tions for the maximum likelihood estimate of the random function, $W(t)$, which was previously obtained by Youla follows.[11] There also results another system of equations determining that estimate of the random function $W(t)$ which is equivalent to Youla's system of equations.

The general method discussed in the preceding section also yields the solution of many other problems of optimum system theory and similar problems of many other branches of science.

IV. Concluding Remarks

We see that the general method presented above yields the effective solution of various problems of applied statistical decision theory, particularly, of the statistical theory of signal detection and reproduction.

The algorithm given by the method for obtaining a signal estimate may be used as a base for real system design. The main difficulty which arises in the practical realization of such systems is generally the absence of necessary data characterizing the *a priori* distribution of the signal parameter U [*i.e.*, the probability density $f(u)$]. To avoid this difficulty, the same algorithm may be applied to obtain the signal parameter U estimate in each cycle of the system acting, and to construct for each cycle an estimate for $f(u)$ using estimates of U obtained in all previous cycles. Using such estimates of $f(u)$ for each cycle of the system acting, we obtain a "self-learning" system which will be nearer and nearer with each new cycle to the true optimum system corresponding to the true probability distribution of the vector U.[12]

[11] D. Youla, "The use of the method of maximum likelihood in estimating continuous-modulated intelligence which has been corrupted by noise," IRE Trans. on Information Theory, vol. IT-3, pp. 90–105; March, 1954.

[12] K. Winkelbauer, "Experience in Games of Strategy and In Statistical Decision," *Trans. of the First Prague Conf. on Information Theory, Statistical Decision Functions and Random Processes*, Liblice, November 28–30, 1956, Czech. Acad. of Sc., Prague, pp. 297–354, 1957.

Quantizing for Minimum Distortion*

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Summary—**This paper discusses the problem of the minimization of the distortion of a signal by a quantizer when the number of output levels of the quantizer is fixed. The distortion is defined as the expected value of some function of the error between the input and the output of the quantizer. Equations are derived for the parameters of a quantizer with minimum distortion. The equations are not soluble without recourse to numerical methods, so an algorithm is developed to simplify their numerical solution. The case of an input signal with normally distributed amplitude and an expected squared error distortion measure is explicitly computed and values of the optimum quantizer parameters are tabulated. The optimization of a quantizer subject to the restriction that both input and output levels be equally spaced is also treated, and appropriate parameters are tabulated for the same case as above.**

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In many data-transmission systems, analog input signals are first converted to digital form at the transmitter, transmitted in digital form, and finally reconstituted at the receiver as analog signals. The resulting output normally resembles the input signal but is not precisely the same since the quantizer at the transmitter produces the same digits for all input amplitudes which lie in each of a finite number of amplitude ranges. The receiver must assign to each combination of digits a single value which will be the amplitude of the reconstituted signal for an original input anywhere within the quantized range. The difference between input and output signals, assuming errorless transmission of the digits, is the quantization error. Since the digital transmission rate of any system is finite, one has to use a quantizer which sorts the input into a finite number of ranges, N. For a given N, the system is described by specifying the end points, x_k, of the N input ranges, and an output level, y_k, corresponding to each input range. If the amplitude probability density of the signal which is the quantizer input is given, then the quantizer output is a quantity whose amplitude probability density may easily be determined as a function of the x_k's and y_k's. Often it is appropriate to define a distortion measure for the quantization process, which will be some statistic of the quantization error. Then one would like to choose the N y_k's and the associated x_k's so as to minimize the distortion. If we define the distortion, D, as the expected value of $f(\epsilon)$, where f is some function (differentiable), and ϵ is the quantization error, and call the input amplitude probability density $p(x)$, then

$$D = E[f(s_{\text{in}} - s_{\text{out}})] = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} f(x - y_i)p(x)\,dx$$

where $x_{N+1} = \infty$, $x_1 = -\infty$, and the convention is that an input between x_i and x_{i+1} has a corresponding output y_i.

