Propositional Temporal Logics: Decidability and Completeness

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Abstract

A sound and complete axiomatic system and a tableau-based decision procedure are presented for propositional temporal logic over linear and discrete time models. The axiomatic system and decision procedure are presented for the full logic, including the past operators, but contain a clear identification of the parts whose omission yields axiomatization and a decision procedure for the future fragment. The paper summarizes work of over 20 years and is intended to provide a definitive reference to the version of propositional temporal logic used for the specification and verification of reactive systems.

Keywords: completeness, decision procedure, deductive proof system, past operators, temporal logic.

1 Introduction

Much was written about the logic of time (especially about the future) by ancient writers such as Aristotle, by medieval ones such as William of Ockham or Peter Auriole, and by modern logicians, such as A.N. Prior ([31]).

One of the most important applications of temporal logic is its use for specification and verification of reactive systems (recently extended to deal also with timed and hybrid systems), which was first proposed in [27]. Similar suggestions were made about the same time by [15], but these developed a calculus of their own and did not explicitly refer to any of the known temporal logics.

Some of the earlier applications of temporal logic for the specification and verification of concurrent programs are reported in [27, 21, 9, 26, 13, 16, 17, 5], and many others. These articles use temporal logic for the specification of program properties, such as termination, correctness, safety, deadlock freedom, accessibility, responsiveness, and fair scheduling.

The literature contains many versions of temporal logic which differ from one another in several respects, including syntactic and semantic differences. Their structure may be either linear or branching. Their sizes may be finite, infinite at one or both directions. The granularity of the underlying time mode may be either discrete or dense, where the density might have the characteristics of the rational numbers, real numbers, etc. A more classical criterion distinguishes between propositional, first-order, or higher-order logics.

An important development in the theory of temporal logic was the introduction of the *until* and *since* operators into linear-time temporal logic by Kamp in [12], where the extended logic is shown to be more expressive than the logic with $\Box$ and $\Diamond$ alone, regardless of the density of the models. In his thesis, Kamp also proved the expressive completeness (relative to the first-order monadic theory of linear orders) of propositional linear temporal logic with both *until* and *since* operators.

The proposal in [27] was a little vague about the price temporal logic recommended for use. This oversight was corrected in [8], which focuses on a linear temporal logic over discrete time models with a first state but no last state (i.e., the models are infinite sequences of states), using the *next-time* and the (reflexive) *until* as the basic temporal operators. Applications and justifications of these choices were presented in [21], which extends the propositional logic of [8] into a first-order one. A sound and complete axiomatic system for the propositional logic was presented in [8], where the logic was shown to be decidable and expressively complete.

Two reasons are given in [8, 21] for the switch from a fully symmetric time models, infinite in both directions, such as the ones studied in [31, 12, 32, 3]), to models with a definite starting point, and the corresponding restriction to future temporal operators. The first is that of *minimality* by showing that the future fragment of the logic as is expressive as the full logic over models with a first state. The restriction to such models is justified by the intended application to the description of the computations of programs which all have an initial state.

Completeness of the axiomatic system was shown in [8] by resorting to filtration techniques, while decidability was claimed based on the reduction to the first-order theory of linear order. While these are theoretically valid arguments, they are computationally infeasible, in particular the reduction to the first-order language which is known to have a non-elementary complexity.

An improved joint approach to the establishment of both decidability and completeness of a proposed axiomatic system is based on the construction of a *tableau* for a temporal formula. The use of semantic tableaux in first-order logic as a basis for a semi-decision procedure and for proving completeness of the predicate calculus is discussed in [34]. Semantic tableaux for modal logics (of which temporal logic is a special case) are presented in [10]. Fischer and Ladner proposed in [7] a decision procedure for propositional dynamic logic (PDL), based on the observation that, if a PDL formula has a model, it has a model which can be presented as a structure containing worlds (corresponding to our atoms), in which each world is uniquely identified by the interpretation it assigns to a set of formulae that are defined as the closure of the original formula. Thus, they have coined the notion of "the Fischer-Ladner closure" of a given modal formula, and identified the existence of a tableau model, if the formula is satisfiable, but did not propose a systematic construction of the tableau.

Such a systematic construction was proposed by [30]. Also, [2] and [4] present a tableau-based decision algorithm for the branching-time logic CTL (called UB in [2]). It is straightforward to derive from a branching-time decision procedure an algorithm for checking satisfiability of linear-time formulae. Such an algorithm is presented in [29, 35, 25]. Extending the approach to a single algorithm which checks validity and constructs a proof (thus establishing completeness), in case the formula is valid, has been proposed in [29, 19].

A major reversal of our view about the essentiality of the past operators is presented in [20], which reintroduced the past fragment for the sake of clear and uniform specifications and accordingly extended the proof system of [8] to a complete proof system for the logic that includes both future and past operators.

Several reasons were presented as motivation for this change of mind. The first observation is that many statements that arise naturally in specifications, are easier to express using the past operators. For example, the fact that every ψ is preceded by a φ is easily expressed by the past-augmented formula $\square(\psi \longrightarrow \diamondminus \varphi)$. In this formula, $\diamondminus$ is the past version of the future $\diamond$. See [14] for examples of such statements. Past operators were also used by [13] to specify a buffer.

Another strong evidence for the importance of the past operators is their natural use for *compositional verification*, as observed and substantiated in [1] and [28]. In addition, we show in [20] how the past operators help to clarify several issues. In particular, they lead to a natural definition of the basic notions of safety and progress properties. In fact, it is only with the help of the past operators that we can construct the hierarchy of temporal properties, corresponding to the first 2.5 levels in the Borel hierarchy, as reported in [24], and relate it to the similar hierarchy established in the study of ω-automata, such as the one studied in [11].

Once we consider a logic with past operators over linear models with a first state, there are two different ways of defining the notion of satisfiability of a formula φ by a model $\sigma : s_0, s_1, \ldots$. The first interpretation is the *floating* interpretation by which $\sigma \models \varphi$ if φ holds at *every* state of σ. The second is the *anchored* interpretation, by which $\sigma \models \varphi$ if φ hold at the *first* state of σ. Our approach in [20] uses the floating interpretation, but it has been later found that, in order to properly define the temporal hierarchy and relate it to the corresponding hierarchy of ω-automata, the anchored interpretation is superior.

In this paper we present the culmination of all of these developments over the last 20 years, by presenting a propositional temporal logic with the full set of operators under the anchored interpretation, proposing an axiomatic system for this logic and establishing decidability of the logic and completeness of the axiomatic system by an improved tableau construction.

To satisfy all needs, we present the logic, the axiomatic system, and the main algorithms in a modular way. The presentation is done for the complete PTL, including the past operators. However, all clauses and paragraphs pertaining to the past are explicitly marked in the margins or by underlines, so that their removal yields a full and complete treatment of the future fragment of PTL.

Undoubtedly, this paper builds upon a long development of ideas and methods, but contains several meaningful improvements over all of its predecessors, many of which were never fully reported as a journal article. Following is a list of some of the unique contributions of this paper compared to previous works, such as [20]:

- This paper is based on the *anchored* interpretation of PTL-satisfaction, which was found to be more appropriate for specification and property classification.
- An improved axiomatic system with fewer past axioms.
- An improved tableau construction with less reliance on the *next* operator, and minimal reference to the past.
- A clean separation between the past and the future operators, so that the omission of all references to the past yields a full and complete treatment of the future

fragment.

2 The Language PTL

We present here a Propositional Temporal Logic with future operators, and then we allow to include also operators referring to a bounded past. Propositional Temporal Logic (in short, PTL) consists of propositions $p, q, \ldots$, representing abstractions of first order (state) formulae, the boolean connectives $\neg$ and $\vee$, the basic future temporal operators: $\bigcirc$- Next, $\mathcal{U}$- Until, and the basic past temporal operators: $\circledsim$- Weak Previous, $\mathcal{S}$- Since.

2.1 Syntax and Semantics of PTL

Let $\mathcal{P}$ be a set of *propositions.*
Formulae of PTL are defined recursively as follows:

- The symbols *true* and *false* are formulae.
- Every proposition $p \in \mathcal{P}$ is a formula.
- If φ and ψ are formulae, then $\neg\varphi$, $\varphi\vee\psi$, $\bigcirc\,\varphi$, $\underline{\circledsim\,\varphi}$, $\varphi\,\mathcal{U}\psi$ $\underline{\text{and } \varphi\,\mathcal{S}\psi}$ are formulae.

When it is clear from the context, we call a PTL formula just a formula. A PTL formula that does not contain any temporal operator, i.e., an operator that belongs to the set $\{\bigcirc, \underline{\circledsim}, \mathcal{U}, \underline{\mathcal{S}}\}$, is called a *state formula.* Clearly, a state formula belongs to *Propositional Logic* (PL).

We add to the language PTL the connectives $\wedge$, $\rightarrow$, $\leftrightarrow$, that can be defined in terms of the basic boolean operators $\neg$ and $\vee$.

The symbols $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are called the *negation, disjunction, conjunction, implication* and *equivalence* operators, respectively.

Additional future and past temporal operators are defined as follows:

- *Previously*: $\ominus\,\varphi = \neg\circledsim\neg\varphi$.
- *Eventually*: $\Diamond\,\varphi = (true)\,\mathcal{U}\varphi$.
  Once: $\diamondminus\,\varphi = (true)\,\mathcal{S}\varphi$.
- *Henceforth*: $\Box\,\varphi = \neg\Diamond\neg\varphi$.
  So-far: $\boxminus\,\varphi = \neg\diamondminus\neg\varphi$.
- *Waiting-for, Weak Until*: $\varphi\,\mathcal{W}\psi = (\Box\,\varphi)\vee(\varphi\,\mathcal{U}\psi)$.
  Back-to, Weak Since: $\varphi\,\mathcal{B}\psi = (\boxminus\,\varphi)\vee(\varphi\,\mathcal{S}\psi)$.

Clearly, we prefer to omit parentheses from the formulae. In a formula with no parentheses, the order of operator precedence is given by the following table, where a higher precedence number represents a higher binding strength.

Precedence	Operators
6	$\neg, \bigcirc, \ominus, \tilde{\ominus}$
5	$\square, \Diamond, \square, \diamondminus, \boxminus$
4	$\mathcal{U}, \mathcal{W}, \mathcal{S}, \mathcal{B}$
3	$\wedge$
2	$\vee$
1	$\rightarrow, \leftrightarrow$

For example, the formula $\square\, p \rightarrow \bigcirc \neg q \mathcal{U} r$ should be interpreted as $(\square\, p) \rightarrow ((\bigcirc(\neg q))\mathcal{U} r)$.

Definition 2.1 *Let φ be a formula.*

- *The* length *of the formula φ, $|\varphi|$, is defined recursively as follows:*
  - *If $\varphi \in \mathcal{P} \bigcup \{true,$ false$\}$ then $|\varphi| = 1$.*
  - *If φ is the formula $\neg\psi$, $\bigcirc\psi$, or $\tilde{\ominus}\psi$ then $|\varphi| = 1 + |\psi|$.*
  - *If φ is the formula $\psi_1 \vee \psi_2$, $\psi_1 \mathcal{U} \psi_2$, or $\psi_1 \mathcal{S} \psi_2$ then $|\varphi| = |\psi_1| + 1 + |\psi_2|$.*
- *The* decomposition tree *of the formula φ, $\mathcal{T}_\varphi$, is a tree, whose nodes are labeled by formulae, as follows:*
  - *The root of $\mathcal{T}_\varphi$ is labeled by φ.*
  - *A node in $\mathcal{T}_\varphi$ that is labeled by true,* false*, or some proposition, is a leaf of $\mathcal{T}_\varphi$.*
  - *A node in $\mathcal{T}_\varphi$ that is labeled by one of the formulae $\{\neg\psi,\ \bigcirc\psi,\ \tilde{\ominus}\psi\}$ has one child labeled by ψ.*
  - *A node in $\mathcal{T}_\varphi$ that is labeled by one of the formulae $\{\psi_1 \vee \psi_2,\ \psi_1 \mathcal{U} \psi_2,\ \psi_1 \mathcal{S} \psi_2\}$ has two labeled children, labeled by ψ_1, ψ_2, respectively.*

  Clearly, $\mathcal{T}_\varphi$ is unique.
- *A* subformula *of φ is a formula that labels one of the node in $\mathcal{T}_\varphi$.*

Definition 2.2 *Let S be a set of* states, *and $\mathcal{I}$ be an* interpretation *$\mathcal{I} : \mathcal{P} \mapsto 2^S$ mapping each proposition $p \in \mathcal{P}$ to the set of states $\mathcal{I}(p)$ in which p is true.*

- *A* computation, *σ, is an infinite sequence of states*

$$\sigma :\ s_0, s_1, \ldots\ ,$$

  where $s_i \in S$.
- *A* model *is a pair $(\sigma, \mathcal{I})$, consisting of a computation σ, and an interpretation $\mathcal{I}$.*

We interpret PTL formulae over triples $(\sigma, \mathcal{I}, j)$, where $(\sigma, \mathcal{I})$ is a model and j is some position within the computation σ. The *satisfiability relation* $\models$ is defined recursively as follows:
Let $(\sigma, \mathcal{I})$ be some given model, and let j be a position within σ.

