 2006

Shannon's Noisy Coding Theorem for the BSC

- A BSC with error probability $f < 1/2$ has capacity $1 - H_2(f)$.
- **Theorem**: For any $\eta > 0$ and $\epsilon > 0$, there is a code (of some length N) whose rate, R, is at least $C - \eta$, and for which the probability that nearest neighbor decoding will fail is less than ϵ.
- Last class we started to give a proof of this, which shows that a *randomly chosen code* performs quite well and hence that there must be *specific codes* which also perform quite well (although the proof does not construct such a code).



Proving the Theorem with Random Codes

- Here's the process (equivalent to generating a random code):
  1. We randomly pick *one* codeword, $\mathbf{x}$, which the sender transmits.
  2. The channel randomly generates an error pattern, $\mathbf{n}$, that is added to $\mathbf{x}$ to give the received data, $\mathbf{y}$.
     Let the number of transmission errors (ie, ones in $\mathbf{n}$) be w.
  3. We now randomly pick the other $M-1$ codewords. If the Hamming distance from $\mathbf{y}$ of all these codewords is greater than w, nearest-neighbor decoding will make the correct choice.
- If the probability that this process leads to a decoding error is $< \epsilon$, then there must be some specific code with error probability $< \epsilon$.
- For large N, the fraction of errors w/N in the transmission is very close to the crossover probability f of the channel:

$$P(f - \beta < w/N < f + \beta) \;\geq\; 1 - \epsilon/2$$

Decoding with Typical Error Patterns

- The probability that the codeword nearest to $\mathbf{y}$ is the correct decoding will be at least as great as the probability that the following sub-optimal decoder decodes correctly:

  If there is exactly one codeword $\mathbf{x}^*$ for which $\mathbf{n} = \mathbf{y} - \mathbf{x}^*$ has a "typical" number of ones, then decode to $\mathbf{x}^*$, otherwise declare that decoding has failed.
- This sub-optimal decoder can fail in two ways:
  – The correct decoding, $\mathbf{x}$, may correspond to an error pattern, $\mathbf{n} = \mathbf{y} - \mathbf{x}$, that is not "typical".
  – Some other codeword, $\mathbf{x}'$, may exist for which the error pattern $\mathbf{n}' = \mathbf{y} - \mathbf{x}'$ is typical.
- The total probability of decoding failure is less than the sum of the probabilities of failing in these two ways.
  We will try to limit each of these to $\epsilon/2$.

Bounding the Probability of Failure

- We can choose N large enough to ensure the actual error pattern will be non-typical with probability less than $\epsilon/2$:
$$P(f-\beta < w/N < f+\beta) \;\geq\; 1-\epsilon/2$$
- We now need to limit the probability that some other codeword also corresponds to one of the J typical error patterns. J is bounded by
$$J \;<\; 2^{N(H_2(f)+\beta\log_2((1-f)/f))}$$
- For a random codeword $\mathbf{x}'$ (other than the one actually transmitted), the difference $(\mathbf{x}'-\mathbf{y})$ will contain 0s and 1s that are independent and equally likely. It will be "typical" with probability
$$J/2^N \;<\; 2^{-N(1-H_2(f)-\beta\log_2((1-f)/f))}$$
- The probability that *any* of the $M-1$ incorrect codewords will differ from $\mathbf{y}$ by a typical error pattern is bounded by M times this. We need this bound to be less than $\epsilon/2$, ie
$$M\,2^{-N(1-H_2(f)-\beta\log_2((1-f)/f))} \;<\; \epsilon/2$$

Finishing the Proof

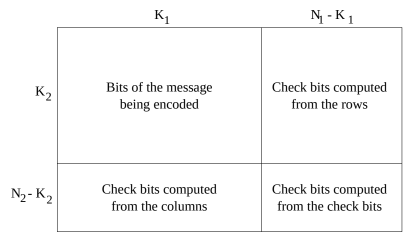
- Finally, we need to pick β, M, and N so that the two types of error have probabilities less than $\epsilon/2$, and the rate, R is at least $C-\eta$.
- We will let $M = 2^{\lceil (C-\eta)N\rceil}$, and make sure N is large enough that $R = \lceil (C-\eta)N\rceil/N < C$ [and $P(f-\beta < w/N < f+\beta) \;\geq\; 1-\epsilon/2$].
- With this value of M, we need
$$2^{\lceil (C-\eta)N\rceil}\; 2^{-N(1-H_2(f)-\beta\log_2((1-f)/f))} < \epsilon/2$$
$$\Rightarrow 2^{-N(1-H_2(f)-\lceil (C-\eta)N\rceil/N-\beta\log_2((1-f)/f))} < \epsilon/2$$
- The channel capacity is $C = 1-H_2(f)$, so that $1-H_2(f)-\lceil (C-\eta)N\rceil/N \;=\; C-R$ is positive.
- For a sufficiently small value of β (which we can get with large N), $1-H_2(f)-\lceil (C-\eta)N\rceil/N-\beta\log_2((1-f)/f)$ will also be positive. With this β and a large enough N, the probabilities of both types of error will be less than $\epsilon/2$, so the total error probability will be less than ϵ.

Good Codes and Minimum Distance

- Recall that for a code to be guaranteed to correct up to t errors, it's minimum distance must be at least $2t+1$.
- What's the minimum distance for the random codes used to prove the noisy coding theorem?
- A random N-bit code is very likely to have minimum distance $d \leq N/2$ — if we pick two codewords randomly, about half their bits will differ. So these codes are likely *not guaranteed* to correct patterns of $N/4$ or more errors.
- A BSC with error probability f will produce about Nf errors. So for $f > 1/4$, we expect to get more errors than the code is guaranteed to correct. Yet we know these codes are good!
- **Conclusion:** A code may be able to correct *almost all* patterns of t errors even if it can't correct *all* such patterns.

