(Extended Abstract)
Integral Analysis of Evaluation-Based Root Isolation

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Abstract

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function. Subdivision methods are widely used for isolating the roots of f in a given interval. In this paper we consider **evaluation-based subdivision** which uses simpler primitives than well-known subdivision methods such as Sturm or Descartes methods. Evaluation-based algorithms are not restricted to polynomials, and can be seen as 1-dimensional analogues of the Plantinga-Vegter meshing algorithm.

We provide an novel complexity analysis of such algorithms. Our approach can be viewed as a kind of continuous amortization.

(1) First we give a general framework for performing such analysis. This leads to an adaptive upper bound on the complexity of evaluation-based algorithms, based on an integral formula.

(2) Next we consider the benchmark case of a square-free integer polynomial f of degree d and logarithmic height L. We give a priori worst-case upper bounds of the form $O(d^2 L)$ (assuming for simplicity $L \geq \log d$). These results exploit the evaluation analogues of the Mahler-Davenport bounds.

1 Introduction

A basic problem in the computational geometry of surfaces is meshing of implicit surfaces. This asks for an isotopic ε-approximation $\widetilde{S}$ of a surface S in $\mathbb{R}^n$ given by an equation $f = 0$ where $f : \mathbb{R}^n \to \mathbb{R}$. See [4] for a survey of the recent literature on meshing. When f is a polynomial, there are algebraic methods for solving this problem. Numeric/geometric methods based on **subdivision** are widely used by practitioners because they are easier to implement than algebraic methods. They have adaptive complexity which can be quite efficient on most inputs. A main example of subdivision methods is the Marching Cube. This is a simple algorithm, whose main primitive is the evaluation of the sign of f at vertices of the subdivision. Such non-algebraic algorithms are usually incomplete [32]. Hence, hybrid methods that combine algebraic primitives with subdivision are often used.

The first subdivision method that is provably complete for non-singular surfaces is from Plantinga-Vegter [22, 23]. They provided algorithms in 2 and 3-D, i.e., for $f : \mathbb{R}^n \to \mathbb{R}$ where $n = 2$ or 3. No complexity analysis for these algorithms are known. In this paper, we analyze the complexity of the 1-D version of their algorithm.

The 1-D version amounts to real root isolation (and refinement). There are many well-known subdivision algorithms in this case. What is interesting is the computational model here: Plantinga-Vegter is based on evaluation of functions, like Marching Cubes. But it also uses evaluation of interval versions of a function and its derivatives. We call such algorithms **evaluation-based**. In contrast, subdivision methods such as Sturm [11, 25, 17] and Descartes [8, 12, 2], use the more sophisticated primitives which seem to restrict f to polynomials. But evaluation-based methods are more widely applicable (e.g., f could be analytic). Note that Descartes Method can be developed into concrete algorithms such as the Bisection Algorithm of Collins-Akritas [8] or continued fraction algorithm [1, 29, 28]. The Bernstein polynomial approach [16, 21] may also be viewed as a variant of Descartes method [12].

In [6], we extrapolated the Plantinga-Vegter algorithm to the 1-D case; for reference, call it the EVAL algorithm. Mitchell [18] seems to be the first to formulate this algorithm based on an algorithm of Moore [19]. For correctness, Mitchell and Plantinga-Vegter need only minimal constraints on the interval arithmetic in their algorithms. But in order to carry out a complexity analysis, we need some idea of how tight the interval functions are. For this, we assume the **centered form** of interval functions [24].

The **adaptive complexity** of subdivision algorithms is a topic of growing interest. But what is the proper measure of adaptivity? Most measures in the literature are based on the condition number. For instance, Mourrain and Pavone [20] use this measure to bound the complexity of Bernstein-type subdivision for isolating multivariate zeros. Condition-number approaches to complexity are extensively used in the Smale school [3]. Another such concept is precision sensitivity [27], the bit-version of output sensitivity which is well-known in computational geometry. In this paper we introduce integral measures, viewed as a kind of **continuous amortization** argument.

Amortization is a standard analysis technique in discrete algorithms [9]. In the continuous domain, Davenport [10] first gave an amortization argument which yielded the optimal recursion tree complexity for Sturm method. Recently, amortization arguments are used in [11] (for Sturm method) and [12] (for Descartes method). All these complexity bounds are dependent on the Mahler-Davenport root separation bounds [10, 33]. In the present paper, bounds analogous to Mahler-Davenport type bounds appear, but in the form of evaluation bounds rather than root-separation.