- $(\sigma, \mathcal{I}, j) \models$ *true* and $(\sigma, \mathcal{I}, j) \not\models$ *false*.
- For a proposition p, $(\sigma, \mathcal{I}, j) \models p$ iff $s_j \in \mathcal{I}(p)$.
- $(\sigma, \mathcal{I}, j) \models \neg\varphi$ iff $(\sigma, \mathcal{I}, j) \not\models \varphi$.
- $(\sigma, \mathcal{I}, j) \models \varphi \vee \psi$ iff $(\sigma, \mathcal{I}, j) \models \varphi$ or $(\sigma, \mathcal{I}, j) \models \psi$.
- $(\sigma, \mathcal{I}, j) \models \bigcirc\varphi$ iff $(\sigma, \mathcal{I}, j+1) \models \varphi$.

- $(\sigma,\mathcal{I},j) \models \varphi\mathcal{U}\psi$ iff for some k, $j \leq k$, $(\sigma,\mathcal{I},k) \models \psi$ and for every i, $j \leq i < k$, $(\sigma,\mathcal{I},i) \models \varphi$.
- $(\sigma,\mathcal{I},j) \models \circleddash\!\!\!\!\sim \varphi$ iff $j=0$ or $j>0$ and $(\sigma,\mathcal{I},j-1) \models \varphi$.
- $(\sigma,\mathcal{I},j) \models \varphi\mathcal{S}\psi$ iff for some k, $0 \leq k \leq j$, $(\sigma,\mathcal{I},k) \models \psi$ and for every i, $k < i \leq j$. $(\sigma,\mathcal{I},i) \models \varphi$

Remark 2.3 *Note that* $(\sigma,\mathcal{I},j) \models \ominus \varphi$ *iff* $j>0$ *and* $(\sigma,\mathcal{I},j-1) \models \varphi$.

Definition 2.4 *Let* φ,ψ *be some formulae.*

- *Let* $(\sigma,\mathcal{I})$ *be some model. We say that the model* $(\sigma,\mathcal{I})$ satifies *the formula* φ, *or the formula* φ *is* satisfied *in the model* $(\sigma,\mathcal{I})$ *if* $(\sigma,\mathcal{I},0) \models \varphi$.
- *The formula* φ *is* satisfiable *if there exists a model satisfying it.*
- *The formula* φ *is* valid *if it is satisfied in all models.*

Remark 2.5 *Note that the formula* $\Box \varphi$ *states the invariance of the formula* φ *from the present point of some computation satisfying it.*

Let us define the operator *entailment*, $\Rightarrow$, as follows:

$$(\varphi \Rightarrow \psi) = \Box(\varphi \longrightarrow \psi).$$

Clearly, φ entails ψ over a computation if φ implies ψ at all positions. The operator *strong equivalence*, $\Longleftrightarrow$, is defined as follows:

$$(\varphi \Longleftrightarrow \psi) = \Box(\varphi \longleftrightarrow \psi).$$

φ is strongly equivalent to ψ over a computation if φ is equivalent to ψ at all positions.

Remark 2.6 *Note that the "strong" temporal operators entail the "weak" temporal operators, i.e., the following entailments are valid:*

- $\ominus \varphi \Rightarrow \circleddash\!\!\!\!\sim \varphi$.
- $\varphi\mathcal{U}\psi \Rightarrow \varphi\mathcal{W}\psi$.
- $\varphi\mathcal{S}\psi \Rightarrow \varphi\mathcal{B}\psi$.

2.2 Examples

Following are some examples of PTL formulae and their intuitive interpretations. The interpretation characterizes the computations such that the considered formula holds at some point (position) within the computation.

- It is easy to see that
$$\Diamond(\varphi \land \psi) \to (\Diamond \varphi \land \Diamond \psi)$$
is satisfied in every position, while the inverse implication
$$(\Diamond \varphi \land \Diamond \psi) \to \Diamond(\varphi \land \psi)$$
is not necessarily satisfied at every point.

- The formula $\varphi \rightarrow \diamondsuit\, \psi$ expresses an eventuality property, namely, it guarantees that if φ is presently true, then ψ will eventually become true.
- The formula $\varphi \Rightarrow \diamondsuit\, \psi$ expresses a repeated eventuality property (also called response property), which guarantees that any φ is eventually followed by a ψ.
- The formula $\varphi \Rightarrow \square\, \psi$ states that if φ ever becomes true, then ψ will be true at that instant and ever after.
- The formula $\diamondsuit(\varphi \wedge \bigcirc \neg\varphi)$ states that there will be a future instant such that φ is true at that instance and false at the next.
- $\diamondsuit\, \square\, \varphi$ states that at some future instant φ will become permanently true.
- $\square\, \diamondsuit\, \varphi$ states that every future instant is followed by or coincides with a later one in which φ is true. This implies that φ holds at infinitely many positions.
- The formula $\square\, \diamondsuit\, \varphi \rightarrow \square\, \diamondsuit\, \psi$, expresses the property ensuring that if there are infinitely many positions satisfying φ, then there are infinitely many positions satisfying ψ. This formula can be used to represent a situation of response to persistence, in which the response ψ is guaranteed only if the request φ is repeated sufficiently many times (infinitely many in the limit).

Precedence properties can be described by formulae like:

- $\neg\varphi\, \mathcal{W} \psi$ says that if φ ever happens, its first occurrence is preceded by (or coincides with) ψ.
- $\varphi \Rightarrow \neg\psi\, \mathcal{W} \chi$ expresses a continuous precedence property, ensuring that from any occurrence of φ, the first following occurrence of ψ (if any) must be preceded by an occurrence of of χ.

The following formulae present typical applications of the past operators.

- The formula $\varphi \Rightarrow \diamondminus\, \psi$ describes the fact that every φ is preceded by a ψ.
- The formula $\circleddash$ *false* characterizes the first position in a computation. It is a valid formula, since it states that all computations have a starting point.

2.3 Anchored vs. Floating Versions of PTL

As seen above, we define a formula to be valid if it holds at the first position of all models. A somewhat different approach has been taken in [23, 18], where a formula is defined to be valid if it holds at all positions of all models. Similarly, a formula is considered to be satisfiable if it holds in some position of some computation. Our view assigns special significance to the initial state. We refer to our version of the logic as the *anchored* version of PTL. In comparison, we may refer to the approach, that treats all states in the computation uniformly, as the *floating* version of PTL.

There is a straightforward translation between the two versions that can be described as follows:
Let φ be a PTL formula.

- φ is valid in the anchored version of PTL if and only if $\circleddash$ *false* $\rightarrow \varphi$ is valid in the floating version of PTL.
- φ is valid in the floating version of PTL if and only if $\square\, \varphi$ is valid in the anchored version of PTL.

3 A Decision Procedure for Validity

In this section an exponential time decision procedure for validity is presented. We show a tableau algorithm for the validity problem for PTL. We take the negation of the considered formula and check it for satisfiability, using a semantic tableaux.

The semantic tableaux for PL formulae are very simple: we decompose the considered formula to its subformulae according to the different possibilities that come from disjunctions (see for example, [34]). This decomposition is progressing in a tree-like tableaux, where each branch is terminated either by a leaf that contains some subformula and its negation, which means that this branch does not present a model, or by a leaf that contains a set of non-contradictory propositions or negations of propositions, that represent some model.

The situation in PTL is a little bit more complicated because , in theory, we have to analyze every formula in an infinite sequence of states according to its occurrence in some position, and its influence on the suffix that begins at that position. Indeed, every temporal formula can be decomposed to one formula that can be analyzed in the current state and to a second formula that is "pushed" to the next state or to the previous state. For example, the formula $\varphi\mathcal{U}\psi$ is analyzed as follows: There is one possibility that ψ holds now, and the other possibility is that φ holds now, and $\varphi\mathcal{U}\psi$ holds in the next state. This observation suggests that we have to construct some mechanism to control the repeated appearances of the formulae, trying to identify periodic situations in finite time.

3.1 Closure and Atoms

Let φ be the PTL formula whose satisfiability we want to check. In checking satisfiability of φ, we always assume that the set of propositions $\mathcal{P}$ includes all the propositions appearing in φ.

Definition 3.1 *The* closure *of* φ, $\mathrm{CL}(\varphi)$, *is the set of all subformulae of* φ *and their negations. We identify* $\neg\neg\psi$ *with* ψ *and* $\neg true$ *with* $false$.

Lemma 3.2 *For every* PTL *formula* φ, $\mathrm{CL}(\varphi) \leq 2\cdot|\varphi|$.

PROOF. We prove the lemma by induction on the structure of φ.

- If φ is a proposition p or its negation, $|CL(\varphi)| = |\{p, \neg p\}| = 2 \leq 2\cdot|p|$.
- If φ is the formula $\neg\psi$, $CL(\varphi) = CL(\neg\psi) = CL(\psi)$.

  By the induction hypothesis we obtain that $|CL(\varphi)| \leq 2\cdot|\psi| \overset{|\varphi|=1+|\psi|}{\leq} 2\cdot|\varphi|$.
- If φ is the formula $\psi_1 \vee \psi_2$,
  $CL(\varphi) = CL(\psi_1 \vee \psi_2) = \{\psi_1 \vee \psi_2, \neg(\psi_1 \vee \psi_2)\} \bigcup CL(\psi_1) \bigcup CL(\psi_2)$.
  By the induction hypothesis we obtain that
  $|CL(\varphi)| \leq 2 + 2\cdot|\psi_1| + 2\cdot|\psi_2| \overset{|\varphi|=1+|\psi_1|+|\psi_2|}{=} 2\cdot|\varphi|$.
- If φ is the formula $\bigcirc\psi$, $CL(\varphi) = CL(\bigcirc\psi) = \{\bigcirc\psi, \neg\bigcirc\psi\} \bigcup CL(\psi)$.

  By the induction hypothesis, we obtain that $|CL(\varphi)| \leq 2 + 2\cdot|\psi| \overset{|\varphi|=1+|\psi|}{=} 2\cdot|\varphi|$.
- If φ is the formula $\circleddash\psi$, the details of the proof are similar to those of $\bigcirc\psi$.

- If φ is the formula $\psi_1 \mathcal{U} \psi_2$,
  $CL(\varphi) = CL(\psi_1 \mathcal{U} \psi_2) = \{\psi_1 \mathcal{U} \psi_2, \neg(\psi_1 \mathcal{U} \psi_2)\} \bigcup CL(\psi_1) \bigcup CL(\psi_2)$.
  By the induction hypothesis we obtain that
  $$|CL(\varphi)| \leq 2 + 2 \cdot |\psi_1| + 2 \cdot |\psi_2| \overset{|\varphi|=1+|\psi_1|+|\psi_2|}{=} 2 \cdot |\varphi|.$$
- If φ is the formula $\psi_1 \mathcal{S} \psi_2$, the details of the proof are similar to those of $\psi_1 \mathcal{U} \psi_2$. ■

Definition 3.3 *An* Atom A *is a subset of* $\mathrm{CL}(\varphi)$ *such that*

- *For every* $\psi \in \mathrm{CL}(\varphi)$,
  $\psi \in A$ *iff* $\neg\psi \notin A$.
- *For every* $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$,
  $\psi_1 \vee \psi_2 \in A$ *iff* $\psi_1 \in A$ *or* $\psi_2 \in A$.
- *For every* $\psi_1 \mathcal{U} \psi_2 \in \mathrm{CL}(\varphi)$,
  If $\psi_1 \mathcal{U} \psi_2 \in A$ *then* $\psi_1 \in A$ *or* $\psi_2 \in A$.
  If $\psi_2 \in A$ *then* $\psi_1 \mathcal{U} \psi_2 \in A$.
- *For every* $\psi_1 \mathcal{S} \psi_2 \in \mathrm{CL}(\varphi)$,
  If $\psi_1 \mathcal{S} \psi_2 \in A$ *then* $\psi_1 \in A$ *or* $\psi_2 \in A$.
  If $\psi_2 \in A$ *then* $\psi_1 \mathcal{S} \psi_2 \in A$.

Note that intuitively, an atom is a consistent subset of $CL(\varphi)$. We denote the set of all atoms by $\mathcal{A}_\varphi$. Clearly, $|\mathcal{A}_\varphi| \leq 2^{2 \cdot |\varphi|}$.

Definition 3.4 *An atom* A *is* initial *if:*

1. *A does not contain any formula of the form* $\ominus \varphi$.
2. *For every* $\psi_1 \mathcal{S} \psi_2 \in \mathrm{CL}(\varphi)$, *if* $\psi_1 \mathcal{S} \psi_2 \in A$ *then* $\psi_2 \in A$.

We denote the set of all initial atoms that contain the formula ψ by $Init_\psi$.

