CSC310 – Information Theory Sam Roweis

Lecture 6:

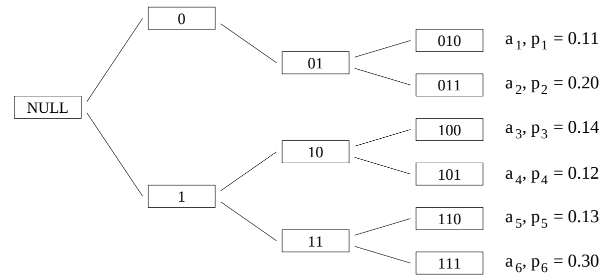
September 28, 2005

Some Observations about Instantaneous Codes 1

- Suppose we have an instantaneous code for symbols $a_1, \ldots, a_I$, with probabilities $p_1, \ldots, p_I$ and codeword lengths $l_1, \ldots, l_I$.
- Under each of the following conditions, we can find a better instantaneous code, i.e. one with smaller expected codeword length:
  1. If $p_1 < p_2$ and $l_1 < l_2$: Swap the codewords for a_1 and a_2.
  2. If there is a codeword of the form xby, where x and y are strings of zero or more bits, and b is a single bit, but there are no codewords of the form $xb'z$, where z is a string of zero or more bits, and $b' \neq b$:
     Change all the codewords of the form xby to xy. (This improves things if none of the p_i are zero, and never makes things worse.)

The Improvements in Terms of Trees 2

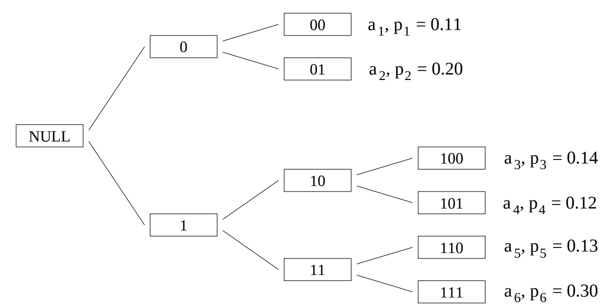
- We can view these improvements in terms of the trees for the codes. Here's an example:



- Two codewords have the form $01\ldots$ but none have the form $00\ldots$ (ie, there's only one branch out of the 0 node).
- We can therefore improve the code by deleting the surplus node.

Continuing to Improve the Example 3

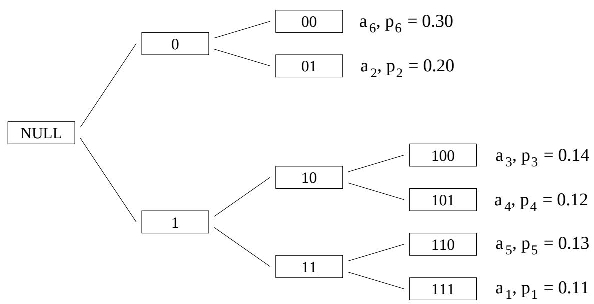
- The result is the code shown below:



- Now we note that a_6, with probability 0.30, has a longer codeword than a_1, which has probability 0.11. We can improve the code by swapping the codewords for these symbols.

The State After These Improvements

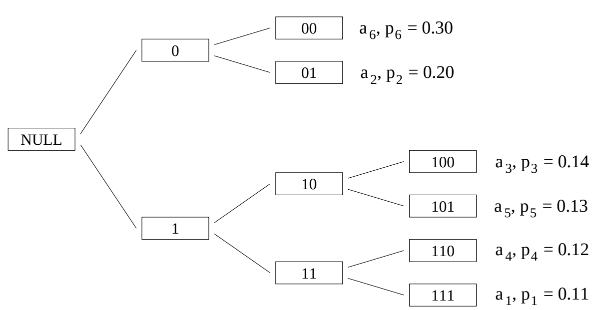
- Here's the code after this improvement:



- In general, after such improvements:
  The most improbable symbol will have the longest codeword and there will be at least one other codeword of this length — its "sibling" in the tree. The second-most improbable symbol will also have a codeword of the longest length.

A Final Rearrangement

- The codewords for the most improbable and second-most improbable symbols must have the same length.
- The most improbable symbol's codeword also has a "sibling" of the same length.
- We can swap codewords to make this sibling be the codeword for the second-most improbable symbol. For the example, the result is:



Reminder: Block Codes for Achieving the Entropy

- Last class we proved that Huffman codes are the optimal single symbol codes (plus a warning: top-down splitting does not work).
- We also proved Shannon's first theorem by showing that if we encode long enough blocks we can get the average per-symbol entropy as close as we want to the entropy of the source.
- Our proof used *lossless codes* of *variable length* (some blocks had codes longer than other blocks). For ease, we used Shannon-Fano codes, but we could also have used Huffman Codes or any other symbol other code which is guaranteed to get within a constant of the entropy.
- There is another way to compress down to the entropy using long blocks; that is to use *lossy codes* of *fixed length*.