Subdivision methods for root isolation may be classified by their "stopping predicates". The Sturm predicate is based on Sturm sequences, and Descartes predicate is based on the Descartes rule of sign. In the evaluation-based method, we use an extremely simple principle: *in an interval* (a, b)

where $f(a)f(b) < 0$*, there exists* $c \in [a, b]$ *such that* $f(c) = 0$. This is known as Bolzano Theorem, a special case of the Intermediate Value Theorem. For this reason, the evaluation-based method could be called the "Bolzano method". These predicates represent a progression of decreasing strength:

$$STURM > DESCARTES > BOLZANO \tag{1}$$

Sturm is the strongest predicate and is algebraic in nature (it only works for polynomials). Bolzano is weakest but is more general, being purely numerical in nature. The computational complexity of the predicates also decreases in this sequence. This may work to the overall advantage of simpler predicates. Descartes method is empirically known to be faster than Sturm method (see [13, 26]). The difference is attributable to the cheaper primitives of Descartes method since the number of subdivisions in Sturm method is minimal among all subdivision methods. In [6] we offer evidence that evaluation-based methods might similarly be competitive with Descartes method.

For the purposes of complexity analysis, however, we find a reverse ordering in (1): the simpler predicates are harder to analyze. It is standard to judge these algorithms using the **benchmark problem** of isolating all the real roots of an integer polynomial of degree d and logarithmic height L. *What is the size of the subdivision tree in terms of* d *and* L*?* Davenport [10] proves that the tree size is $O(d(L + \log d))$ for the Sturm predicate. The corresponding bound for Descartes method is also $O(d(L+\log d))$ [12] but more subtle to show. In this paper, our analysis shall indicate to what extent EVAL can match these bounds.

¶1. Overview of Paper. In the rest of this Section, we provide some additional literature background. In Section 2, we describe the Vegter-Plantinga computational model and the algorithm EVAL. In Section 3, we describe the general framework of "stopping functions" for analyzing the complexity of EVAL. In Section 4, we illustrate the general framework with explicit stopping functions. In Section 5, we give our main result, giving an a priori complexity bound of $O(d^2L)$ on EVAL. In Section 6, we bound the gamma integral which is a component of the main bound. We conclude in Section 7. An appendix gives missing proofs and additional material: a local stopping function that remains to be analyzed, analysis of an "ideal" integral bound, and a bound for the values of polynomials evaluated at algebraic points.

¶2. Additional Background. Root isolation has a large literature; we touch on a few results.

It appears that evaluation-based methods, in order to be complete, are necessarily tied to interval arithmetic. Other examples of evaluation-based root isolation are based on interval forms of the Newton operator. Moore, Krawcyk and others have provided such algorithms [19]. Mitchell [18] presented a form of EVAL. His version is incomplete because he implicitly adopted the numerical analyst's view of fixed precision arithmetic. Mitchell notes that his algorithm is simpler than the Newton-based method of Moore [19]. Kearfott [15, 14] has provided empirical evaluation of Newton-type subdivision algorithms, and also provided a complexity analysis.

The Descartes/Bernstein method for root isolation has been extensively studied in recent years [21]. Rouillier and Zimmermann [26] describe various improvements on the basic algorithm that goes back to Akritas and Collins. The almost optimality of recursion tree size for such algorithms was recently established [12]. This paper describes a unified framework for Descartes method that includes the Bernstein polynomial approach. Evaluation bounds were recently introduced in our work [7] on numerical solution of zero-dimensional triangular polynomial systems.

A major open problem in meshing is to construct subdivision algorithms that can treat singularities (see [4]). Recently, we provided such solutions in the Plantinga-Vegter model: for root isolation (1-D) [6] and for curves (2-D) [5].

2 An Evaluation-based Algorithm.

Many of the results in this paper are applicable to C^1 functions with simple zeros; our results, however, will focus on polynomials and we leave it to the reader to reformulate the results to general C^1 functions. Fix f to be a polynomial of degree d. In the Plantinga-Vegter model, we need the box (i.e., interval) versions of f and its derivatives.

