Theorem: Consider the probability distribution of velocities over a body of gas. Assume that

a. The distribution depends only on speed; it is invariant under direction.

b. The velocities in the $x-$, $y-$, and $z-$dimensions are all independent of one another.

c. The distribution is continuous (this is probably unnecessary, but makes things easier).

Then the distribution is a Gaussian.

Proof:

Let X, Y, Z be random variables denoting the components of velocity. Let $f(X)$ be the distribution of X; by assumption (a) $f(Y)$ and $f(Z)$ are the distributions of Y and Z. Let $H(X, Y, Z)$ be the joint distribution of X, Y, Z.

By assumption (a), if $x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2$ then $H(x_1, y_1, z_1) = H(x_2, y_2, z_2)$ (condition 1).

By assumption (b), $H(x, y, z) = f(x)f(y)f(z)$ for all x, y, z (condition 2).

Note first that for any constants a, b the functions $f(t) = ae^{bt^2}$, $H(x, y, z) = a^3 e^{b(x^2+y^2+z^2)}$ satisfies both conditions above.

Conversely, suppose that f and H satisfy the above conditions.

The following equation holds: for any u, v,

$$f(u)f(v)f(0) = H(u, v, 0) = H(\sqrt{u^2 + v^2}, 0, 0) = f(\sqrt{u^2 + v^2})f(0)f(0)$$

Thus $f(\sqrt{u^2 + v^2}) = f(u)f(v)/f(0)$.
In particular if $u = v$, we have $f(u\sqrt{2}) = f(u)^2/f(0)$.
Equivalently, substituting $v = u\sqrt{2}$, we get $f(v/\sqrt{2}) = \sqrt{f(v) \cdot f(0)}$.

Now let $a = f(0)$. Define $b = \ln(f(1)/a)$. Thus the equation $f(t) = ae^{bt^2}$ is satisfied for both $t = 0$ and $t = 1$. We need to show that it is satisfied for all t.

Suppose that the equation $f(t) = ae^{bt^2}$ is satisfied for some value t.
Let $v = t\sqrt{2}$ and let $u = t/\sqrt{2}$.
Then $f(v) = f(t\sqrt{2}) = f(t)^2/f(0) = (ae^{bt^2})^2/a = ae^{2bt^2} = ae^{bv^2}$.
And $f(u) = f(t/\sqrt{2}) = (f(t)f(0))^{1/2} = (a^2e^{bt^2})^{1/2} = ae^{bt^2/2} = ae^{bu^2}$.

Next, suppose that u and v both satisfy the equation and let $w = \sqrt{u^2 + v^2}$. Then $f(w) = f(u)f(v)/f(0) = ae^{bu^2} \cdot ae^{bv^2}/a = ae^{b(u^2+v^2)} = ae^{bw^2}$.

Thus, the equation $f(t) = ae^{bt^2}$ holds for arbitrarily large values of t and for arbitrarily small values of t and it is preserved when two values are combined using the operator $\sqrt{u^2 + v^2}$. Since any positive real value can be approximated with arbitrary precision by enough combinations of that operator applied to small enough arguments, the equation holds over a dense subset of the reals. By continuity, it holds everywhere on the reals.

Finally the constraint $a = 1/\sqrt{2\pi}\sigma$, $b = -1/2\sigma^2$ follows from the requirement that the integral of f is 1.

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