If we wish to minimize D for fixed N, we get necessary conditions by differentiating D with respect to the x_i's and y_i's and setting derivatives equal to zero:

$$\frac{\partial D}{\partial x_j} = f(x_j - y_{j-1})p(x_j) - f(x_j - y_j)p(x_j) = 0 \qquad j = 2, \cdots, N \tag{1}$$

$$\frac{\partial D}{\partial y_j} = -\int_{x_j}^{x_{j+1}} f'(x - y_j)p(x)\,dx = 0 \qquad j = 1, \cdots, N \tag{2}$$

(1) becomes (for $p(x_j) \neq 0$)

$$f(x_j - y_{j-1}) = f(x_j - y_j) \qquad j = 2, \cdots, N \tag{3}$$

(2) becomes

$$\int_{x_j}^{x_{j+1}} f'(x - y_j)p(x)\,dx = 0 \qquad j = 1, \cdots, N. \tag{4}$$

We may ask when these are sufficient conditions. The best answer one can manage in a general case is that if all the second partial derivatives of D with respect to the x_i's and y_i's exist, then the critical point determined by conditions (3) and (4) is a minimum if the matrix whose ith row and jth column element is

$$\left. \frac{\partial^2 D}{\partial p_i\,\partial p_j} \right|_{\text{critical point}},$$

where the p's are the x's and y's, is positive definite. In a specific case, one may determine whether or not the matrix is positive definite or one may simply find all the critical points (*i.e.*, those satisfying necessary conditions) and evaluate D at each. The absolute minimum must be at one of the critical points since "end points" can be easily ruled out.

The sort of f one would want to use would be a good metric function, *i.e.*, $f(x)$ is monotonically nondecreasing

$$f(0) = 0$$

$$f(x) = f(-x).$$

If we require that $f(x)$ be *monotonically increasing* (with x) then (1) implies

$$|x_j - y_{j-1}| = |x_j - y_j| \qquad j = 2, \cdots, N$$

which implies (since y_{j-1} and y_j should not coincide) that

$$x_j = (y_j + y_{j-1})/2 \qquad j = 2, \cdots, N$$

(x_j is halfway between y_j and y_{j-1}).

We now take a specific example of $f(x)$ to further illuminate the situation.

Let $f(x) = x^2$

(3) implies

$$x_j = (y_j + y_{j-1})/2 \quad \text{or} \quad y_j = 2x_j - y_{j-1} \qquad j = 2, \cdots, N, \tag{5}$$

(4) implies

$$\int_{x_j}^{x_{j+1}} (x - y_j)p(x)\,dx = 0 \qquad j = 1, \cdots, N. \tag{6}$$

That is, y_j is the centroid of the area of $p(x)$ between x_j and x_{j+1}.

Because of the complicated functional relationships which are likely to be induced by $p(x)$ in (6), this is not a set of simultaneous equations we can hope to solve with any ease. Note, however, that if we choose y_1 correctly we can generate the succeeding x_j's and y_j's by (5) and (6), the latter being an implicit equation for x_{j+1} in terms of x_j and y_j.

A method of solving (5) and (6) is to pick y_1, calculate the succeeding x_j's and y_j's by (5) and (6) and then if y_N is the centroid of the area of $p(x)$ between x_N and ∞, y_1 was chosen correctly. (Of course, a different choice is appropriate to each value of N.) If y_N is not the appropriate centroid, then of course y_1 must be chosen again. This search may be systematized so that it can be performed on a computer in quite a short time.[1]

This procedure has been carried out numerically on the IBM 709 for the distribution $p(x) = 1/\sqrt{2\pi}\, e^{-x^2/2}$, under the restriction that $x_{N/2+1} = 0$ for N even, and $y_{(N+1)/2} = 0$ for N odd. This procedure gives symmetric results, *i.e.*,