¶3. Box Functions. For any set $S \subseteq \mathbb{R}$, let $\square S$ denote the set of closed intervals in S. If $I = [a, b]$, denote the **midpoint** of I $m(I) = (a + b)/2$ and the **width** of I $w(I) = b - a$. A **partition** of I is a finite subset $P \subseteq \square I$ such that the union of the intervals in P is equal to I, and any two intervals in P have disjoint interiors. The **size** of P is the number of intervals in P, $\#(P)$.

Our partitions of I mostly come from repeated bisections: for any interval $X = [a, b]$, the term **children** of X refers to the two intervals $[a, m(X)], [m(X), b]$. Note $\{X\}$ and $\{[a, m(X)], [m(X), b]\}$ are both partitions of X. In general, if P is a partition of I, and $X \in P$, then to **bisect** X **in** P means to replace X by its two children in P. As a result $\#(P)$ increases by 1. A partition of I that arises from repeated bisections of the initial $\{I\}$ is called a **subdivision** of I.

For any interval X, define

$$K_X = K_X(f) := \max_{a \in X} \sum_{i=1}^{d} \frac{|f^{(i)}(a)|}{i!} (w(X))^{i-1}. \tag{2}$$

Also, write K'_X for $K_X(f')$ where $f' = f^{(1)}$ is the first derivative. Note that $X \subseteq Y$ implies $K_X \le K_Y$. We may call K_X the **Lipschitz constant** for X, as it is easily seen that $|f(a) - f(b)| \le K_X|a - b|$ for $a, b \in X$.

A **box function** for f **over** I is a function of the form

$$\square f : \square I \to \square \mathbb{R}$$

such that all $X \in \square I$, we have $f(X) \subseteq \square f(X)$. Here, $f(X)$ denotes the set extension of f where $f(S) = \{f(a) : a \in S\}$ for any set $S \subseteq \mathbb{R}$.

We use a particular box function, defined as follows:

$$\square f(X) := \sum_{i=0}^{d} \frac{|f^{(i)}(m(X))|}{i!} \left(\frac{w(X)}{2}[-1, 1]\right)^i. \tag{3}$$

This is the **centered form** box function (see [24]). It satisfies the following properties:

PROPOSITION 1. *Let $Y \subseteq X$ be an interval, then:*
(i) $w(\square f(Y)) \le K_X \cdot w(Y)$.
(ii) $w(\square f(Y)) - w(f(Y)) \le K_X \cdot w(Y)^2$.

Property (ii) is called quadratic convergence for $\square f$. However, we do not use this property.

¶4. The Evaluation Algorithm. We now present the Evaluation Algorithm (EVAL) taken from [6]. Given an interval I, EVAL will isolate all the real roots of $f(x)$ in I. The idea is to maintain a subdivision P of I. Initially, $P = \{I\}$. The algorithm operates in two phases.

Phase 1: Repeatedly bisect each $X \in P$ until each interval X in P is **terminal**. By this, we mean that one of the following two conditions hold for X:

$$C_0(X) \quad : \quad 0 \notin \square f(X) \tag{4}$$
$$C_1(X) \quad : \quad 0 \notin \square f'(X) \tag{5}$$

Note that we need the box functions for f' as well as for f.

Phase 2: Let P_I denote the subdivision of I at the end of Phase 1. For each $X \in P_I$, we either discard or retain X. If $C_0(X)$ holds, we discard X. If $C_1(X)$ holds, we evaluate the signs of f at the two end points of X. If f have different sign change at these 2 points, we retain X, else we discard it. We output the set P'_I of retained intervals. Proving correctness of this algorithm amounts to showing that P'_I is set of isolating intervals for the roots of f in I.

This simplified description is correct provided the midpoints of each bisected interval is not a root of f. Otherwise, we can give simple modifications (see [6]). When f is an integer polynomial, EVAL can be implemented exactly using bigfloats.

Our goal is to find an upper bound for the size $\#(P_I)$ of P_I. This size is one more than the number of bisection steps. The starting point for our analysis is a simple observation:

LEMMA 2. *If* $a \in Y \subseteq X$ *and* $0 \in \Box f(Y)$ *then* $w(Y) \geq |f(a)|/K_X$.