Another Way to Compress Down to the Entropy

- We get a similar result by supposing that we will always encode N symbols into a block of exactly NR bits (fixed length code).
  Can we do this in a way that is very likely to be decodable?
- Yes, for large values of N. The Law of Large Numbers (LLN) tells us that the sequence of symbols to encode, $a_{i_1}, \ldots, a_{i_N}$, is very likely to be a "typical" one, for which

$$\frac{1}{N}\log_2(1/(p_{i_1}\cdots p_{i_N})) \;=\; \frac{1}{N}\sum_{j=1}^{N}\log_2(1/p_{i_j})$$

  is very close to the expectation of $\log_2(1/p_i)$, which is the entropy, $H(X) = \sum_i p_i \log_2(1/p_i)$. (See Section 4.3 of MacKay's book.)
- So if we encode all the sequences in this *typical set* in a way that can be decoded, the code will almost always be uniquely decodable.

How Big is the Typical Set? 8

- Let's define "typical" sequences as ones where

$$(1/N)\log_2(1/(p_{i_1}\cdots p_{i_N})) \;\le\; H(X) + \eta/\sqrt{N}$$

  The probability of any such typical sequence will satisfy

$$p_{i_1}\cdots p_{i_N} \;\ge\; 2^{-NH(X)-\eta\sqrt{N}}$$

- We scale the margin allowed above $H(X)$ as $1/\sqrt{N}$ since that's how the standard deviation of an average scales. LLN (Chebychev's inequality) then tells us that most sequences will satisfy this condition, for some large enough value of η.
- The total probability for all such sequences can't be greater than one, so the number of "typical" sequences can't be greater than

$$2^{NH(X)+\eta\sqrt{N}}$$

Encoding sequences in the typical set 9

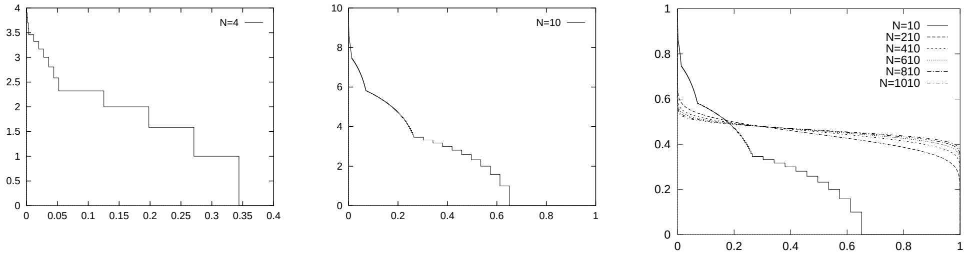
- The number of "typical" sequences can't be greater than

$$2^{NH(X)+\eta\sqrt{N}}$$

- We will be able to encode these sequences in NR bits if $NR \ge NH(X) + \eta\sqrt{N}$. (Using any arbitrary code in which we assign each typical sequence to one of the 2^{NR} codes.)
  If $R > H(X)$, this will be true if N is sufficiently large.
- How often will a sequence of length N fail to be in the typical set? To answer this, we need to know how many sequences live in the upper "tail" of the distribution of $(1/N)\log_2(1/(p_{i_1}\cdots p_{i_N}))$.
- We can define $H_\delta(X^N)$ to be average codeword length needed for the typical set to leave out only a fraction δ of possible sequences. Formally, it is the logarithm of the minimum number of sequences in the N^{th} extension of X whose probabilities sum to at least $1-\delta$.

Example 10

- Example: Consider flipping a coin with $p_{heads} = 0.1$.
- Here are the plots of δ vs. H_δ.



- For large N, H_δ becomes almost independent of δ.

Another Statement of Shannon's Theorem 11

- Let X be an ensemble with entropy $H(X) = H$ bits.
- Given $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer N_0 such that for $N > N_0$,

$$\left| \frac{1}{N} H_\delta(X^N) - H \right| < \epsilon.$$

- Both sides of the inequality are interesting. The first part tells us that even if the probability of error δ is extremely small, the average number of bits per symbol $\frac{1}{N}H_\delta(X^N)$ needed to specify a long N-symbol string with vanishingly small error probability does not have to exceed $H + \epsilon$ bits. We need to have only a tiny tolerance for error, and the number of bits required drops significantly from $H_0(X)$ to $(H + \epsilon)$.

- Let X be an ensemble with entropy $H(X) = H$ bits.
- Given $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer N_0 such that for $N > N_0$,
$$\left| \frac{1}{N} H_\delta(X^N) - H \right| < \epsilon.$$
- What happens if we are yet more tolerant to compression errors? The second part tells us that even if δ is very close to 1, so that errors are made most of the time, the average number of bits per symbol needed must still be at least $H - \epsilon$ bits.
- These two extremes tell us that regardless of our specific allowance for error, the number of bits per symbol needed is H bits; no more and no less.

Shannon's Noiseless Coding Theorem is mathematically satisfying. From a practical point of view, though, we still have two problems:

- How can we compress data to nearly the entropy *in practice*? The number of possible blocks of size N is I^N — huge when N is large. And N sometimes must be large to get close to the entropy by encoding blocks of size N.
  Solution: Instead of symbol codes or block codes, we will introduce a more powerful set of codes called *stream codes*. The most important example is known as *arithmetic coding* (coming next).
- Where do the symbol probabilities $p_1, \ldots, p_I$ come from? And are symbols really independent, with known, constant probabilities? This is the problem of *source modeling*.
  Solution: *adaptive methods*, which update their estimates of the source model as they encode more and more data.
  (We'll see these shortly.)