3.2 Tableau Construction

Definition 3.5 *A* structure for the formula φ *is a graph* $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$ *whose nodes are all the atoms and whose edges are defined by the relation* $\mathcal{R}_\varphi$*:*

$$(A, B) \in \mathcal{R}_\varphi \iff \begin{cases} \textit{For every } \bigcirc \psi \in \mathrm{CL}(\varphi), \\ \quad \bigcirc \psi \in A \textit{ iff } \psi \in B. \\ \underline{\textit{For every } \tilde{\ominus} \psi \in \mathrm{CL}(\varphi),} \\ \quad \underline{\psi \in A \textit{ iff } \tilde{\ominus} \psi \in B.} \\ \textit{For every } \psi_1 \mathcal{U} \psi_2 \in \mathrm{CL}(\varphi), \\ \quad \textit{if } \psi_1 \mathcal{U} \psi_2, \neg\psi_2 \in A \textit{ then } \psi_1 \mathcal{U} \psi_2 \in B. \\ \quad \textit{if } \psi_1 \mathcal{U} \psi_2 \in B, \psi_1 \in A \textit{ then } \psi_1 \mathcal{U} \psi_2 \in A. \\ \underline{\textit{For every } \psi_1 \mathcal{S} \psi_2 \in \mathrm{CL}(\varphi),} \\ \quad \underline{\textit{if } \psi_1 \mathcal{S} \psi_2, \neg\psi_2 \in B \textit{ then } \psi_1 \mathcal{S} \psi_2 \in A.} \\ \quad \underline{\textit{if } \psi_1 \mathcal{S} \psi_2 \in A, \psi_1 \in B \textit{ then } \psi_1 \mathcal{S} \psi_2 \in B.} \end{cases}$$

Let $A, B \in \mathcal{A}_\varphi$. If $(A, B) \in \mathcal{R}_\varphi$ then we say that the atom B is a *successor* of the atom A. If there is a finite path from the atom A to the atom B, including a path of length 0, i.e., $A = B$, we say that B is a *descendant* of A.

The following two lemmas describe interesting properties concerning the propagation of formulae between an atom and its descendants.

Lemma 3.6 *Let A and B be atoms such that B is a successor of A. Let $\bigcirc \psi$ and $\ominus \psi'$ be formulae in* CL(φ).

1. $\neg \bigcirc \psi \in A$ *iff* $\neg\psi \in B$.
2. $\psi' \in A$ *iff* $\ominus \psi' \in B$.
3. $\neg\psi' \in A$ *iff* $\neg \ominus \psi' \in B$.

Proof. 1. Assume $\neg \bigcirc \psi \in A$. From the definition of an atom we know that either $\psi \in B$ or $\neg\psi \in B$. If $\psi \in B$, then by the fact that $\bigcirc \psi \in CL(\varphi)$ and the definition of $\mathcal{R}_\varphi$, we obtain that $\bigcirc \psi$ must belong to A, a fact that together with our assumption contradicts the definition of an atom.
Assume $\neg\psi \in B$. From the atom definition we know that $\bigcirc \psi \in A$ or $\neg \bigcirc \psi \in A$. If $\bigcirc \psi \in A$, then by the definition of $\mathcal{R}_\varphi$, we obtain that $\psi \in B$, a fact that together with our assumption contradicts the atom definition.

2. Since $\ominus \psi' \in CL(\varphi)$, we know by the definition of $\ominus$ that $\neg \widetilde{\ominus} \neg\psi'$, $\widetilde{\ominus} \neg\psi'$, $\neg\psi'$, $\psi' \in CL(\varphi)$.
Assume $\psi' \in A$. From the definition of an atom we know that either $\widetilde{\ominus} \neg\psi' \in B$ or $\neg \widetilde{\ominus} \neg\psi' \in B$, i.e., $\ominus \psi' \in B$. If $\widetilde{\ominus} \neg\psi' \in B$, then by the definition of $\mathcal{R}_\varphi$, we obtain that $\neg\psi'$ must belong to A, a fact that together with our assumption contradicts the definition of an atom.
Assume $\ominus \psi' \in B$, i.e., $\neg \widetilde{\ominus} \neg\psi' \in B$. From the atom definition we know that $\psi' \in A$ or $\neg\psi' \in A$. If $\neg\psi' \in A$, then by the definition of $\mathcal{R}_\varphi$, we obtain that $\widetilde{\ominus} \neg\psi' \in B$, a fact that together with our assumption contradicts the atom definition.

3. The proof that $\neg\psi' \in A$ iff $\neg \ominus \psi' \in B$ consists on the fact proven in 2. The proof is similar to the proof of 1. ∎

Lemma 3.7 *Let $A_0, A_1, \ldots, A_n$, $n \geq 0$, be a finite path of atoms in $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$ (i.e., A_n is a descendant of A_0).*

- *If $\psi_1 \mathcal{U} \psi_2 \in A_0$ and for every atom A_i, $0 \leq i \leq n-1$, $\neg\psi_2 \in A_i$, then*
  1. *For every i, $0 \leq i \leq n$, $\psi_1 \mathcal{U} \psi_2 \in A_i$.*
  2. *For every i, $0 \leq i \leq n-1$, $\psi_1 \in A_i$.*
- *If $\psi_1 \mathcal{S} \psi_2 \in A_n$ and for every atom A_i, $1 \leq i \leq n$, $\neg\psi_2 \in A_i$, then*
  1. *For every i, $0 \leq i \leq n$, $\psi_1 \mathcal{S} \psi_2 \in A_i$.*
  2. *For every i, $1 \leq i \leq n$, $\psi_1 \in A_i$.*

Proof. The lemma can be proved by induction on n. The proof follows directly from the definitions of atoms and the relation $\mathcal{R}_\varphi$ connecting them. ∎

Given a set of atoms $\mathcal{C} \subseteq \mathcal{A}_\varphi$, the *subgraph* of $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$ *specified by* $\mathcal{C}$ is the graph consisting of the nodes $\mathcal{C}$ and all the $\mathcal{R}_\varphi$ edges that connect them.

Definition 3.8

- *A pre-model is an infinite path of atoms in the structure $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$, $\pi = A_0, A_1, \ldots$, such that A_0 is initial.*
- *Let π be a pre-model. We define by $Fin(\pi)$ the subgraph of $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$ specified by all the atoms that appear infinitely many times in π.*

- *A pre-model* for φ, $\pi = A_0, A_1, \ldots$, *is a pre-model where* $A_0 \in Init_\varphi$.

The procedure for checking satisfiability of a formula φ attempts to find a model for φ in the structure $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$. Clearly, the candidates to be models in the structure for φ are the pre-models for φ.

For an arbitrary model, $(\sigma, \mathcal{I})$, we denote by $\pi_{(\sigma,\mathcal{I})}$ the pre-model given by $A_0, A_1, \ldots$ where $A_i = \{\psi \in CL(\varphi) \mid (\sigma, \mathcal{I}, i) \models \psi\}$, $0 \leq i$. It is easy to check that each A_i is an atom and that $\pi_{(\sigma,\mathcal{I})}$ is indeed a pre-model.

On the other hand, the atoms in a pre-model can be viewed as states by taking the natural interpretation $\mathcal{I}_\varphi$, which for each $p \in \mathcal{P}$ is defined as $\mathcal{I}_\varphi(p) = \{A_i \mid p \in A_i\}$. Thus, any pre-model π corresponds to a model $(\pi, \mathcal{I}_\varphi)$.

Clearly, If $(\sigma, \mathcal{I})$ is a model for φ, then $\pi_{(\sigma,\mathcal{I})}$ is a pre-model for φ. However, it is not generally true that if π is a pre-model for φ, then $(\pi, \mathcal{I}_\varphi)$ is a model for φ. We may say that a pre-model for φ is *almost* a model for φ, except for the until formulae that do not necessarily get their right semantics.

3.3 Fulfilling Paths and Satisfiability

Definition 3.9 *A pre-model,* $\pi = A_0, A_1, \ldots$, *is a* fulfilling path *if for every* i, $0 \leq i$, *and every* $\psi_1 \mathcal{U} \psi_2 \in A_i$, *there exists some* $j \geq i$ *such that* $\psi_2 \in A_j$.
A fulfilling path for φ *is a fulfilling path which is also a pre-model for* φ.

The symmetric requirement for the behavior of the operator $\mathcal{S}$ (since) is always ensured in a pre-model, as is established by the following lemma.

Lemma 3.10 *Let* $\pi = A_0, A_1, \ldots$ *be a pre-model. Then, for every* i, $0 \leq i$, *and every* $\psi_1 \mathcal{S} \psi_2 \in A_i$, *there exists some* j, $0 \leq j \leq i$, *such that* $\psi_2 \in A_j$.

PROOF. Suppose the lemma is not true, i.e., there exist some i, $0 \leq i$ and some formula $\psi_1 \mathcal{S} \psi_2$ such that $\psi_1 \mathcal{S} \psi_2 \in A_i$, and for every j, $0 \leq j \leq i$, $\psi_2 \notin A_j$ and therefore $\neg\psi_2 \in A_j$; particularly, $\neg\psi_2 \in A_0$. From lemma 3.7 we obtain that $\psi_1 \mathcal{S} \psi_2 \in A_0$. Hence, we have that $\psi_1 \mathcal{S} \psi_2, \neg\psi_2 \in A_0$ which contradicts the fact that in a fulfilling path A_0 is initial. ∎

Theorem 3.11 *Let* φ *be a* PTL *formula.* φ *is satisfiable if and only if there exists a fulfilling path for* φ *in* $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$.

PROOF. If φ is satisfiable and $(\sigma, \mathcal{I})$ is a model for φ, then it is easy to check that the pre-model $\pi_{(\sigma,\mathcal{I})}$, is a fulfilling path for φ.
Assume $\pi = A_0, A_1 \ldots$ is a fulfilling path for φ. We claim that $(\pi, \mathcal{I}_\varphi)$ is a model for φ.
Claim: For every $\psi \in CL(\varphi)$, and every j, $0 \leq j$, $\psi \in A_j$ if and only if $(\pi, \mathcal{I}_\varphi, j) \models \psi$.
We prove this claim by induction on the structure of the formula ψ.

- ψ is the proposition p:
  $(\pi, \mathcal{I}_\varphi, j) \models p \overset{\text{semantics def.}}{\Longleftrightarrow} A_j \in \mathcal{I}_\varphi(p) \overset{\mathcal{I}_\varphi\text{def.}}{\Longleftrightarrow} p \in A_j$.
- ψ is the formula $\neg\chi$:
  $\psi \in A_j \Longleftrightarrow \neg\chi \in A_j \overset{\text{atom } def.}{\Longleftrightarrow} \chi \notin A_j \overset{\text{ind. } hyp.}{\Longleftrightarrow} (\pi, \mathcal{I}_\varphi, j) \not\models \chi \Longleftrightarrow (\pi, \mathcal{I}_\varphi, j) \models \neg\chi \Longleftrightarrow (\pi, \mathcal{I}_\varphi, j) \models \psi$.

- ψ is the formula $\chi_1 \vee \chi_2$:
  $\psi \in A_j \iff \chi_1 \vee \chi_2 \in A_j \overset{\text{atom } def.}{\iff} \chi_1$ or $\chi_2 \in A_j \overset{\text{ind. } hyp.}{\iff} (\pi, \mathcal{I}_\varphi, j) \models \chi_1$ or $(\pi, \mathcal{I}_\varphi, j) \models \chi_2 \iff (\pi, \mathcal{I}_\varphi, j) \models \psi$.
- ψ is the formula $\bigcirc \chi$:
  $\psi \in A_j \iff \bigcirc \chi \in A_j \overset{\mathcal{R}_\varphi \text{def.}}{\iff} \chi \in A_{j+1} \overset{\text{ind. } hyp.}{\iff} (\pi, \mathcal{I}_\varphi, j+1) \models \chi \iff (\pi, \mathcal{I}_\varphi, j) \models \bigcirc \chi \iff (\pi, \mathcal{I}_\varphi, j) \models \psi$.
- ψ is the formula $\circleddash \chi$:
  $\psi \in A_j \iff \circleddash \chi \in A_j \overset{\mathcal{R}_\varphi \text{def.}}{\iff} j=0$ or $\chi \in A_{j-1} \overset{\text{ind. } hyp.}{\iff} j=0$ or $(\pi, \mathcal{I}_\varphi, j-1) \models \chi \iff (\pi, \mathcal{I}_\varphi, j) \models \circleddash \chi \iff (\pi, \mathcal{I}_\varphi, j) \models \psi$.
- ψ is the formula $\chi_1 \mathcal{U} \chi_2$:
  From the fact that π is a fulfilling path and from lemma 3.7, we obtain that there exists some $k \geq j$ such that for every $i,\ j \leq i < k,\ \chi_1 \mathcal{U} \chi_2,\ \chi_1,\ \neg\chi_2 \in A_i$ and $\chi_1 \mathcal{U} \chi_2,\ \chi_2 \in A_k$.
  For every $i,\ j \leq i < k,\ \chi_1,\ \neg\chi_2 \in A_i$ and $\chi_2 \in A_k \overset{\text{ind. hyp.}}{\iff}$ For every $i,\ j \leq i < k,\ (\pi, \mathcal{I}_\varphi, i) \models \chi_1$ and $(\pi, \mathcal{I}_\varphi, k) \models \chi_2 \overset{\text{semantics def.}}{\iff} (\pi, \mathcal{I}_\varphi, j) \models \chi_1 \mathcal{U} \chi_2 \iff (\pi, \mathcal{I}_\varphi, j) \models \psi$.
- ψ is the formula $\chi_1 \mathcal{S} \chi_2$:
  From the fact that π is a fulfilling path and lemma 3.10 we obtain that there exists some $k \leq j$ such that for every $i,\ k < i \leq j,\ \chi_1 \mathcal{S} \chi_2,\ \chi_1\ , \neg\chi_2 \in A_i$ and $\chi_1 \mathcal{S} \chi_2\ , \chi_2 \in A_k$.
  For every $i,\ k < i \leq j,\ \chi_1,\ \neg\chi_2 \in A_i$ and $\chi_2 \in A_k \overset{\text{ind. hyp.}}{\iff}$ For every $i,\ k < i \leq j,\ (\pi, \mathcal{I}_\varphi, i) \models \chi_1$ and $(\pi, \mathcal{I}_\varphi, k) \models \chi_2 \overset{\text{semantics def.}}{\iff} (\pi, \mathcal{I}_\varphi, j) \models \chi_1 \mathcal{S} \chi_2 \iff (\pi, \mathcal{I}_\varphi, j) \models \psi$.