Proof. Since $\{0, f(a)\} \subseteq \Box f(Y)$, we have $w(\Box f(Y)) \geq |f(a)|$. By Proposition 1(i), $w(Y) \geq w(\Box f(Y))/K_X$ and hence $w(Y) \geq |f(a)|/K_X$. **Q.E.D.**

3 General Framework of Stopping Functions.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. If X is any interval, we will call X **big** (relative to g) if

$$w(X) \geq \frac{1}{2} \max_{a \in X}\{g(a)\}. \tag{6}$$

For convenience, say X is **large** (relative to g) if $w(X) \geq \max_{a\in X}\{g(a)\}$. Clearly, if X is large, then X is big, and both of the children of X are also big. A partition P of I is **big** if each $X \in P$ is big. Our key definition is this: call g a **stopping function** (over an interval I) if for any interval $X \subseteq I$ that is *not* large relative to g must be terminal. The following is immediate:

LEMMA 3. *If* g_1, g_2 *are stopping functions over* I*, then so is* $\max\{g_1, g_2\}$.

We remark that this simple device of using $\max\{g_1, g_2\}$ is critical for achieving complexity bounds; it acts as a regularizing device when we integrate.

LEMMA 4. *Let* P *be a big partition of* I *relative to stopping function* g*. Then the size of* P *is at most*

$$S := 2\int_I \frac{1}{g(a)} da \tag{7}$$

In addition, if g *is never zero in* I*, then the integral* S *is finite.*

Proof. If g is never zero, $1/g$ is continuous and never infinity. As I is compact and $1/g$ is continuous, $1/g$ is bounded in I and so the integral is finite. S in the lemma can be rewritten as

$$S = \sum_{X\in P} \int_X \frac{2}{g(a)} da. \tag{8}$$

It remains to show that this sum is at least n. It suffices to show that each summand is at least 1. For any $X \in P$, if we choose $c = \arg\max_{a\in X}\{g(a)\}$. Then we have

$$\int_X \frac{2}{g(a)} da \geq \int_X \frac{2}{g(c)} da = w(X) \cdot \frac{2}{g(c)} \geq 1, \tag{9}$$

where the last step uses the fact that X is big. **Q.E.D.**

Theorem 5. *Let P_I be the partition of I at the end of Phase 1 of EVAL. Then*

$$\#(P_I) \le \max\left\{1, \int_I \frac{2}{g(a)} da\right\}. \tag{10}$$

Proof. If $\#(P_I) = 1$, the result is trivial. Assume $\#(P_I) > 1$. We may assume that I is large since otherwise, I is terminal. EVAL maintains a partition of P which is initially $\{I\}$. Consider the loop invariant that P is big. In each iteration, a large interval $J \in P$ is replaced by its two children, both big. In addition, the algorithm does not divide intervals that are not large. Thus the invariant is preserved and the final subdivision P_I is big. Then Lemma 4 implies that $\#(P_I) \le \int_I \frac{2}{g(a)} da$.
Q.E.D.

Thus we see the utility of a stopping function g: it is an analysis tool for bounding the complexity of EVAL. We will investigate possible g's and discuss the information that each provides.

4 Global Stopping Function

So far, we have not seen any explicit stopping functions. We now give an example:

¶5. Global Lipschitz Constants. Our first example will use "global" Lipschitz constants K_I and K'_I. Its main merit lies in its simplicity. The key definition is this: let $X \in \square I$. Write

$$f_X(a) := \max\left\{\frac{|f(a)|}{K_X}, \frac{|f'(a)|}{K'_X}\right\}. \tag{11}$$

We choose $g = f_I$ (i.e., $X = I$ in (11)) as our stopping function. To show that g is a stopping function, we must show that if X is not large (relative to g) then X is terminal. The next lemma proves this result.

Lemma 6. *The functions $\frac{|f(a)|}{K_I}$, $\frac{|f'(a)|}{K'_I}$ and f_I are stopping functions over I.*

Theorem 7. *Let P_I be the partition of I at the end of Phase 1 of the Evaluation Algorithm. Then*

$$\#(P) \le \max\left\{1, \int_I \frac{2da}{f_I(a)}\right\}.$$

and this integral is finite.

Proof. We already know that f_I is a stopping function over I (Lemma 6). The result follows from Theorem 5 if f_I is never 0. f_I is never 0 since f is square free and f and f' do not share any roots.
Q.E.D.

The bound using f_I is not very satisfactory, because it does not take into consideration local conditions. See the appendix for a local version K_a ($a \in I$) of the Lipschitz constants and the corresponding stopping function.