[1] Obtaining *explicit* solutions to the quantizer problem for a nontrivial $p(x)$ is easily the most difficult part of the problem. The problem may be solved analytically where $p(x) = 1/\sqrt{2\pi}\, e^{-x^2/2}$ only for $N = 1$, $N = 2$. For $N = 1$, $x_1 = -\infty$, $y_1 = 0$, $x_2 = +\infty$. For $N = 2$, $x_1 = -\infty$, $y_1 = -\sqrt{2/\pi}$, $x_2 = 0$, $y_2 = \sqrt{2/\pi}$, $x_3 = +\infty$, ($\sqrt{2/\pi}$ is the centroid of the portion of $1/\sqrt{2\pi}\, e^{-x^2/2}$ between the origin and $+\infty$.) For $N \geqq 3$, some sort of numerical estimation is required. A somewhat different approach, which yields results somewhat short of the optimum, is to be found in V. A. Gamash, "Quantization of signals with non-uniform steps," *Electrosvyaz*, vol. 10, pp. 11–13; October, 1957.

if a signal amplitude x is quantized as y_k, then $-x$ is quantized as $-y_k$. The answers appear in Table I on page 11.

An attempt has been made to determine the functional dependence of the distortion on the number of output levels. A log-log plot of the distortion vs the number of output levels is in Fig. 1. The curve is not a straight line. The tangent to the curve at $N = 4$ has the equation $D = 1.32\ N^{-1.74}$ and the tangent at $N = 36$ has the equation $D = 2.21\ x^{-1.96}$. One would expect this sort of behavior for large N. When N is large, the amplitude probability density does not vary appreciably from one end of a single input range to another, except for very large amplitudes, which are sufficiently improbable so that their influence is slight. Hence, most of the output levels are very near to being the means of the end points of the corresponding input ranges. Now, the best way of quantizing a uniformly distributed input signal is to space the output levels uniformly and to put the end points of the input ranges halfway between the output levels, as in Fig. 2, shown for $N = 1$. The best way of producing a quantizer with $2N$ output levels for this distribution is to divide each input range in half and put the new output levels at the midpoints of these ranges, as in Fig. 3. It is easy to see that the distortion in the second case is $\frac{1}{4}$ that in the first. Hence, $D = kN^{-2}$ where k is some constant. In fact, k is the variance of the distribution.

If this sort of equal division process is performed on each input range of the optimum quantizer for a normally distributed signal with N output levels where N is large then again a reduction in distortion by a factor of 4 is expected. Asymptotically then, the equation for the tangent to the curve of distortion vs the number of output levels should be $D = kN^{-2}$ where k is some constant.

Commercial high-speed analog-to-digital conversion equipment is at present limited to transforming equal input ranges to outputs midway between the ends of the input ranges. In many applications one would like to know the best interval length to use, *i.e.*, the one yielding minimum distortion for a given number of output levels, N. This is an easier problem than the first, since it is only two-dimensional (for $N \geqq 2$), *i.e.*, D is a function of the common length r of the intervals and of any particular output level, y_k. If the input has a symmetric distribution and a symmetric answer is desired, the problem becomes one dimensional. If $p(x)$ is the input amplitude probability density and $f(x)$ is the function such that the distortion D is $E[f(s_{out} - s_{in})]$, then, for an even number $2N$ of outputs,

$$D = 2 \sum_{i=1}^{N-1} \int_{(i-1)r}^{ir} f\left(x - \left[\frac{2i-1}{2}\right]r\right)p(x)\,dx + 2\int_{(N-1)r}^{\infty} f\left(x - \left[\frac{2N-1}{2}\right]r\right)p(x)\,dx. \quad (7)$$

For a minimum we require

$$\frac{dD}{dr} = -\sum_{i=1}^{N-1}(2i-1)\int_{(i-1)r}^{ir} f'\left(x - \left[\frac{2i-1}{2}\right]r\right)p(x)\,dx - (2N-1)\int_{(N-1)r}^{\infty} f'\left(x - \left[\frac{2N-1}{2}\right]r\right)p(x)\,dx = 0. \quad (8)$$

A similar expression exists for the case of an odd number of output levels. In either case the problem is quite susceptible to machine computation when $f(x)$, $p(x)$ and N are specified. Results have been obtained for $f(x) = x^2$, $p(x) = 1/\sqrt{2\pi}\, e^{-x^2/2}$, $N = 2$ to 36. They are indicated in Table II on page 12.