Since π is a fulfilling path for φ, $\varphi \in A_0$. From the claim we obtain that $(\pi, \mathcal{I}_\varphi, 0) \models \varphi$, which means that $(\pi, \mathcal{I}_\varphi)$ is a model for φ. ∎

From the last theorem we conclude that it is sufficient to look for a fulfilling path for φ in $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$ in order to determine the satisfiability of φ.

Let $\mathcal{C} \subseteq \langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$ be a subgraph. $\mathcal{C}$ is a *strongly connected* subgraph if for any two *different* nodes, $A, B \in \mathcal{C}$, $A \neq B$, there exists a path in $\mathcal{C}$ leading from A to B.

It is not difficult to see that for every pre-model π, the subgraph $Fin(\pi)$ is strongly connected.

We show that instead of searching fulfilling paths, which can be infinite objects, it is sufficient to investigate the *strongly connected subgraphs* (*scs's*, for short) of $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$ and the finite paths leading to them. Actually, we examine only the maximal *scs's* (*mscs's*, for short), i.e., strongly connected subgraphs that are not properly contained in a larger *scs*.

Definition 3.12 *Let $\mathcal{C}$ be an* scs *in $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$. $\mathcal{C}$ is* self fulfilling *if for every formula $\psi_1 \mathcal{U} \psi_2 \in A \in \mathcal{C}$, there exists an atom $B \in \mathcal{C}$ such that $\psi_2 \in B$.*

The following lemma claims that being self fulfilling is a monotone property, i.e.,

Lemma 3.13 *Let $\mathcal{C}$ and $\mathcal{C}'$ be* scs's *such that $\mathcal{C} \subseteq \mathcal{C}'$. If $\mathcal{C}$ is self fulfilling then so is $\mathcal{C}'$.*

PROOF. It is sufficient to consider the case $\mathcal{C} \subset \mathcal{C}'$, in which $|\mathcal{C}'|>1$.
Assume $\psi_1 \mathcal{U} \psi_2 \in A \in \mathcal{C}'$. Since $\mathcal{C}'$ is an *scs*, there is a descendent of A, B, in $\mathcal{C}$ (possibly $B = A$). By lemma 3.7 and the definition of $\mathcal{R}_\varphi$, we know that either $\psi_2 \in C$ where C is an atom belonging to the path from A to B, or $\psi_1 \mathcal{U} \psi_2 \in B$. In the latter case, since $\mathcal{C}$ is self fulfilling, there exists some atom $D \in \mathcal{C}$ such that $\psi_2 \in D$. In any case, we are assured of an atom $E \in \mathcal{C}'$ such that $\psi_2 \in E$ (E being C or D). ∎

We observe that a fulfilling path for φ must start at an initial atom and eventually remain contained in a self fulfilling *scs*. Our search for a fulfilling path proceeds by successively removing from $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$ atoms that cannot participate in a fulfilling path. Fortunately, we do not have to examine every atom separately, but may remove a complete *mscs*, called useless, at a time.

Definition 3.14 *Let $\mathcal{C}$ be an* mscs *in $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$. $\mathcal{C}$ is* useless *if it falls into one of the following categories:*

1. *$\mathcal{C}$ is not reachable from an atom belonging to $Init_\varphi$.*
2. *$\mathcal{C}$ has no outgoing $\mathcal{R}_\varphi$ edges and is not self fulfilling.*

3.4 Satisfiability Checking

Perform the following procedure:

$\langle \mathcal{A}_0, \mathcal{R}_0 \rangle := \langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$
$i := 0$
while $\langle \mathcal{A}_i, \mathcal{R}_i \rangle$ is not empty and contains some useless *mscs* **do**
  let $\mathcal{C}$ be a useless *mscs* in $\langle \mathcal{A}_i, \mathcal{R}_i \rangle$
  $i := i+1$
  let $\langle \mathcal{A}_i, \mathcal{R}_i \rangle$ be a new structure where
   $\mathcal{A}_i = \mathcal{A}_{i-1} - \mathcal{C}$
   $\mathcal{R}_i = \mathcal{R}_{i-1} \cap (\mathcal{A}_i \times \mathcal{A}_i)$
if the final structure is not empty **then**
  report success
else report failure

The final structure obtained by this removal process is called the tableau for φ and is denoted by $\langle \mathcal{A}^*_\varphi, \mathcal{R}^*_\varphi \rangle$. The verification of the algorithm is based on the following property:

Lemma 3.15 *π is a fulfilling path in $\langle \mathcal{A}_i, \mathcal{R}_i \rangle$ if and only if it is a fulfilling path in $\langle \mathcal{A}_{i+1}, \mathcal{R}_{i+1} \rangle$.*

PROOF. This property is justified by the following observations:

1. As we mentioned before, a fulfilling path for φ must start at an initial atom and eventually remain contained in a self fulfilling *scs*.
2. The property of being self fulfilling is monotone, see lemma 3.13.
3. By removing a useless *mscs* from $\langle \mathcal{A}_i, \mathcal{R}_i \rangle$ we cannot damage any fulfilling path.

∎

Theorem 3.16 *The formula φ is satisfiable if and only if the algorithm above reports success.*

PROOF. φ is satisfiable $\overset{\text{Thm. 3.11}}{\Longleftrightarrow}$ there exists a fulfilling path for φ in $\langle \mathcal{A}_0, \mathcal{R}_0 \rangle \overset{\text{Lem. 3.15}}{\Longleftrightarrow}$ there exists a fulfilling path for φ in $\langle \mathcal{A}^*_\varphi, \mathcal{R}^*_\varphi \rangle \overset{\text{Def. of fulfilling path and mscs}}{\Longleftrightarrow} \langle \mathcal{A}^*_\varphi, \mathcal{R}^*_\varphi \rangle$ contains a finite path $\pi = A_0, \ldots, A_k$ where $A_0 \in Init_\varphi$ and the atom A_k is contained in a self fulfilling *mscs* $\overset{\text{absence of useless mscs's}}{\Longleftrightarrow} \langle \mathcal{A}^*_\varphi, \mathcal{R}^*_\varphi \rangle$ is not empty $\Longleftrightarrow$ the algorithm succeeds. ∎

Complexity of the Algorithm The following points lead to an overall time bound of $2^{O(|\varphi|)}$.

- As we already observed $|\mathcal{A}_\varphi| \leq 2^{2\cdot|\varphi|}$; Hence, $|\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle| \leq 2^{c|\varphi|}$.
- The decomposition of $\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle$ into *mscs's*, can be done within time bound of $O(|\langle \mathcal{A}_\varphi, \mathcal{R}_\varphi \rangle|)$ (see [6]).
- All the checkings, required by the algorithm, can be done in time linear in $|\mathcal{A}_\varphi| \cdot |\varphi|$.

The following fact is proved in [33].

Theorem 3.17 *The satisfiability problem for* PTL *is PSPACE-complete.*

3.5 Discussion

A version of PTL which does not assume that the past is bounded is also possible. The approach we have taken here can be easily extended to cover this version. We have to consider sequences that may be infinite at both directions (i.e., indexed by positive and negative integers). In the decision procedure we have to define a criterion of self fulfillment of parentless *mscs's* that requires the presence of the formula ψ in the component for each formula $\varphi \mathcal{S} \psi$ that appears in the component.

4 Completeness of a Deductive Proof System for PTL

We present here a complete deductive system for proving the validity of PTL formulae.

The presented formal proof for completeness uses standard method for showing the completeness of a deductive system: If φ is a valid PTL formula, then to show that $\vdash \varphi$, we investigate the process of checking satisfiability of the formula $\neg\varphi$, over its semantic tableaux.

4.1 A Deductive Proof System for PTL

We propose the following deductive system for establishing the validity of PTL formulae.

The Future Axiom Schemes

Following are eight scheme axioms dealing with the future operators.

$$
\begin{array}{ll}
\text{F0.} & \vdash \Box\varphi \rightarrow \varphi \\
\text{F1.} & \vdash \bigcirc\neg\varphi \Longleftrightarrow \neg\bigcirc\varphi \\
\text{F2.} & \vdash \bigcirc(\varphi \rightarrow \psi) \Rightarrow (\bigcirc\varphi \rightarrow \bigcirc\psi) \\
\text{F3.} & \vdash \Box(\varphi \rightarrow \psi) \Rightarrow (\Box\varphi \rightarrow \Box\psi) \\
\text{F4.} & \vdash \Box\varphi \rightarrow \Box\bigcirc\varphi \\
\text{F5.} & \vdash (\varphi \Rightarrow \bigcirc\varphi) \rightarrow (\varphi \Rightarrow \Box\varphi) \\
\text{F6.} & \vdash \varphi\mathcal{U}\psi \Longleftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi\mathcal{U}\psi)) \\
\text{F7.} & \vdash \varphi\mathcal{U}\psi \Rightarrow \Diamond\psi
\end{array}
$$

Note that –

- Axiom F0 states that if φ holds at all positions of a model, then in particular it holds at the first position.
- Axiom F1 states that the next operator, $\bigcirc$, is self-dual.
- Axiom F2 states that if in the next instant φ implies ψ then if in the next instant φ holds then so does ψ.
- Axiom F3 is the analogue of F2 for the $\Box$ operator.
- Axiom F4 states that if φ is true at all future instants it must be true at every next instant.
- Axiom F5 is a "computational induction" axiom; it states that if a property is inherited over one step transition, it is invariant over any suffix sequence whose first state satifies φ.
- Axiom F6 characterizes the until operator by distributing its effect into what is implied for the present and what is implied for the next instant.
- Axiom F7 states that $\varphi\mathcal{U}\psi$ implies that ψ will eventually happen.

The Past Axiom Schemes

Following are five scheme axioms dealing with the past operators. Four of them are symmetric to four future axioms.

$$
\begin{array}{ll}
\text{P1.} & \vdash \ominus\varphi \Rightarrow \widetilde{\ominus}\varphi \\
\text{P2.} & \vdash \widetilde{\ominus}(\varphi \rightarrow \psi) \Rightarrow (\widetilde{\ominus}\varphi \rightarrow \widetilde{\ominus}\psi) \\
\text{P3.} & \vdash \Box\varphi \rightarrow \Box\widetilde{\ominus}\varphi \\
\text{P4.} & \vdash \varphi\mathcal{S}\psi \Longleftrightarrow \psi \vee (\varphi \wedge \ominus(\varphi\mathcal{S}\psi)) \\
\text{P5.} & \vdash \widetilde{\ominus}\, false
\end{array}
$$

Note that

- Axiom P1 establishes the connection between the weak-previous operator and the previous operator. It defines $\widetilde{\ominus}$ as the weaker version of $\ominus$.
- The axioms P2, P3, P4 have similar descriptions to the axioms F2, F4,F6.
- Axiom P5 does not resemble a corresponding future axiom. It states that the first position of every sequence satisfies $\widetilde{\ominus}$ *false*.

- It is not necessary to require $\varphi \mathcal{S} \psi \Rightarrow \diamondminus \psi$ which is the past counterpart of F7, as an axiom, since we show later that it is a theorem of the deductive system.

The axioms we have considered so far consist of either future formulae or their past counterparts, which retain in some cases the operator $\square$. Following is one scheme axiom that mix future with past.