5 An Integral Bound based on Refined Stopping Function

Our ultimate goal is to analyze the complexity of EVAL for the **benchmark problem** where we want to isolate all the real roots of $f \in \mathbb{Z}[x]$, a square-free integer polynomial of degree d and height $\|f\| < 2^L$. The height $\|f\|$ is defined as the maximum absolute value of the coefficients of f. The logarithmic height is defined to be $\log \|f\|$. In particular, we want *a priori* complexity bounds in terms on d and L (see introduction). To achieve this, we exploit the freedom of our stopping function framework to introduce other stopping functions that are more amenable to analysis.

A general remark is that such a priori bounds is a worst-case *non-adaptive* bound; they do not replace the utility of the integral bounds (such as Theorem 7) which are adapted to the individual f and I.

For simplicity, we shall make two mild assumptions in the rest of this extended abstract: (a) $L \geq \log d$ (cf. [12]). (b) f' and f'' are relatively prime. Removal of (a) only complicates the statement of bounds, but not the proof. Removal of (b) requires more cases to consider, but no essentially new ideas. These assumptions will be removed in the full paper.

Our goal is to give an a priori upper bound on EVAL for the benchmark problem of isolating all the zeros of f. For this purpose, we may assume that the input $I = [a, b]$ where $|a|, |b| \leq 2^L$ and a, b are integers, since all real zeros of f lies in this range [33]. Our main result is the following:

Theorem 8 (Main Result). *The number of bisections performed by EVAL on input f and an interval I is $O(d^2L)$.*

Note that a bound of the form $O(dL)$ would be optimal [12]. We will exploit the "gamma function" that is central in Smale's theory of point estimates [3, 30]. This is defined as

$$\gamma(x) = \gamma_f(x) := \max_{i\geq 2} \left(\frac{|f^{(i)}(x)|}{i!|f'(x)|} \right)^{1/(i-1)}. \tag{12}$$

Intuitively, the inverse of $\gamma(x)$ is the radius of Newton convergence of f at x. Write $\gamma'(x)$ for $\gamma_{f'}(x)$ (so $\gamma'(x)$ should not be confused with the derivative of $\gamma(x)$ which is not used).

Lemma 9. *Let $b \in J$ such that $w(J) \leq \frac{1}{2\gamma(b)}$. Then $K_J \leq 2d|f'(b)|$.*

This is proved by replacing each $f^{(i)}(a)$ in the definition of K_J by its Taylor expansion at b.

Let

$$G(a) := \min \left\{ \frac{1}{2\gamma(a)}, \frac{|f(a)|}{2d|f'(a)|} \right\}. \tag{13}$$

Let $G'(a)$ denote the function analogous to $G(a)$ where, in the above definition, f is replaced by f', f' by f'', and γ by γ'. Again, $G'(a)$ is not the derivative of G.

Lemma 10. *G is a stopping function.*

Proof. Suppose J is not large relative to G. This means there exists $b \in J$ such that $w(J) < G(b)$. We must show that J is terminal. It suffices to show that $C_0(J)$ holds. Since $w(J) < G(b)$, we have

$$w(J) < \frac{|f(b)|}{2d|f'(b)|} \leq \frac{|f(b)|}{K_J}$$

where the second inequality follows from Lemma 9, using the fact that $2w(J) < 1/\gamma(b)$. The conclusion that $C_0(J)$ holds now follows from Lemma 2. **Q.E.D.**

Lemma 11. *If $w(J) \geq \frac{G(a)}{2}$ for all $a \in J$ then*

$$2\int_J \frac{da}{G(a)} \geq 1$$

Proof. By Lemma 10, $G(a)$ is a stopping function for J. Since $\{J\}$ is big partition of J relative to G, our desired result follows from Lemma 4. **Q.E.D.**