A log-log plot of distortion vs number of output levels appears in Fig. 1. This curve is not a straight line. The tangent to the curve at $N = 36$ has the equation $D = 1.47\ N^{-1.74}$. A log-log plot of output level spacing vs number of outputs for the equal spacing which yields lowest distortion is shown in Fig. 4. This curve is also not a straight line. Lastly, a plot of the ratio of the distortion for the optimum quantizer to that for the optimum equally spaced level quantizer can be seen in Fig. 5.

Key to the Tables

The numbering system for the table of output levels, y_i, and input interval end points, x_i, for the minimum mean-squared error quantization scheme for inputs with a normal amplitude probability density with standard deviation unity and mean zero is as follows:

> For the number of output levels, N, even, x_1 is the first end point of an input range to the right of the origin. An input between x_i and x_{i+1} produces an output y_i.
>
> For the number of output levels, N, odd, y_1 is the smallest non-negative output. An input between x_{i-1} and x_i produces an output y_i.

This description, illustrated in Fig. 6, is sufficient because of the symmetry of the quantizer. The expected squared error of the quantization process and informational entropy of the output of the quantizer are also tabulated for the optimal quantizers calculated.[2] (If p_k is the probability of the kth output, then the informational entropy is defined as $-\sum_{k=1}^{N} p_k \log_2 p_k$.)

Table II also pertains to a normally distributed input with standard deviation equal to unity. The meaning of the entries is self-explanatory.

[2] The values of informational entropy given show the minimum average number of binary digits required to code the quantizer output. It can be seen from the tables that this number is always a rather large fraction of $\log_2 N$, and in most cases quite near 0.9 $\log_2 N$. In the cases where $N = 2^n$, n an integer, a simple n binary digit code for the outputs of the quantizer makes near optimum use of the digital transmission capacity of the system.

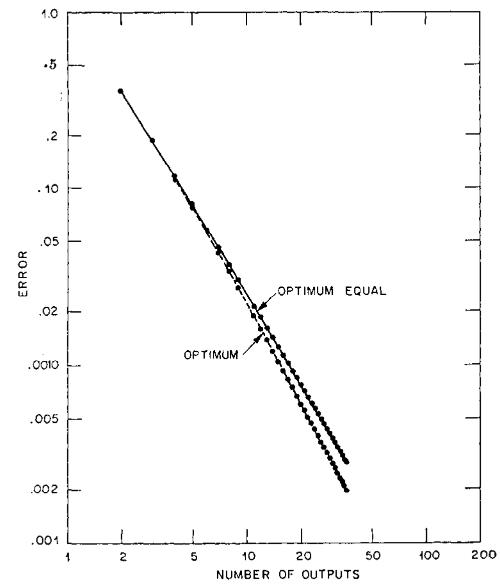

Fig. 1—Mean squared error vs number of outputs for optimum quantizer and optimum equally spaced level quantizer. (Minimum mean squared error for normally distributed input with $\sigma = 1$.)

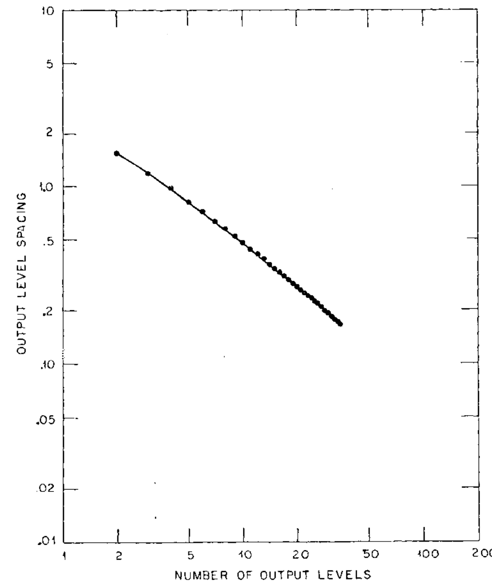

Fig. 4—Output level spacing vs number of output levels for equal optimum case. (Minimum mean squared error for normally distributed input with $\sigma = 1$.)