The Mixed Axiom Scheme

$$\text{M8.} \vdash \varphi \Rightarrow \bigcirc \circleddash \varphi$$

Note that the axiom M8 states that if φ holds at a position j, then going forwards one step to position $j+1$ and then backwards one step, ends up in a position (obviously j again) that also satisfies φ.

Inference Rules

INS *instantiation*

If φ is a substitution instance of a propositional tautology then $\vdash \square \varphi$.

MP *modus ponens*

$$\frac{\vdash \varphi \rightarrow \psi, \vdash \varphi}{\vdash \psi}$$

Note that in INS we also include temporal instances of tautologies; we may substitute an arbitrary PTL formula for a proposition letter in obtaining an instance.

Theorem 4.1 *The system is sound for* PTL*; i.e., for every* PTL *formula, if* $\vdash \varphi$ *then* φ *is valid.*

PROOF. The result follows immediately from the definition of the semantics and checking that every axiom and every inference rule preserves validity. ∎

We present in the next subsection some theorems and derived rules of the system.

4.2 Theorems and Rules of The Deductive System

We present here some theorems and derived rules of the system. We chose to present only the theorems and rules that are needed in the rest of the paper. The reader interested in other theorems or rules can find a comprehensive list of theorems and derived rules together with proofs in [22] for the floating version of PTL, and in [23] for the anchored version of PTL.

We present here only the list of the theorems and the rules. Their detailed proofs appear in the appendix.

Rule PAR (particularization) $\square \varphi \vdash \varphi$

Rule E-MP (entailment modus ponens)
$(\varphi_1 \wedge \ldots \wedge \varphi_n) \Rightarrow \psi,\ \Box\,\varphi_1,\ \ldots\ ,\ \Box\,\varphi_n\ \vdash\ \Box\,\psi$

Rule E-TRNS (entailment transitivity) $\varphi \Rightarrow \psi,\ \psi \Rightarrow \chi\ \vdash\ \varphi \Rightarrow \chi$

Rule R1 $\Box\,\varphi\ \vdash\ \psi \Rightarrow \varphi$

Rule R2 $\varphi \Rightarrow \psi\ \vdash\ \neg\psi \Rightarrow \neg\varphi$

Rule R3 $\varphi \Longleftrightarrow \psi\ \vdash\ \varphi \Rightarrow \psi,\ \psi \Rightarrow \varphi$

Rule R4 $\varphi \Rightarrow \chi,\ \psi \Rightarrow \chi\ \vdash\ \varphi \vee \psi \Rightarrow \chi$

Rule R5 $\varphi \Rightarrow \psi,\ \varphi \Rightarrow \chi\ \vdash\ \varphi \Rightarrow \psi \wedge \chi$

Rule R6 $\varphi \Rightarrow \psi,\ \psi \Rightarrow \varphi\ \vdash\ \varphi \Longleftrightarrow \psi$

Rule R7 $\varphi \Rightarrow (\psi \rightarrow \chi)\ \vdash\ \varphi \wedge \psi \Rightarrow \chi$

Rule CI (computational induction) $\varphi \Rightarrow \bigcirc\,\varphi\ \vdash\ \varphi \Rightarrow \Box\,\varphi$

Rule $\text{\textcircled{$\sim$}}$G ($\text{\textcircled{$\sim$}}$ generalization) $\Box\,\varphi\ \vdash\ \Box\,\text{\textcircled{$\sim$}}\,\varphi$

Rule $\bigcirc$G ($\bigcirc$ generalization) $\Box\,\varphi\ \vdash\ \Box\,\bigcirc\,\varphi$

Rule $\text{\textcircled{$\sim$}}$M ($\text{\textcircled{$\sim$}}$ is monotonic) $\varphi \Rightarrow \psi\ \vdash\ \text{\textcircled{$\sim$}}\,\varphi \Rightarrow \text{\textcircled{$\sim$}}\,\psi$

Theorem T1 $\ominus\,\varphi \Longleftrightarrow \neg\,\text{\textcircled{$\sim$}}\,\neg\varphi$

Rule $\bigcirc$M ($\bigcirc$ is monotonic) $\varphi \Rightarrow \psi\ \vdash\ \bigcirc\,\varphi \Rightarrow \bigcirc\,\psi$

Theorem T2 $\ominus\,\varphi \Rightarrow \neg\,\text{\textcircled{$\sim$}}\,false$

Theorem T3 $\bigcirc(\varphi \wedge \psi) \Longleftrightarrow (\bigcirc\,\varphi \wedge \bigcirc\,\psi)$

Theorem T4 $\Box\,\varphi \Rightarrow \bigcirc\,\Box\,\varphi$

Rule $\Box$G ($\Box$ generalization) $\Box\,\varphi\ \vdash\ \Box\,\Box\,\varphi$

Rule $\Box\,M$ ($\Box$ is monotonic) $\varphi \Rightarrow \psi\ \vdash\ \Box\,\varphi \Rightarrow \Box\,\psi$

Theorem T5 $(\Box\,\varphi \vee \Box\,\psi) \Rightarrow \Box(\varphi \vee \psi)$

Theorem T6 $\Box(\varphi \wedge \psi) \Longleftrightarrow (\Box\,\varphi \wedge \Box\,\psi)$

4.3 Completeness Proof

We will show that the presented deductive system consisting of axioms F0-F7, P1, P2, P3, P4, P5, M8 and the inference rules INS and MP is complete. We use here a standard proof of completeness to the presented deductive system: Let φ be some valid PTL formula. To show that $\vdash \varphi$, we investigate the process of checking satisfiability of the formula $\neg\varphi$, over its semantic tableaux. We try to prove in the deductive system, for every atom A declared as useless by the algorithm, $\vdash \neg\varphi \longrightarrow \Box\neg\widehat{A}$, where $\widehat{A}$ is the conjunction of all the formulae contained in A.

Theorem 4.2 *The system is complete for* PTL*; i.e., for every* PTL *formula, if φ is valid then $\vdash \varphi$.*

PROOF. Suppose the formula φ is valid. Then $\neg\varphi$ is not satisfiable. Let us assume that we apply the satisfiabilty checking procedure, described in section 3 to $\neg\varphi$. In the application we construct a sequence of graphs

$$\langle \mathcal{A}_{\neg\varphi}, \mathcal{R}_{\neg\varphi} \rangle = \langle \mathcal{A}_0, \mathcal{R}_0 \rangle, \ldots, \langle \mathcal{A}_i, \mathcal{R}_i \rangle, \ldots, \langle \mathcal{A}_n, \mathcal{R}_n \rangle = \left\langle \mathcal{A}^*_{\neg\varphi}, \mathcal{R}^*_{\neg\varphi} \right\rangle$$

which eventually converges to an empty structure. ∎

For an atom A we denote $\widehat{A} = \bigwedge_{\psi \in A} \psi$.

The following list of claims establishes that φ is provable in our system.

Claim 4.3

$$\vdash \bigvee_{A \in \mathcal{A}_0} \widehat{A}$$

PROOF. A straightforward result from Propositional Logic; Recall that $\mathcal{A}_0$ are all the atoms. ∎

Claim 4.4 *Let A be some atom. If A is not initial then*

$$\vdash \widehat{A} \Rightarrow \neg \circledsim false.$$

PROOF. Recall that if A is not initial then

1. It contains some formula of the form $\ominus\psi$, or
2. It contains some formula $\psi\,\mathcal{S}\chi$ such that $\chi \notin A$. Since A is an atom, $\neg\chi \in A$. By Propositional Logic and INS we can prove that $\vdash \widehat{A} \Rightarrow \neg\chi$, and by P4 we can prove that $\widehat{A} \Rightarrow \ominus(\psi\,\mathcal{S}\chi)$.

In both cases the claim follows from T2. ∎

Claim 4.5

$$\vdash \bigvee_{A \in \mathcal{A}_0,\ A \text{ is initial}} \widehat{A}.$$

PROOF. From Claim 4.3, Claim 4.4, PAR and P5. ∎

Claim 4.6 *For every* $\psi \in \mathrm{CL}(\varphi)$,

$$\vdash \psi \leftrightarrow \bigvee_{A \in \mathcal{A}_0 \bigcap Init_\psi} \widehat{A}$$

PROOF. From Claim 4.3, Claim 4.5 and the fact that for every atom A, $\neg\psi \in A$ if and only if $\psi \notin A$. ∎

Claim 4.7 *Let A be some atom.*

$$\vdash \widehat{A} \Rightarrow \bigcirc(\bigvee_{B \in \mathcal{A}_0} \widehat{B}).$$

PROOF. Let us denote the formula $\bigvee_{B \in \mathcal{A}_0} \widehat{B}$ by α.

1.	α	Propositional Tautology (Claim 4.3)
2.	$\Box \alpha$	INS Claim 4.3
3.	$\Box \bigcirc \alpha$	$\bigcirc$G 1
4.	$\widehat{A} \Rightarrow \bigcirc \alpha$	R1 3
5.	$\widehat{A} \Rightarrow \bigcirc(\bigvee_{B \in \mathcal{A}_0} \widehat{B})$	3

∎

Claim 4.8 *Let A, B be some atoms such that $(A, B) \notin \mathcal{R}_0$.*

$$\vdash \widehat{A} \Rightarrow \bigcirc \neg \widehat{B}.$$

PROOF. Consider the different reasons for $(A, B) \notin \mathcal{R}_0$.

***Case 1*:** There is a formula $\bigcirc \psi \in CL(\varphi)$ such that $\bigcirc \psi \in Aa$ and $\psi \notin B$, i.e., $\neg\psi \in B$.
Proof: see appendix B.

***Case 2*:** There is a formula $\bigcirc \psi \in CL(\varphi)$ such that $\bigcirc \psi \notin A$, i.e., $\neg \bigcirc \psi \in A$ and $\psi \in B$.
Proof: see appendix B.

***Case 3*:** There is a formula $\circleddash \psi \in CL(\varphi)$ such that $\psi \in A$ and $\circleddash \psi \notin B$, i.e., $\neg \circleddash \psi \in B$.
Proof: see appendix B.

***Case 4*:** There is a formula $\circleddash \psi \in CL(\varphi)$ such that $\psi \notin A$, i.e., $\neg\psi \in A$ and $\circleddash \psi \in B$.
Proof: see appendix B.

***Case 5*:** There is a formula $\psi_1 \mathcal{U} \psi_2 \in CL(\varphi)$ such that $\psi_1 \mathcal{U} \psi_2$, $\neg\psi_2 \in A$ and $\psi_1 \mathcal{U} \psi_2 \notin B$, i.e., $\neg(\psi_1 \mathcal{U} \psi_2) \in B$.
Proof: see appendix B.

***Case 6*:** There is a formula $\psi_1 \mathcal{U} \psi_2 \in CL(\varphi)$ such that $\psi_1 \mathcal{U} \psi_2 \in B$, $\psi_1 \in A$ and $\psi_1 \mathcal{U} \psi_2 \notin A$, i.e., $\neg(\psi_1 \mathcal{U} \psi_2) \in A$.
Proof: From F6 it can be proven that

$$\vdash \neg(\psi_1 \mathcal{U} \psi_2) \Rightarrow (\neg\psi_1 \vee \neg \bigcirc(\psi_1 \mathcal{U} \psi_2)).$$

Since we can prove that $\vdash \widehat{A} \Rightarrow \psi_1$, we can also prove that

$$\vdash \widehat{A} \Rightarrow \neg \bigcirc(\psi_1 \mathcal{U} \psi_2)$$

which implies by F1 that

$$\vdash \widehat{A} \Rightarrow \bigcirc \neg \widehat{B}.$$

***Case 7*:** There is a formula $\psi_1 \mathcal{S} \psi_2 \in CL(\varphi)$ such that $\psi_1 \mathcal{S} \psi_2,\ \neg\psi_2 \in B$ and $\psi_1 \mathcal{S} \psi_2 \notin A$, i.e., $\neg(\psi_1 \mathcal{S} \psi_2) \in A$.

Proof: From the fact that $\neg(\psi_1 \mathcal{S} \psi_2) \in A$ and axiom M8, we can prove

$$\vdash \widehat{A} \Rightarrow \bigcirc \circleddash \neg(\psi_1 \mathcal{S} \psi_2),$$

and from $\circledS$M, T1 and $\bigcirc$M, we can be proven that

$$\vdash \widehat{A} \Rightarrow \bigcirc \neg \circledS (\psi_1 \mathcal{S} \psi_2).$$

By the fact that $\psi_1 \mathcal{S} \psi_2,\ \neg\psi_2 \in B$ and P4, we can prove that

$$\vdash \widehat{B} \Rightarrow \circleddash(\psi_1 \mathcal{S} \psi_2),$$

and by P1

$$\vdash \widehat{B} \Rightarrow \circledS(\psi_1 \mathcal{S} \psi_2)$$

or

$$\vdash \neg \circledS(\psi_1 \mathcal{S} \psi_2) \Rightarrow \neg \widehat{B}.$$

Now, it is easy to derive by $\bigcirc$M that

$$\vdash \widehat{A} \Rightarrow \bigcirc \neg \widehat{B}.$$

***Case 8*:** There is a formula $\psi_1 \mathcal{S} \psi_2 \in CL(\varphi)$ such that $\psi_1 \mathcal{S} \psi_2 \in A$, $\psi_1 \in B$ and $\psi_1 \mathcal{S} \psi_2 \notin B$, i.e., $\neg(\psi_1 \mathcal{S} \psi_2) \in B$.