Product Codes

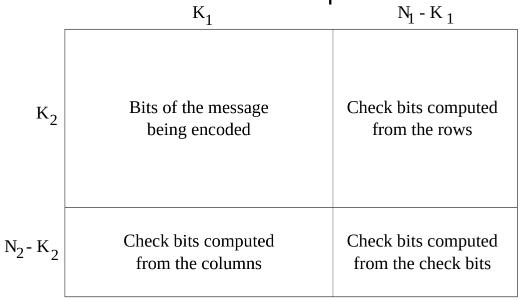
- A *product code* is formed from two other codes $\mathcal{C}_1$, of length N_1, and $\mathcal{C}_2$, of length N_2. The product code has length N_1N_2.
- We can visualize the N_1N_2 symbols of the product code as a 2D array with N_1 columns and N_2 rows.
- Definition of a product code: An array is a codeword of the product code if and only if
  - all its rows are codewords of $\mathcal{C}_1$
  - all its columns are codewords of $\mathcal{C}_2$



- We will assume here that $\mathcal{C}_1$ and $\mathcal{C}_2$ are linear codes, in which case the product code is also linear. (Can you see why?)
- The product codeword (as a matrix) is the **outer product** of the two original codewords (as vectors).

Dimensionality of Product Codes

- Suppose $\mathcal{C}_1$ is an $[N_1, K_1]$ code and $\mathcal{C}_2$ is an $[N_2, K_2]$ code. Then their product will be an $[N_1N_2, K_1K_2]$ code.
- Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are in systematic form. Here's a picture of a codeword of the product code:



- The dimensionality of the product code is not more than K_1K_2, since the message bits in the upper-left determine the check bits.
- We'll see that the dimensionality equals K_1K_2 by showing how to find correct check bits for any message.

Encoding Product Codes

- Here's a procedure for encoding messages with a product code:
  1. Put K_1K_2 message bits into the upper-left K_2 by K_1 corner of the N_2 by N_1 array (using the outer product of the messages).
  2. Compute the check bits for the first K_2 rows, according to $\mathcal{C}_1$.
  3. Compute the check bits for *all* N_1 columns, according to $\mathcal{C}_2$.
- After this, all the columns will be codewords of $\mathcal{C}_2$, since they were given the right check bits in step (3). The first K_2 rows will be codewords of $\mathcal{C}_1$, since they were given the right check bits in step (2). But are the *last* $N_2 - K_2$ rows codewords of $\mathcal{C}_1$?
- Yes! Check bits are linear combinations of message bits. So the last $N_2 - K_2$ rows are linear combinations of earlier rows. Since these rows are in $\mathcal{C}_1$, their combinations are too.

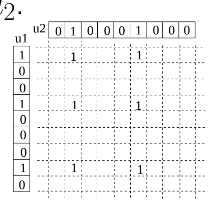
Decoding Product Codes

- Products of even small codes have lots of check bits, so decoding directly may be infeasible.
- But if $\mathcal{C}_1$ and $\mathcal{C}_2$ can easily be decoded, we can decode the product code by first decoding the rows (replacing them with the decoding), then decoding the columns. (Or the other way around.)
- This will usually **not** be a nearest-neighbor decoder (and hence will be sub-optimal, assuming a BSC and equally-likely messages).

Minimum Distance of Product Codes

- If $\mathcal{C}_1$ has minimum distance d_1 and $\mathcal{C}_2$ has minimum distance d_2, then the minimum distance of their product is d_1d_2.
- **Proof:**
  Let $\mathbf{u}_1$ be a codeword of $\mathcal{C}_1$ of weight d_1 and $\mathbf{u}_2$ be a codeword of $\mathcal{C}_2$ of weight d_2. Build a codeword of the product code by putting $\mathbf{u}_1$ in row i of the array if $\mathbf{u}_2$ has a 1 in position i. Put zeros elsewhere. This codeword has weight d_1d_2.

  The new codeword is the outer product of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.



- Furthermore, any non-zero codeword must have at least this weight. It must have at least d_2 rows that aren't all zero, and each such row must have at least d_1 ones in it.

How Good are Products of Codes?

- Let $\mathcal{C}$ be an $[N, K]$ code of minimum distance d (guaranteed to correct $t = \lfloor (d-1)/2 \rfloor$ errors).
- How good is the code obtained by taking the product of $\mathcal{C}$ with itself p times?

  Length: $N_p = N^p$
  Rate: $R_p = K^p/N^p = (K/N)^p \to 0$
  Distance: $d_p = d^p$
  Relative distance: $\rho_p = d_p/N_p = (d/N)^p \to 0$
- The code can correct up to about $d_p/2$ errors, corresponding to a proportion of errors of $\rho_p/2$.
- For a BSC with error probability f, we expect that for large N, the proportion of erroneous bits in a block will be very close to f. (The Law of Large Numbers once again.)

Why use Products of Codes?

- The analysis above shows that for large N, these product codes are both *unlikely* to correct all errors, and also that they have a low rate (approaching zero)!
- Futhermore, they are hard to decode in an exact (maximum likelihood) way, so we have to use an approximate decoder.
- So why would we ever use them?
- One advantage of product codes: They can correct some *burst errors* — errors that come together, rather than independently.

0	1	?	0	0	1	0	0	0	?	?	?	?	?	?	0	0	0	0	1	0	0	0	1	?	0	0