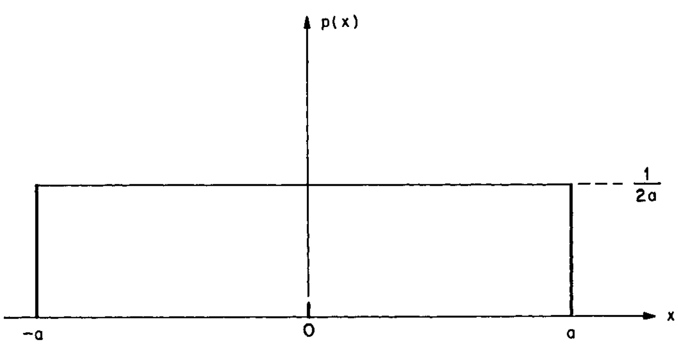

Fig. 2—Optimum quantization for the uniformly distributed case, $N = 1$. (Short strokes mark output levels and long strokes mark end points of corresponding input ranges.)

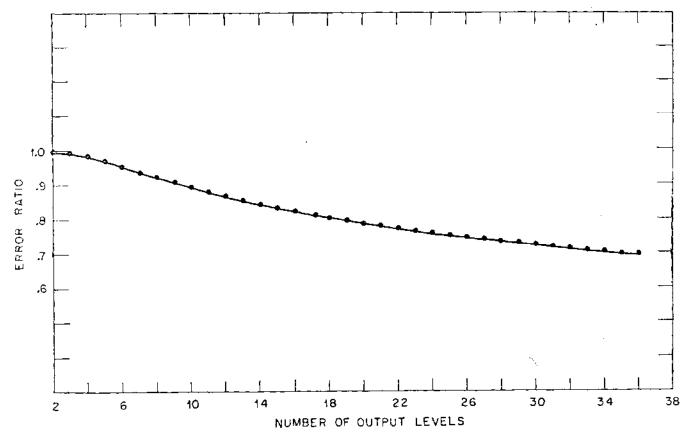

Fig. 5—Ratio of error for optimum quantizer to error for optimum equally spaced level quantizer vs number of outputs. (Minimum mean squared error for normally distribured input with $\sigma = 1$).

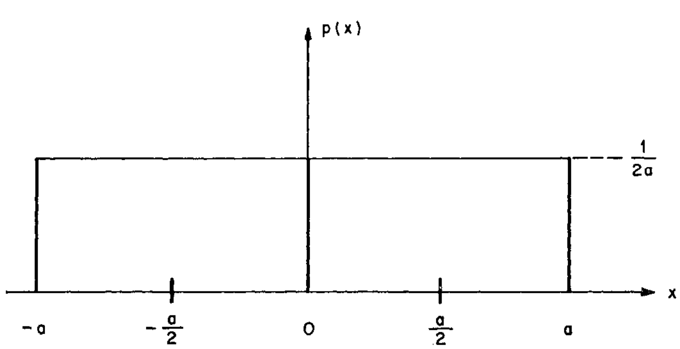

Fig. 3—Optimum quantization for the uniformly distributed case, $N = 2$. (Short strokes mark output levels and long strokes mark end points of corresponding input ranges.)

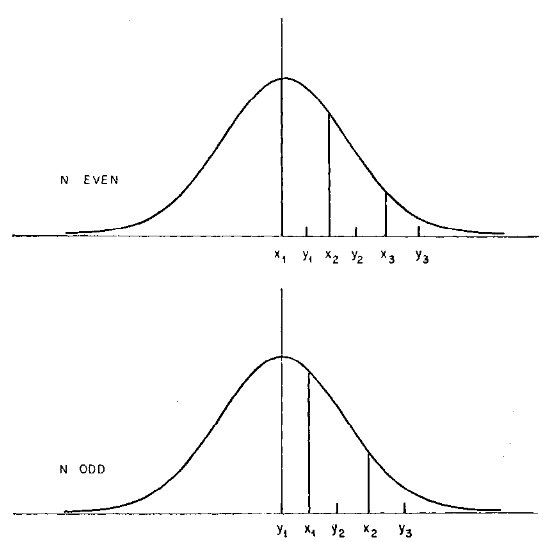

Fig. 6—Labeling of input range end points and output levels for the optimum quantizer. (Short strokes mark output levels and long strokes mark input range end points.)