Proof: From the fact that $\psi_1 \mathcal{S} \psi_2 \in A$ and axiom M8, we can prove

$$\vdash \widehat{A} \Rightarrow \bigcirc \circleddash(\psi_1 \mathcal{S} \psi_2).$$

From P4 and the fact that $\psi_1,\ \neg(\psi_1 \mathcal{S} \psi_2) \in B$, we can prove that

$$\vdash \widehat{B} \Rightarrow \neg \circleddash(\psi_1 \mathcal{S} \psi_2)$$

or

$$\vdash \circleddash(\psi_1 \mathcal{S} \psi_2) \Rightarrow \neg \widehat{B},$$

and by $\bigcirc$M we can prove that

$$\vdash \widehat{A} \Rightarrow \bigcirc \neg \widehat{B}.$$

∎

Claim 4.9 *Let A be some atom.*

$$\vdash \widehat{A} \Rightarrow \bigcirc\Big(\bigvee_{B \in \mathcal{A}_0 \text{ and } (A,B) \in \mathcal{R}_0} \widehat{B}\Big).$$

PROOF. From Claim 4.7, Claim 4.8 by R5, E-TRNS and T3. ∎

Remark 4.10 *Note that in the case that the atom A has not any successor then*

$$\vdash \widehat{A} \Rightarrow \bigcirc\, false.$$

Intuitively, this means that A cannot appear in a fulfilling path.

Let us denote by $\mathcal{R}_i^*$ the reflexive transitive closure of $\mathcal{R}_i$, $1 \leq i \leq n$.

Claim 4.11 *Let A be some atom.*

$$\vdash \widehat{A} \Rightarrow \Box(\bigvee_{B \in \mathcal{A}_0 \ and \ (A,B) \in \mathcal{R}_0^*} \widehat{B}).$$

PROOF. Let us denote by α the formula

$$\bigvee_{B \in \mathcal{A}_0 \ and \ (A,B) \in \mathcal{R}_0^*} \widehat{B}.$$

Let B be some atom such that $(A,B) \in \mathcal{R}_0^*$. Clearly, if C is an atom such that $(B,C) \in \mathcal{R}_0$ then $(A,C) \in \mathcal{R}_0^*$; hence, it can be proven that $\vdash \bigvee_{C \in \mathcal{A}_0 \ and \ (B,C) \in \mathcal{R}_0} \widehat{C} \Rightarrow \alpha$.

1.	$\bigvee_{C \in \mathcal{A}_0 \ and \ (B,C) \in \mathcal{R}_0} \widehat{C} \Rightarrow \alpha$	given
2.	$\bigcirc(\bigvee_{C \in \mathcal{A}_0 \ and \ (B,C) \in \mathcal{R}_0} \widehat{C}) \Rightarrow \bigcirc\, \alpha$	$\bigcirc$M 1
3.	$\widehat{B} \Rightarrow \bigcirc(\bigvee_{C \in \mathcal{A}_0 \ and \ (B,C) \in \mathcal{R}_0} \widehat{C})$	Claim 4.9
4.	$\widehat{B} \Rightarrow \bigcirc\, \alpha$	E-TRNS 3, 2
5.	$\bigvee_{B \in \mathcal{A}_0 \ and \ (A,B) \in \mathcal{R}_0^*} \widehat{B} \Rightarrow \bigcirc\, \alpha$	R4 4
6.	$\alpha \Rightarrow \bigcirc\, \alpha$	5
7.	$\alpha \Rightarrow \Box\, \alpha$	CI 6
8.	$\widehat{A} \rightarrow \alpha$	$\mathcal{R}_0$ is reflexive
9.	$\widehat{A} \Rightarrow \alpha$	INS 8
10.	$\widehat{A} \Rightarrow \Box\, \alpha$	E-TRNS 8, 7
11.	$\widehat{A} \Rightarrow \Box(\bigvee_{B \in \mathcal{A}_0 \ and \ (A,B) \in \mathcal{R}_0^*} \widehat{B})$	10 ∎

Remark 4.12 *Note that in the case that the atom A has no descendants then*

$$\vdash \widehat{A} \Rightarrow \Box\, false.$$

Intuitively, this means that A cannot appear in a fulfilling path.

Claim 4.13

$$\vdash \neg\varphi \longrightarrow \Box(\bigvee_{\substack{A \in \mathcal{A}_0 \ and \ A \ is \ reachable \\ from \ some \ atom \ in \ Init_{\neg\varphi}}} \widehat{A}).$$

PROOF.

1. $\neg\varphi \rightarrow \bigvee_{A\in\mathcal{A}_0 \bigcap Init_{\neg\varphi}} \widehat{A}$ — from Claim 4.6

   For every $A \in \mathcal{A}_0 \bigcap Init_{\neg\varphi}$,

2. $\widehat{A} \Rightarrow \Box(\bigvee_{B\in\mathcal{A}_0 \ and \ (A,B)\in\mathcal{R}_0^*} \widehat{B})$ — Claim 4.11
3. $\widehat{A} \rightarrow \Box(\bigvee_{B\in\mathcal{A}_0 \ and \ (A,B)\in\mathcal{R}_0^*} \widehat{B})$ — PAR 2
4. $\neg\varphi \rightarrow \bigvee_{A\in\mathcal{A}_0 \bigcap Init_{\neg\varphi}} \Box(\bigvee_{B\in\mathcal{A}_0 \ and \ (A,B)\in\mathcal{R}_0^*} \widehat{B})$ — from 1, 3
5. $\bigvee_{A\in\mathcal{A}_0 \bigcap Init_{\neg\varphi}} \Box(\bigvee_{B\in\mathcal{A}_0 \ and \ (A,B)\in\mathcal{R}_0^*} \widehat{B}) \rightarrow$
   $\Box(\bigvee_{A\in\mathcal{A}_0 \ and \ A \ is \ reachable \ from \ some \ atom \ in \ Init_{\neg\varphi}} \widehat{A})$ — From T5
6. $\neg\varphi \longrightarrow \Box(\bigvee_{A\in\mathcal{A}_0 \ and \ A \ is \ reachable \ from \ some \ atom \ in \ Init_{\neg\varphi}} \widehat{A})$ — from 4, 5

∎

We now wish to generalize claim 4.11 and claim 4.13 to $\mathcal{R}_i$, $1 \le i \le n$. We will prove by induction on i, $0 \le i \le n$ the following five claims:

Claim 4.14 *For every atom $A \notin \mathcal{A}_i$,*

$$\vdash \neg\varphi \longrightarrow \Box\, \neg\widehat{A}.$$

Claim 4.14 says that a removed atom can be proven to be contradictory in a computation beginning with a state satisfying $\neg\varphi$.

Claim 4.15 *Let A be some atom in $\mathcal{A}_i$.*

$$\vdash \neg\varphi \longrightarrow \left(\widehat{A} \Rightarrow \bigcirc\Big(\bigvee_{B\in\mathcal{A}_i \ and \ (A,B)\in\mathcal{R}_i} \widehat{B}\Big)\right).$$

Claim 4.16 *Let A be some atom in $\mathcal{A}_i$.*

$$\vdash \neg\varphi \longrightarrow \left(\widehat{A} \Rightarrow \Box\Big(\bigvee_{B\in\mathcal{A}_i \ and \ (A,B)\in\mathcal{R}_i^*} \widehat{B}\Big)\right).$$

Claim 4.17

$$\vdash \neg\varphi \leftrightarrow \bigvee_{A\in\mathcal{A}_i \bigcap Init_{\neg\varphi}} \widehat{A}.$$

Claim 4.18

$$\vdash \neg\varphi \longrightarrow \Box\Big(\bigvee_{\substack{A \in \mathcal{A}_i \ and \ A \ is \ reachable \\ from \ some \ atom \ in \ Init_{\neg\varphi}}} \widehat{A}\Big).$$

For $i=0$ claim 4.14 is vacuously true; claims 11, 12 can be proven from claims 7, 8, respectively by $\Box$G, by R1 and PAR; claim 4.17 is claim 4.4 and claim 4.18 is claim 4.13.

Consider the passage from i to $i+1$. It is effected by removing a strongly connected component $\mathcal{C}$ from $\langle \mathcal{A}_i, \mathcal{R}_i \rangle$. Consider the two different reasons for $\mathcal{C}$ being declared useless in $\langle \mathcal{A}_i, \mathcal{R}_i \rangle$:

1. $\mathcal{C}$ is not reachable from an atom belonging to $Init_{\neg\varphi}$.
   Let A be some atom in $\mathcal{C}$.
   - Since $\mathcal{C}$ is not reachable from an atom belonging to $Init_{\neg\varphi}$, A itself is not reachable from an atom belonging to $Init_{\neg\varphi}$. Hence, it can be proven by Propositional Logic that
   $$\vdash \bigvee_{\substack{B \in \mathcal{A}_i \text{ and } B \text{ is reachable} \\ \text{from some atom in } Init_{\neg\varphi}}} \widehat{B} \longrightarrow \neg \widehat{A}.$$
   - Let us denote the formula
   $$\bigvee_{\substack{B \in \mathcal{A}_i \text{ and } B \text{ is reachable} \\ \text{from some atom in } Init_{\neg\varphi}}} \widehat{B}$$
   by α.

1.	$\alpha \longrightarrow \neg\widehat{A}$	Propositional Tautology
2.	$\Box(\alpha \longrightarrow \neg\widehat{A})$	INS 1
3.	$\Box(\alpha \longrightarrow \neg\widehat{A}) \Rightarrow (\Box\,\alpha \longrightarrow \Box\,\neg\widehat{A})$	F3
4.	$\Box(\alpha \longrightarrow \neg\widehat{A}) \longrightarrow (\Box\,\alpha \longrightarrow \Box\,\neg\widehat{A})$	PAR 3
5.	$\Box\,\alpha \longrightarrow \Box\,\neg\widehat{A}$	MP 4, 2
6.	$\neg\varphi \longrightarrow \Box\,\alpha$	Claim 4.18 (ind. hyp.)
7.	$\neg\varphi \longrightarrow \Box\,\neg\widehat{A}$	from 6,5

2. $\mathcal{C}$ has no outgoing edges and is not self fulfilling.
   Let A be some atom in $\mathcal{C}$.
   - In this case we know that there is a formula $\psi_1 \mathcal{U} \psi_2 \in CL(\varphi)$ such that every atom B in $\mathcal{C}$ contains $\psi_1 \mathcal{U} \psi_2$ and $\neg\psi_2$, hence we can prove that
   $$\vdash \widehat{A} \rightarrow \psi_1 \mathcal{U} \psi_2$$
   and
   $$\vdash \widehat{B} \rightarrow \neg\psi_2.$$
   By INS we have
   $$\vdash \widehat{A} \Rightarrow \psi_1 \mathcal{U} \psi_2$$
   and
   $$\vdash \widehat{B} \Rightarrow \neg\psi_2.$$
   - Since $\mathcal{C}$ has no outgoing edges, any atom B such that $(A, B) \in \mathcal{R}_i^*$ is necessarily contained in $\mathcal{C}$. It follows that we can prove by claim 4.16 that
   $$\vdash \neg\varphi \longrightarrow \left(\widehat{A} \Rightarrow (\psi_1 \mathcal{U} \psi_2 \wedge \Box\,\neg\psi_2)\right).$$

- From the last theorem and F7, which is

$$\vdash \psi_1 \mathcal{U} \psi_2 \Rrightarrow \Diamond \psi_2$$

we can prove that

$$\vdash \neg\varphi \longrightarrow \left(\widehat{A} \Rrightarrow \mathit{false}\right)$$

which implies

$$\vdash \neg\varphi \longrightarrow \Box(\neg\widehat{A}).$$

This establishes that claim 4.14 holds also for $\mathcal{A}_{i+1}$.

Proof of claim 4.15 for $\mathcal{R}_{i+1}$:
Consider an atom $A \in \mathcal{A}_{i+1}$.

