What you need is that your brain is open. – Paul Erdős

1. In a BST assume we have a function $TD[x]$ that gives the number of descendents (including $x$ itself) of node $x$. Now we apply $DELETE[z]$. Assume that $z$ is childless. Give a procedure that will update the function $TD$ in time $O(H)$.

2. Determine an LCS of 10010101 and 010110110 using the algorithm studied.

3. Write all the parenthesizations of $ABCDE$. Associate them in a natural way with (setting $n = 5$) the terms $P(i)P(n - i)$, $i = 1, 2, 3, 4$ given in the recursion for $P(n)$.

4. Let $x_1, \ldots, x_m$ be a sequence of distinct real numbers. For $1 \leq i \leq m$ let $INC[i]$ denote the length of the longest increasing subsequence ending with $x_i$. Let $DEC[i]$ denote the length of the longest decreasing subsequence ending with $x_i$. Caution: The subsequence must use $x_i$. For example, 20, 30, 4, 50, 10. Now $INC[5] = 2$ because of 4, 10 – we do not count 20, 30, 50.

   (a) Find an efficient method for finding the values $INC[i]$, $1 \leq i \leq n$. (You should find $INC[i]$ based on the previously found $INC[j]$, $1 \leq j < i$. Your algorithm should take time $O(i)$ for each particular $i$ and thus $O(n^2)$ overall.)

   (b) Let $LIS$ denote the length of the longest increasing subsequence of $x_1, \ldots, x_m$. Show how to find $LIS$ from the values $INC[i]$. Your algorithm, starting with the $INC[i]$, should take time $O(n)$. Similarly, let $DIS$ denote the length of the longest decreasing subsequence of $x_1, \ldots, x_m$. Show how to find $DIS$ from the values $DEC[i]$.

   (c) Suppose $i < j$. Prove that it is impossible to have $INC[i] = INC[j]$ and $DEC[i] = DEC[j]$. (Hint: Show that if $x_i < x_j$ then $INC[j] \geq INC[i] + 1$.)

   (d) Deduce (assume (4c)) the following celebrated result (called the Monotone Subsequence Theorem) of Paul Erdős and George Szekeres: Let $m = ab + 1$. Then any sequence $x_1, \ldots, x_m$ of distinct distinct

\[\text{Fundamental Algorithms, Assignment 6} \]
\[\text{Due Monday, March 9, 8a.m. via Gradescope} \]
real numbers either $LIS > a$ or $DIS > b$. (Idea: Assume not and look at the pairs $(INC[i], DEC[i])$.) Paul Erdős was a great twentieth century mathematician, whose work remains highly influential in many areas.

5. Find an optimal parenthesization of a matrix-chain product whose sequence of dimensions is 5, 10, 3, 12, 5, 50, 6.

6. Some exercises on logarithms:

   (a) Write $\lg(4^n/\sqrt{n})$ in simplest form. What is its asymptotic value.

   (b) Which is bigger, $5^{313340}$ or $7^{271251}$? Give reason. (You can use a calculator but you can’t use any numbers bigger than $10^9$.)

   (c) Simplify $n^2 \lg(n^2)$ and $\lg^2(n^3)$.

   (d) Solve (for $x$) the equation $e^{-x^2/2} = \frac{1}{n}$.

   (e) Write $\log_n 2^n$ and $\log_n n^2$ in simple form.

   (f) What is the relationship between $\lg n$ and $\log_3 n$?

   (g) Assume $i < n$. How many times need $i$ be doubled before it reaches (or exceeds) $n$?

   (h) Write $\lg[n^n e^{-n} \sqrt{2\pi n}]$ precisely as a sum in simplest form. What is it asymptotic to as $n \to \infty$? What is interesting about the bracketed expression?

Usually when we hear or read something new, we just compare it to our own ideas. If it is the same, we accept it and say that it is correct. If it is not, we say it is incorrect. In either case, we learn nothing.

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