TABLE I
Parameters for the Optimum Quantizer

	$N = 1$		$N = 2$		$N = 3$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	—	0.0	0.0	0.7980	0.0	1.224
2					0.6120	
Error	1.000		0.3634		0.1902	
Entropy	0.0		1.000		1.536	

	$N = 4$		$N = 5$		$N = 6$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.0	0.4528	0.3823	0.0	0.0	0.3177
2	0.9816	1.510	1.244	0.7646	0.6589	1.000
3				1.724	1.447	1.894
Error	0.1175		0.07994		0.05798	
Entropy	1.911		2.203		2.443	

	$N = 7$		$N = 8$		$N = 9$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.2803	0.0	0.0	0.2451	0.2218	0.0
2	0.8744	0.5606	0.5006	0.7560	0.6812	0.4436
3	1.611	1.188	1.050	1.344	1.198	0.9188
4		2.033	1.748	2.152	1.866	1.476
						2.255
Error	0.04400		0.03454		0.02785	
Entropy	2.647		2.825		2.983	

	$N = 10$		$N = 11$		$N = 12$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.0	0.1996	0.1837	0.0	0.0	0.1684
2	0.4047	0.6099	0.5599	0.3675	0.3401	0.5119
3	0.8339	1.058	0.9656	0.7524	0.6943	0.8768
4	1.325	1.591	1.436	1.179	1.081	1.286
5	1.968	2.345	2.059	1.693	1.534	1.783
6				2.426	2.141	2.499
Error	0.02293		0.01922		0.01634	
Entropy	3.125		3.253		3.372	

	$N = 13$		$N = 14$		$N = 15$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.1569	0.0	0.0	0.1457	0.1369	0.0
2	0.4760	0.3138	0.2935	0.4413	0.4143	0.2739
3	0.8126	0.6383	0.5959	0.7505	0.7030	0.5548
4	1.184	0.9870	0.9181	1.086	1.013	0.8512
5	1.623	1.381	1.277	1.468	1.361	1.175
6	2.215	1.865	1.703	1.939	1.776	1.546
7		2.565	2.282	2.625	2.344	2.007
8						2.681
Error	0.01406		0.01223		0.01073	
Entropy	3.481		3.582		3.677	

	$N = 16$		$N = 17$		$N = 18$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.0	0.1284	0.1215	0.0	0.0	0.1148
2	0.2582	0.3881	0.3670	0.2430	0.2306	0.3464
3	0.5224	0.6568	0.6201	0.4909	0.4653	0.5843
4	0.7996	0.9424	0.8875	0.7493	0.7091	0.8339
5	1.099	1.256	1.178	1.026	0.9680	1.102
6	1.437	1.618	1.508	1.331	1.251	1.400
7	1.844	2.069	1.906	1.685	1.573	1.746
8	2.401	2.733	2.454	2.127	1.964	2.181
				2.781	2.504	2.826
Error	0.009497		0.008463		0.007589	
Entropy	3.765		3.849		3.928	

	$N = 19$		$N = 20$		$N = 21$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.1092	0.0	0.0	0.1038	0.09918	0.0
2	0.3294	0.2184	0.2083	0.3128	0.2989	0.1984
3	0.5551	0.4404	0.4197	0.5265	0.5027	0.3994
4	0.7908	0.6698	0.6375	0.7486	0.7137	0.6059
5	1.042	0.9117	0.8661	0.9837	0.9361	0.8215
6	1.318	1.173	1.111	1.239	1.175	1.051
7	1.634	1.464	1.381	1.524	1.440	1.300
8	2.018	1.803	1.690	1.857	1.743	1.579
9	2.550	2.232	2.068	2.279	2.116	1.908
10		2.869	2.594	2.908	2.635	2.324
11						2.946
Error	0.006844		0.006203		0.005648	
Entropy	4.002		4.074		4.141	