1.	$\neg\varphi \longrightarrow \left(\widehat{A} \Rrightarrow \bigcirc(\bigvee_{B\in\mathcal{A}_i \text{ and } (A,B)\in\mathcal{R}_i} \widehat{B})\right)$	claim 4.15 for $\mathcal{R}_i$ (ind. hyp.)
	For every atom B such that $(A,B) \in \mathcal{R}_i - \mathcal{R}_{i+1}$,	
2.	$\neg\varphi \longrightarrow \Box\left(\widehat{A} \longrightarrow \bigcirc(\bigvee_{B\in\mathcal{A}_i \text{ and } (A,B)\in\mathcal{R}_i} \widehat{B})\right)$	Definition of $\Rrightarrow$
3.	$\neg\varphi \longrightarrow \Box\,\neg\widehat{B}$	claim 4.14 for $\mathcal{R}_{i+1}$ (ind. hyp.)
4.	$\neg\varphi \longrightarrow \Box \bigcirc \neg\widehat{B}$	from 3 by F4
5.	$\neg\varphi \longrightarrow \bigwedge_{B\in\mathcal{A}_i \text{ and } (A,B)\in\mathcal{R}_i-\mathcal{R}_{i+1}} \Box \bigcirc \neg\widehat{B}$	from 4
6.	$\neg\varphi \longrightarrow \Box(\bigwedge_{B\in\mathcal{A}_i \text{ and } (A,B)\in\mathcal{R}_i-\mathcal{R}_{i+1}} \bigcirc \neg\widehat{B})$	from 5 by T6
7.	$\neg\varphi \longrightarrow \Box \bigcirc(\bigwedge_{B\in\mathcal{A}_i \text{ and } (A,B)\in\mathcal{R}_i-\mathcal{R}_{i+1}} \neg\widehat{B})$	from 6 by T3
8.	$\neg\varphi \longrightarrow \Box\left(\widehat{A} \longrightarrow \left(\bigcirc(\bigvee_{B\in\mathcal{A}_i \text{ and } (A,B)\in\mathcal{R}_i} \widehat{B})\right.\right.$	
	$\left.\left.\wedge \bigcirc(\bigwedge_{B\in\mathcal{A}_i \text{ and } (A,B)\in\mathcal{R}_i-\mathcal{R}_{i+1}} \neg\widehat{B})\right)\right)$	from 2, 7 by T6
9.	$\neg\varphi \longrightarrow \Box\left(\widehat{A} \longrightarrow \bigcirc(\bigvee_{B\in\mathcal{A}_i \text{ and } (A,B)\in\mathcal{R}_{i+1}} \widehat{B})\right)$	from 8 by T3
10.	$\neg\varphi \longrightarrow \Box\left(\widehat{A} \Rrightarrow \bigcirc(\bigvee_{B\in\mathcal{A}_i \text{ and } (A,B)\in\mathcal{R}_{i+1}} \widehat{B})\right)$	Definition of $\Rrightarrow$

This establishes claim 4.15 for $\mathcal{R}_{i+1}$. Repeating the proof of claim 4.11, but using $\mathcal{R}^*_{i+1}$ instead of $\mathcal{R}^*_0$, we can derive claim 4.16 from claim 4.15. Claim 4.17 is a straightforward propositional logic result of claim 4.17 for i (the induction hypothesis) and claim 4.14. Claim 4.18 can be proven from claims 12 and 13 by F0 and T5.

We show now that φ is provable by the proof system. Since $\neg\varphi$ is unsatisfiable, we know that $\langle \mathcal{A}_n, \mathcal{R}_n \rangle = \langle \mathcal{A}^*_{\neg\varphi}, \mathcal{R}^*_{\neg\varphi} \rangle$ is empty. Consequently, by claim 4.17 for $\mathcal{A}_n$ we have

$$\vdash \neg\varphi \longleftrightarrow false$$

which leads to

$$\vdash \neg\neg\varphi$$

and to

$$\vdash \varphi.$$

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A Proofs of Theorems and Rules of The Deductive System

We present here the detailed proofs of the theorems and derived rules that appeared in subsection 4.2.

Rule PAR (particularization) $\Box\varphi \vdash \varphi$

PROOF.

1.	$\Box\varphi \rightarrow \varphi$	F0
2.	$\Box\varphi$	Premise
3.	φ	MP 1, 2

∎

Rule E-MP (entailment modus ponens)
$(\varphi_1 \wedge \ldots \wedge \varphi_n) \Rightarrow \psi,\ \Box\varphi_1,\ \ldots,\ \Box\varphi_n \vdash \Box\psi$

PROOF. We prove the rule for $n = 1$.

1.	$\Box(\varphi \rightarrow \psi) \Rightarrow (\Box\varphi \rightarrow \Box\psi)$	F3
2.	$\Box(\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi))$	Definition of $\Rightarrow$
3.	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$	PAR 2
4.	$\varphi \Rightarrow \psi$	Premise
5.	$\Box(\varphi \rightarrow \psi)$	Definition of $\Rightarrow$
6.	$\Box\varphi \rightarrow \Box\psi$	MP 3, 5
7.	$\Box\varphi$	Premise
8.	$\Box\psi$	MP 6, 7

∎

Rule E-TRNS (entailment transitivity) $\varphi \Rightarrow \psi,\ \psi \Rightarrow \chi \vdash \varphi \Rightarrow \chi$

PROOF.

1.	$\varphi \Rightarrow \psi$	Premise
2.	$\Box(\varphi \rightarrow \psi)$	Definition of $\Rightarrow$
3.	$\psi \Rightarrow \chi$	Premise
4.	$\Box(\psi \rightarrow \chi)$	Definition of $\Rightarrow$
5.	$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$	Propositional Tautology
6.	$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi) \Rightarrow (\varphi \rightarrow \chi)$	INS 5
7.	$\Box(\varphi \rightarrow \chi)$	E-MP 6, 2, 4
8.	$\varphi \Rightarrow \chi$	Definition of $\Rightarrow$

∎

Rule R1 $\Box\varphi \vdash \psi \Rightarrow \varphi$

PROOF.

1.	$\varphi \rightarrow (\psi \rightarrow \varphi)$	Propositional Tautology
2.	$\varphi \Rightarrow (\psi \rightarrow \varphi)$	INS 1
3.	$\Box\varphi$	Premise
4.	$\Box(\psi \rightarrow \varphi)$	E-MP 2, 3
5.	$\psi \Rightarrow \varphi$	Definition of $\Rightarrow$

∎

The following rules R2, R3, R4, R5, R6 and R7 have proofs similar to the proof of rule R1 (i.e., proposition al tautology, INS and then E-MP).
Rule R2 $\varphi \Rightarrow \psi \vdash \neg\psi \Rightarrow \neg\varphi$
Rule R3 $\varphi \Longleftrightarrow \psi \vdash \varphi \Rightarrow \psi,\ \psi \Rightarrow \varphi$
Rule R4 $\varphi \Rightarrow \chi,\ \psi \Rightarrow \chi \vdash \varphi \vee \psi \Rightarrow \chi$
Rule R5 $\varphi \Rightarrow \psi,\ \varphi \Rightarrow \chi \vdash \varphi \Rightarrow \psi \wedge \chi$
Rule R6 $\varphi \Rightarrow \psi,\ \psi \Rightarrow \varphi \vdash \varphi \Longleftrightarrow \psi$
Rule R7 $\varphi \Rightarrow (\psi \rightarrow \chi) \vdash \varphi \wedge \psi \Rightarrow \chi$
Rule CI (computational induction) $\varphi \Rightarrow \bigcirc\varphi \vdash \varphi \Rightarrow \Box\varphi$

PROOF.	1.	$\varphi \Rightarrow \bigcirc\varphi$	Premise
	2.	$(\varphi \Rightarrow \bigcirc\varphi) \rightarrow (\varphi \Rightarrow \Box\varphi)$	F5
	3.	$\varphi \Rightarrow \Box\varphi$	MP 2, 1

$\blacksquare$

Rule $\circledsim$G ($\circledsim$ generalization) $\Box\varphi \vdash \Box\circledsim\varphi$

PROOF.	1.	$\Box\varphi \rightarrow \Box\circledsim\varphi$	P3
	2.	$\Box\varphi$	Premise
	3.	$\Box\circledsim\varphi$	MP 1, 2

$\blacksquare$

Rule $\bigcirc$G ($\bigcirc$ generalization) $\Box\varphi \vdash \Box\bigcirc\varphi$

PROOF.	1.	$\Box\varphi \rightarrow \Box\bigcirc\varphi$	F4
	2.	$\Box\varphi$	Premise
	3.	$\Box\bigcirc\varphi$	MP 1, 2

$\blacksquare$

Rule $\circledsim$M ($\circledsim$ is monotonic) $\varphi \Rightarrow \psi \vdash \circledsim\varphi \Rightarrow \circledsim\psi$

PROOF.	1.	$\varphi \Rightarrow \psi$	Premise
	2.	$\Box(\varphi \rightarrow \psi)$	Definition of $\Rightarrow$
	3.	$\Box\circledsim(\varphi \rightarrow \psi)$	$\circledsim$G 2
	4.	$\circledsim(\varphi \rightarrow \psi) \Rightarrow (\circledsim\varphi \rightarrow \circledsim\psi)$	P2
	5.	$\Box(\circledsim\varphi \rightarrow \circledsim\psi)$	E-MP 4, 3
	6.	$\circledsim\varphi \Rightarrow \circledsim\psi$	Definition of $\Rightarrow$

$\blacksquare$

Theorem T1 $\ominus\varphi \Longleftrightarrow \neg\circledsim\neg\varphi$

PROOF. Definition of $\ominus$. $\blacksquare$

Rule $\bigcirc$M ($\bigcirc$ is monotonic) $\varphi \Rightarrow \psi \vdash \bigcirc\varphi \Rightarrow \bigcirc\psi$

PROOF.	1.	$\varphi \Rightarrow \psi$	Premise
	2.	$\Box(\varphi \rightarrow \psi)$	Definition of $\Rightarrow$
	3.	$\Box\bigcirc(\varphi \rightarrow \psi)$	$\bigcirc$G 2
	4.	$\bigcirc(\varphi \rightarrow \psi) \Rightarrow (\bigcirc\varphi \rightarrow \bigcirc\psi)$	F2
	5.	$\Box(\bigcirc\varphi \rightarrow \bigcirc\psi)$	E-MP 4, 3
	6.	$\bigcirc\varphi \Rightarrow \bigcirc\psi$	Definition of $\Rightarrow$

$\blacksquare$

Theorem T2 $\ominus\varphi \Rightarrow \neg\circledsim\mathit{false}$

PROOF.	1.	$\mathit{false} \Rightarrow \neg\varphi$	INS
	2.	$\circledsim\mathit{false} \Rightarrow \circledsim\neg\varphi$	$\circledsim$M 1
	3.	$\neg\circledsim\neg\varphi \Rightarrow \neg\circledsim\mathit{false}$	R2 2
	4.	$\ominus\varphi \Longleftrightarrow \neg\circledsim\neg\varphi$	T1
	5.	$\ominus\varphi \Rightarrow \neg\circledsim\neg\varphi$	R3 4
	6.	$\ominus\varphi \Rightarrow \neg\circledsim\mathit{false}$	E-TRNS 5, 3

$\blacksquare$

Theorem T3 $\bigcirc(\varphi \wedge \psi) \Longleftrightarrow (\bigcirc \varphi \wedge \bigcirc \psi)$

PROOF.

1.	$\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi))$	Propositional Tautology
2.	$\varphi \Rightarrow (\psi \rightarrow \varphi \wedge \psi))$	INS 1
3.	$\bigcirc \varphi \Rightarrow \bigcirc(\psi \rightarrow \varphi \wedge \psi)$	$\bigcirc$M 2
4.	$\bigcirc(\psi \rightarrow \varphi \wedge \psi) \Rightarrow (\bigcirc \psi \rightarrow \bigcirc(\varphi \wedge \psi))$	F2
5.	$\bigcirc \varphi \Rightarrow (\bigcirc \psi \rightarrow \bigcirc(\varphi \wedge \psi))$	E-TRNS 3, 4
6.	$(\bigcirc \varphi \wedge \bigcirc \psi) \Rightarrow \bigcirc(\varphi \wedge \psi)$	R7 5
7.	$\varphi \wedge \psi \rightarrow \varphi$	Propositional Tautology
8.	$\varphi \wedge \psi \Rightarrow \varphi$	INS 7
9.	$\bigcirc(\varphi \wedge \psi) \Rightarrow \bigcirc \varphi$	$\bigcirc$M 8
10.	$\varphi \wedge \psi \rightarrow \psi$	Propositional Tautology
11.	$\varphi \wedge \psi \Rightarrow \psi$	INS 10
12.	$\bigcirc(\varphi \wedge \psi) \Rightarrow \bigcirc \psi$	$\bigcirc$M 11
13.	$\bigcirc(\varphi \wedge \psi) \Rightarrow (\bigcirc \varphi \wedge \bigcirc \psi)$	R5 9,12
14.	$\bigcirc(\varphi \wedge \psi) \Longleftrightarrow (\bigcirc \varphi \wedge \bigcirc \psi)$	R6 6, 13

∎

Theorem T4 $\Box \varphi \Rightarrow \bigcirc \Box \varphi$

PROOF. The proof is derived directly from the definition of $\Box \varphi$ as $\neg \Diamond \neg\varphi$, which is defined as $\neg(true\,\mathcal{U}\,\neg\varphi)$.

1.	$true\,\mathcal{U}\,\neg\varphi \Longleftrightarrow \neg\varphi \vee (true \wedge \bigcirc(true\,\mathcal{U}\,\neg\varphi))$	F6
2.	$\neg(true\,\mathcal{U}\,\neg\varphi) \Rightarrow \varphi \wedge \neg \bigcirc(true\,\mathcal{U}\,\neg\varphi))$	from F6
3.	$\neg(true\,\mathcal{U}\,\neg\varphi) \Rightarrow \neg \bigcirc(true\,\mathcal{U}\,\neg\varphi))$	from 2
4.	$\neg \bigcirc(true\,\mathcal{U}\,\neg\varphi) \Rightarrow \bigcirc \neg(true\,\mathcal{U}\,\neg\varphi)$	from F1
5.	$\neg(true\,\mathcal{U}\,\neg\varphi) \Rightarrow \bigcirc \neg(true\,\mathcal{U}\,\neg\varphi)$	E-TRNS 3, 4
6.	$\Box \varphi \Rightarrow \bigcirc \Box \varphi$	Definition of $\Box$

∎

Rule $\Box$G ($\Box$ generalization) $\Box \varphi \vdash \Box \Box \varphi$

PROOF.

1.	$(\Box \varphi \Rightarrow \bigcirc \Box \varphi) \rightarrow (\Box \varphi \Rightarrow \Box \Box \varphi)$	F5
2.	$\Box \varphi \Rightarrow \bigcirc \Box \varphi$	T4
3.	$\Box \varphi \Rightarrow \Box \Box \varphi$	MP 1,2
4.	$\Box \varphi \longrightarrow \Box \Box \varphi$	PAR 3
5.	$\Box \varphi$	Premise
6.	$\Box \Box \varphi$	MP 4,5

∎

Rule $\Box M$ ($\Box$ is monotonic) $\varphi \Rightarrow \psi \vdash \Box \varphi \Rightarrow \Box \psi$

PROOF.

1.	$\Box(\varphi \rightarrow \psi) \Rightarrow (\Box \varphi \rightarrow \Box \psi)$	F3
2.	$\varphi \Rightarrow \psi$	Premise
3.	$\Box(\varphi \rightarrow \psi)$	Definition of $\Rightarrow$
4.	$\Box \Box(\varphi \rightarrow \psi)$	$\Box$G 3
5.	$\Box(\Box \varphi \rightarrow \Box \psi)$	E-MP 1, 4
6.	$\Box \varphi \Rightarrow \Box \psi$	Definition of $\Rightarrow$

∎

Theorem T5 $(\Box \varphi \vee \Box \psi) \Rightarrow \Box(\varphi \vee \psi)$

PROOF.	1.	$\varphi \to \varphi \vee \psi$	Propositional Tautology
	2.	$\varphi \Rightarrow \varphi \vee \psi$	INS 1
	3.	$\Box\varphi \Rightarrow \Box(\varphi \vee \psi)$	$\Box$M 2
	4.	$\psi \to \varphi \vee \psi$	Propositional Tautology
	5.	$\varphi \Rightarrow \varphi \vee \psi$	INS 4
	6.	$\Box\psi \Rightarrow \Box(\varphi \vee \psi)$	$\Box$M 3
	7.	$(\Box\varphi \vee \Box\psi) \Rightarrow \Box(\varphi \vee \psi)$	R4 3, 6

∎

Theorem T6 $\Box(\varphi \wedge \psi) \Longleftrightarrow (\Box\varphi \wedge \Box\psi)$

PROOF.	1.	$\varphi \to (\psi \to \varphi \wedge \psi))$	Propositional Tautology
	2.	$\varphi \Rightarrow (\psi \to \varphi \wedge \psi))$	INS 1
	3.	$\Box\varphi \Rightarrow \Box(\psi \to \varphi \wedge \psi)$	$\Box$M 2
	4.	$\Box(\psi \to \varphi \wedge \psi) \Rightarrow (\Box\psi \to \Box(\varphi \wedge \psi))$	F3
	5.	$\Box\varphi \Rightarrow (\Box\psi \to \Box(\varphi \wedge \psi))$	E-TRNS 3, 4
	6.	$(\Box\varphi \wedge \Box\psi) \Rightarrow \Box(\varphi \wedge \psi)$	R7 5
	7.	$\varphi \wedge \psi \to \varphi$	Propositional Tautology
	8.	$\varphi \wedge \psi \Rightarrow \varphi$	INS 7
	9.	$\Box(\varphi \wedge \psi) \Rightarrow \Box\varphi$	$\Box$M 8
	10.	$\varphi \wedge \psi \to \psi$	Propositional Tautology
	11.	$\varphi \wedge \psi \Rightarrow \psi$	INS 10
	12.	$\Box(\varphi \wedge \psi) \Rightarrow \Box\psi$	$\Box$M 11
	13.	$\Box(\varphi \wedge \psi) \Rightarrow (\Box\varphi \wedge \Box\psi)$	R5 9,12
	14.	$\Box(\varphi \wedge \psi) \Longleftrightarrow (\Box\varphi \wedge \Box\psi)$	R6 6, 13

∎

B Proof of Claim 4.8

We present here the missing proofs of some cases appearing in claim 4.8 from subsection 4.3.

Claim B.1 *Let A, B be some atoms such that $(A, B) \notin \mathcal{R}_0$.*

$$\vdash \widehat{A} \Rightarrow \bigcirc\neg\widehat{B}.$$

PROOF. Consider the different reasons for $(A, B) \notin \mathcal{R}_0$.
***Case 1*:** There is a formula $\bigcirc\psi \in CL(\varphi)$ such that $\bigcirc\psi \in A$ and $\psi \notin B$, i.e., $\neg\psi \in B$.

PROOF.	1.	$\widehat{A} \Rightarrow \bigcirc\psi$	INS given
	2.	$\psi \Rightarrow \neg\neg\psi$	from PL and INS
	3.	$\bigcirc\psi \Rightarrow \bigcirc\neg\neg\psi$	$\bigcirc$M 2
	4.	$\widehat{A} \Rightarrow \bigcirc\neg\neg\psi$	E-TRNS 1, 3
	5.	$\widehat{B} \Rightarrow \neg\psi$	INS given
	6.	$\neg\neg\psi \Rightarrow \neg\widehat{B}$	R2 5
	7.	$\bigcirc\neg\neg\psi \Rightarrow \bigcirc\neg\widehat{B}$	$\bigcirc$M 6
	8.	$\widehat{A} \Rightarrow \bigcirc\neg\widehat{B}$	E-TRNS 4, 7

∎

***Case 2*:** There is a formula $\bigcirc\psi \in CL(\varphi)$ such that $\bigcirc\psi \notin A$, i.e., $\neg\bigcirc\psi \in A$ and $\psi \in B$.

PROOF.	1.	$\widehat{A} \Rrightarrow \neg \bigcirc \psi$	INS given
	2.	$\bigcirc \neg\psi \Longleftrightarrow \neg \bigcirc \psi$	F1
	3.	$\neg \bigcirc \psi \Rrightarrow \bigcirc \neg\psi$	R3 2
	4.	$\widehat{A} \Rrightarrow \bigcirc \neg\psi$	E-TRNS 1, 3
	5.	$\widehat{B} \Rrightarrow \psi$	INS given
	6.	$\neg\psi \Rrightarrow \neg\widehat{B}$	R2 5
	7.	$\bigcirc \neg\psi \Rrightarrow \bigcirc \neg\widehat{B}$	$\bigcirc$M 6
	8.	$\widehat{A} \Rrightarrow \bigcirc \neg\widehat{B}$	E-TRNS 4, 7

∎

***Case 3*:** There is a formula $\circledsim \psi \in CL(\varphi)$ such that $\psi \in A$ and $\circledsim \psi \notin B$, i.e., $\neg \circledsim \psi \in B$.

PROOF.	1.	$\widehat{A} \Rrightarrow \psi$	INS given
	2.	$\psi \Rrightarrow \bigcirc \ominus \psi$	M8
	3.	$\widehat{A} \Rrightarrow \bigcirc \ominus \psi$	E-TRNS 1, 2
	4.	$\ominus \psi \Rrightarrow \circledsim \psi$	P1
	5.	$\bigcirc \ominus \psi \Rrightarrow \bigcirc \circledsim \psi$	$\bigcirc$M 4
	6.	$\widehat{A} \Rrightarrow \bigcirc \circledsim \psi$	E-TRNS 3, 5
	7.	$\widehat{B} \Rrightarrow \neg \circledsim \psi$	INS given
	8.	$\neg\neg \circledsim \psi \Rrightarrow \neg\widehat{B}$	R2 7
	9.	$\circledsim \psi \Rrightarrow \neg\neg \circledsim \psi$	from PL and INS
	10.	$\circledsim \psi \Rrightarrow \neg\widehat{B}$	E-TRNS 9, 8
	11.	$\bigcirc \circledsim \psi \Rrightarrow \bigcirc \neg\widehat{B}$	$\bigcirc$M 10
	12.	$\widehat{A} \Rrightarrow \bigcirc \neg\widehat{B}$	E-TRNS 6, 11

∎

***Case 4*:** There is a formula $\circledsim \psi \in CL(\varphi)$ such that $\psi \notin A$, i.e., $\neg\psi \in A$ and $\circledsim \psi \in B$.

PROOF.	1.	$\widehat{A} \Rrightarrow \neg\psi$	INS given
	2.	$\neg\psi \Rrightarrow \bigcirc \ominus \neg\psi$	M8
	3.	$\widehat{A} \Rrightarrow \bigcirc \ominus \neg\psi$	E-TRNS 1, 2
	4.	$\ominus \neg\psi \Longleftrightarrow \neg \circledsim \neg\neg\psi$	T1
	5.	$\ominus \neg\psi \Rrightarrow \neg \circledsim \neg\neg\psi$	R3 4
	6.	$\bigcirc \ominus \neg\psi \Rrightarrow \bigcirc \neg \circledsim \neg\neg\psi$	$\bigcirc$M 5
	7.	$\widehat{A} \Rrightarrow \bigcirc \neg \circledsim \neg\neg\psi$	E-TRNS 3, 6
	8.	$\widehat{B} \Rrightarrow \circledsim \psi$	INS given
	9.	$\neg \circledsim \psi \Rrightarrow \neg\widehat{B}$	R2 8
	10.	$\bigcirc \neg \circledsim \psi \Rrightarrow \bigcirc \neg\widehat{B}$	$\bigcirc$M 9
	11.	$\psi \Rrightarrow \neg\neg\psi$	from PL and INS
	12.	$\circledsim \psi \Rrightarrow \circledsim \neg\neg\psi$	$\circledsim$M 11
	13.	$\neg \circledsim \neg\neg\psi \Rrightarrow \neg \circledsim \psi$	R2 12
	14.	$\bigcirc \neg \circledsim \neg\neg\psi \Rrightarrow \bigcirc \neg \circledsim \psi$	$\bigcirc$M 13
	15.	$\widehat{A} \Rrightarrow \bigcirc \neg \circledsim \psi$	E-TRNS 7, 14
	16.	$\widehat{A} \Rrightarrow \bigcirc \neg\widehat{B}$	E-TRNS 15, 10

∎

***Case 5*:** There is a formula $\psi_1 \mathcal{U} \psi_2 \in CL(\varphi)$ such that $\psi_1 \mathcal{U} \psi_2$, $\neg\psi_2 \in A$ and $\psi_1 \mathcal{U} \psi_2 \notin B$, i.e., $\neg(\psi_1 \mathcal{U} \psi_2) \in B$.

PROOF.

1.	$\vdash \widehat{A} \Rightarrow (\psi_1 \mathcal{U} \psi_2 \wedge \neg\psi_2)$	INS given
2.	$\vdash \psi_1 \mathcal{U} \psi_2 \Longleftrightarrow \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \mathcal{U} \psi_2))$	F6
3.	$\vdash \widehat{A} \Rightarrow \bigcirc(\psi_1 \mathcal{U} \psi_2)$	from 1, 2
4.	$\vdash \widehat{B} \Rightarrow \neg(\psi_1 \mathcal{U} \psi_2)$	INS given
5.	$\vdash \neg\neg(\psi_1 \mathcal{U} \psi_2) \Rightarrow \neg\widehat{B}$	R2 4
6.	$\vdash \psi_1 \mathcal{U} \psi_2 \Rightarrow \neg\neg(\psi_1 \mathcal{U} \psi_2)$	from PL and INS
7.	$\vdash \psi_1 \mathcal{U} \psi_2 \Rightarrow \neg\widehat{B}$	E-TRNS 6, 5
8.	$\vdash \bigcirc(\psi_1 \mathcal{U} \psi_2) \Rightarrow \bigcirc\neg\widehat{B}$	$\bigcirc$M 7
9.	$\vdash \widehat{A} \Rightarrow \bigcirc\neg\widehat{B}$	E-TRNS 3, 8

∎

∎