	$N = 22$		$N = 23$		$N = 24$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.0	0.09469	0.09085	0.0	0.0	0.08708
2	0.1900	0.2852	0.2736	0.1817	0.1746	0.2621
3	0.3822	0.4793	0.4594	0.3654	0.3510	0.4399
4	0.5794	0.6795	0.6507	0.5534	0.5312	0.6224
5	0.7844	0.8893	0.8504	0.7481	0.7173	0.8122
6	1.001	1.113	1.062	0.9527	0.9122	1.012
7	1.235	1.357	1.291	1.172	1.119	1.227
8	1.495	1.632	1.546	1.411	1.344	1.462
9	1.793	1.955	1.841	1.681	1.595	1.728
10	2.160	2.366	2.203	2.000	1.885	2.042
11	2.674	2.982	2.711	2.406	2.243	2.444
12				3.016	2.746	3.048
Error	0.005165		0.004741		0.004367	
Entropy	4.206		4.268		4.327	

	$N = 25$		$N = 26$		$N = 27$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.08381	0.0	0.0	0.08060	0.07779	0.0
2	0.2522	0.1676	0.1616	0.2425	0.2340	0.1556
3	0.4231	0.3368	0.3245	0.4066	0.3921	0.3124
4	0.5982	0.5093	0.4905	0.5743	0.5537	0.4719
5	0.7797	0.6870	0.6610	0.7477	0.7202	0.6354
6	0.9702	0.8723	0.8383	0.9289	0.8936	0.8049
7	1.173	1.068	1.025	1.121	1.077	0.9824
8	1.394	1.279	1.224	1.328	1.273	1.171
9	1.641	1.510	1.442	1.556	1.487	1.374
10	1.927	1.772	1.685	1.814	1.727	1.599
11	2.281	2.083	1.968	2.121	2.006	1.854
12	2.779	2.480	2.318	2.514	2.352	2.158
13		3.079	2.811	3.109	2.842	2.547
14						3.137
Error	0.004036		0.003741		0.003477	
Entropy	4.384		4.439		4.491	

	$N = 28$		$N = 29$		$N = 30$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.0	0.07502	0.07257	0.0	0.0	0.07016
2	0.1503	0.2256	0.2182	0.1451	0.1406	0.2110
3	0.3018	0.3780	0.3655	0.2913	0.2821	0.3532
4	0.4556	0.5333	0.5154	0.4396	0.4255	0.4978
5	0.6132	0.6930	0.6693	0.5912	0.5719	0.6460
6	0.7760	0.8589	0.8287	0.7475	0.7225	0.7990
7	0.9460	1.033	0.9956	0.9100	0.8788	0.9586
8	1.126	1.218	1.172	1.081	1.043	1.127
9	1.319	1.419	1.362	1.263	1.217	1.306
10	1.529	1.640	1.570	1.461	1.404	1.501
11	1.766	1.892	1.804	1.680	1.609	1.717
12	2.042	2.193	2.077	1.929	1.840	1.964
13	2.385	2.578	2.417	2.226	2.111	2.258
14	2.871	3.164	2.899	2.609	2.448	2.638
15				3.190	2.926	3.215
Error	0.003240		0.003027		0.002834	
Entropy	4.542		4.591		4.639	

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TABLE I, *Cont'd*

	$N = 31$		$N = 32$		$N = 33$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.06802	0.0	0.0	0.06590	0.06400	0.0
2	0.2045	0.1360	0.1320	0.1981	0.1924	0.1280
3	0.3422	0.2729	0.2648	0.3314	0.3218	0.2567
4	0.4822	0.4115	0.3991	0.4668	0.4530	0.3868
5	0.6254	0.5528	0.5359	0.6050	0.5869	0.5192
6	0.7730	0.6979	0.6761	0.7473	0.7245	0.6547
7	0.9265	0.8481	0.8210	0.8947	0.8667	0.7943
8	1.088	1.005	0.9718	1.049	1.015	0.9392
9	1.259	1.170	1.130	1.212	1.171	1.091
10	1.444	1.347	1.299	1.387	1.338	1.252
11	1.646	1.540	1.482	1.577	1.518	1.424
12	1.875	1.753	1.682	1.788	1.716	1.612
13	2.143	1.997	1.908	2.029	1.940	1.821
14	2.477	2.289	2.174	2.319	2.204	2.060
15	2.952	2.665	2.505	2.692	2.533	2.347
16		3.239	2.977	3.263	3.002	2.718
17						3.285
Error	0.002658		0.002499		0.002354	
Entropy	4.685		4.730		4.773	

	$N = 34$		$N = 35$		$N = 36$	
	x_j	y_j	x_j	y_j	x_j	y_j
$j = 1$	0.0	0.06212	0.06043	0.0	0.0	0.05876
2	0.1244	0.1867	0.1816	0.1209	0.1177	0.1765
3	0.2495	0.3122	0.3036	0.2423	0.2359	0.2952
4	0.3758	0.4394	0.4272	0.3650	0.3552	0.4152
5	0.5043	0.5691	0.5530	0.4895	0.4762	0.5372
6	0.6355	0.7020	0.6819	0.6166	0.5996	0.6620
7	0.7705	0.8391	0.8146	0.7471	0.7261	0.7903
8	0.9104	0.9818	0.9523	0.8820	0.8567	0.9231
9	1.057	1.131	1.096	1.023	0.9923	1.062
10	1.211	1.290	1.248	1.170	1.134	1.207
11	1.375	1.460	1.411	1.327	1.285	1.362
12	1.553	1.646	1.587	1.495	1.445	1.528
13	1.749	1.853	1.781	1.679	1.619	1.710
14	1.971	2.090	2.001	1.883	1.812	1.913
15	2.232	2.375	2.260	2.119	2.030	2.146
16	2.559	2.743	2.584	2.401	2.287	2.427
17	3.025	3.307	3.048	2.767	2.609	2.791
18				3.328	3.070	3.349
Error	0.002220		0.002097		0.001985	
Entropy	4.815		4.856		4.895	

TABLE II

PARAMETERS FOR THE OPTIMUM EQUALLY SPACED LEVEL QUANTIZER

Number Output Levels	Output Level Spacing	Mean Squared Error	Informational Entropy
1	—	1.000	0.0
2	1.596	0.3634	1.000
3	1.224	0.1902	1.536
4	0.9957	0.1188	1.904
5	0.8430	0.08218	2.183
6	0.7334	0.06065	2.409
7	0.6508	0.04686	2.598
8	0.5860	0.03744	2.761
9	0.5338	0.03069	2.904
10	0.4908	0.02568	3.032
11	0.4546	0.02185	3.148
12	0.4238	0.01885	3.253
13	0.3972	0.01645	3.350
14	0.3739	0.01450	3.440
15	0.3534	0.01289	3.524
16	0.3352	0.01154	3.602
17	0.3189	0.01040	3.676
18	0.3042	0.009430	3.746
19	0.2909	0.008594	3.811
20	0.2788	0.007869	3.874
21	0.2678	0.007235	3.933
22	0.2576	0.006678	3.990
23	0.2482	0.006185	4.045
24	0.2396	0.005747	4.097
25	0.2315	0.005355	4.146
26	0.2240	0.005004	4.194
27	0.2171	0.004687	4.241
28	0.2105	0.004401	4.285
29	0.2044	0.004141	4.328
30	0.1987	0.003905	4.370
31	0.1932	0.003688	4.410
32	0.1881	0.003490	4.449
33	0.1833	0.003308	4.487
34	0.1787	0.003141	4.524
35	0.1744	0.002986	4.560
36	0.1703	0.002843	4.594