By a similar argument, $G'(a)$, and hence also $\max\{G(a), G'(a)\}$, are stopping functions. To apply these lemmas, consider the following conceptual procedure. We say "conceptual" because it is not meant to be implemented, but only a tool for analysis.

```
Procedure G:
    Input: interval I
    Output: partition of I
        Start with the partition P = {I}.
        For each J ∈ P,
            if for all a ∈ J, we have
                    w(J) ≥ max {G(a), G'(a)}
                then we split J in P.
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THEOREM 12. *Suppose Procedure G terminates with the partition P and $\#(P) \ge 2$.*
(a) $\#(P)$ is an upper bound on the number of steps taken by the Eval Algorithm on input I.
(b)

$$\#(P) \le 2 \int_I \frac{da}{\max\{G(a), G'(a)\}}.$$

Proof. It is easy to see that the following two properties must hold for each $J \in P$:

(a') There exists $b \in J$ such that $w(J) < \max\{G(b), G'(b)\}$.

(b') For all $a \in J$, we have $2w(J) \ge \max\{G(a), G'(a)\}$.

From (a'), we conclude that either $w(J) < G(b)$ or $w(J) < G'(b)$. Then Lemma 10 implies either $C_0(J)$ or $C_1(J)$. Therefore the Eval Algorithm halts on J. This proves that $\#(P)$ is at least as large as the number of subdivisions steps of the Eval Algorithm on input I. This proves (a).

Now (b) follows from (b'):

$$2 \int_I \frac{da}{\max\{G(a), G'(a)\}} = \sum_{J \in P} 2 \int_J \frac{da}{\max\{G(a), G'(a)\}} \ge \sum_{J \in P} 1 = \#(P). \tag{14}$$

Q.E.D.

¶6. Avoiding Zeros of ff'. By definition, $G(a) \ge 0$ and $G'(a) \ge 0$ for all a. The integral in Theorem 12 is infinite iff both $G(a) = 0$ and $G'(a) = 0$. Now, $G(a) = 0$ iff $f(a) = 0$ or $f'(a) = 0$. Similar, $G'(a) = 0$ iff $f'(a) = 0$ or $f''(a) = 0$. Hence, $G(a) = G'(a) = 0$ iff $f'(a) = 0$. Thus, we want to bound the integral over an interval $I' \subseteq I$ that avoids $\text{Zero}(f')$. It will turn out that we want to avoid $\text{Zero}(f)$ as well. We now outline the strategy to do this.

For each zero $\alpha \in \text{Zero}(f)$, let $\rho(\alpha)$ denote the distance from α to the nearest zero of $\text{Zero}(f)$ different from α. Note that $\rho(\alpha) = 0$ iff α is a multiple zero. But since f is square-free, $\rho(\alpha) > 0$. Similarly, if $\beta \in \text{Zero}(f')$, let $\rho'(\beta)$ be the corresponding function for f'. We need our assumption about the square-freeness of f' to conclude that $\rho'(\beta) > 0$. Since $\text{Zero}(f) \cap \text{Zero}(f')$ is empty, we can merge these two ρ functions into one, $\rho : \text{Zero}(ff') \to \mathbb{R}_{>0}$.

We now provide a refined conceptual two-staged procedure:

Procedure H:

  Input: interval I

  Output: partition P_2 of I

    Start with the partition $P = \{I\}$.

    Stage 1:

      For each $J \in P$, split J in P if

        one of the following conditions hold:

        (a) $\#(J \cap \textsc{Zero}(ff')) > 1$.

        (b) $\#(J \cap \textsc{Zero}(ff')) = 1$, and $w(J) \geq \min\left\{B(\alpha), \frac{\rho(\alpha)}{4d(d-1)}\right\}$

          where $\alpha \in \textsc{Zero}(ff') \cap J$ and $B(\alpha)$ is a technical bound discussed below

    Stage 2:

      For each $J \in P$, partition J using Procedure G.

We consider two partitions of I: Let P_1 be the partition at the end of Stage 1, and P_2 be the partition at the end of Stage 2. An interval $J \in P_1$ is said to be **special** if $\#(J \cap \textsc{Zero}(ff')) = 1$ and **non-special** otherwise. Clearly, there are at most $2d - 1$ special intervals. Let $P_1' \subseteq P_1$ denote the set of non-special intervals of P_1 and $I' = \bigcup P_1'$ is the union of all non-special intervals. The following lemma will be shown below:

Lemma 13. *If $J \in P_1$ is special then it is terminal, i.e., $C_0(J)$ or $C_1(J)$ holds.*

The proof of lemma 13 will need the following property:

$$w(J) < \frac{1}{8\gamma(\alpha)}. \tag{15}$$

Condition (b) for splitting interval J is designed to achieve this. When we stop splitting J then it is clear that

$$w(J) < \frac{\rho(\alpha)}{4d(d-1)}. \tag{16}$$

Then (15) follows from an application of the following bound from [31]:

Proposition 14. $\frac{1}{\gamma(\alpha)} > \frac{2\rho(\alpha)}{d(d-1)}$.

In view of Lemma 13, we have the following bound on the final partition of Procedure H:

$$\begin{aligned}
\#(P_2) &\leq 2d - 1 + \sum_{J \in P_1'} \max\left\{1, \int_J \frac{2da}{\max\{G(a), G'(a)\}}\right\} \\
&\leq \#(P_1) + 2 \sum_{J \in P_1'} \int_J \frac{da}{\max\{G(a), G'(a)\}} \\
&\leq \#(P_1) + 2 \int_{I'} \frac{da}{\max\{G(a), G'(a)\}}. \qquad (17)
\end{aligned}$$

Below we will show that $\#(P_1) = O(dL)$, and in the next section, we show $\int_{I'} \frac{da}{\max\{G(a),G'(a)\}} = O(d^2 L)$ (Theorem 15). This concludes our main theorem.

¶7. Proof that Special Intervals are Terminal. We now prove Lemma 13. Let J be a special interval. Then there is a unique $\alpha \in J \cap \textsc{Zero}(ff')$. There are two cases: when $\alpha \in \textsc{Zero}(f)$, we show that $C_1(J)$ holds, and when $\alpha \in \textsc{Zero}(f')$, we show that $C_0(J)$ holds. We now define the technical bound $B(\alpha)$ in Procedure H. Define

$$B(\alpha) = \begin{cases} \infty & \text{if } \alpha \text{ is zero of } f \\ \sqrt{\frac{|f(\alpha)|}{4(\log d)|f''(\alpha)|}} & \text{if } \alpha \text{ is zero of } f' \end{cases}. \tag{18}$$

CASE $\alpha \in \textsc{Zero}(f)$**:** Since $B(\alpha) = \infty$, it plays no role in the stopping condition (b) of Procedure H. In this case, we know that $w(J) \leq \frac{1}{8\gamma(\alpha)}$ (see (15)). Since there is a zero of f in J, we would like $C_1(J)$ to hold. By Lemma 6, $C_1(J)$ would hold provided

$$w(J) < \frac{|f'(\alpha)|}{K'_J}. \tag{19}$$

Using the bound on $w(J)$ and a Taylor expansion about α, one can show $K'_J \leq \frac{7}{9}\frac{|f'(\alpha)|}{w(J)}$, giving the desired bound.

CASE $\alpha \in \textsc{Zero}(f')$**:** In this case, we know that

$$w(J) \leq \frac{1}{2\gamma'(\alpha)}. \tag{20}$$

because of the f'-analogue of inequality (15). Since there is a zero of f' in J, we would like $C_0(J)$ to hold. By Lemma 6, $C_0(J)$ would hold provided

$$w(J) < \frac{|f(\alpha)|}{K_J} \tag{21}$$

Using the bound on $w(J)$ and a Taylor expansion about α, we can see that $K_J \leq 4(\log d)|f''(\alpha)|w(J)$, giving the desired bound when combined with the additional condition supplied by $B(\alpha)$.

¶8. Bounding the Size of P_1. To bound the size of P_1, it is enough to focus on the (at most) $2d-1$ special intervals. Consider the recursion tree T_1 whose leaves are labeled by P_1. Clearly, $\#(T_1) = 2\#(P_1) - 1$. A leaf is said to be special iff it is labeled by a special interval. Let T'_1 be the result of pruning all non-special leaves from T_1. Every non-special leaf has a sibling which is either special or an interior node, and the root has no sibling. Hence $\#(T'_1) \geq \frac{\#(T_1)-1}{2} = \#(P_1) - 1$. Let $S = \textsc{Zero}(f) \cap I$ and $S' = \textsc{Zero}(f') \cap I$ where I is the input interval. Each leaf of T'_1 is associated with a unique $\alpha \in S \cup S'$; the corresponding interval will be denoted I_α. Clearly

$$\#(T'_1) \leq \sum_{\alpha \in S \cup S'} \lg(w(I)/w(I_\alpha))$$

where $\lg = \log_2$. Without loss of generality, we may assume $w(I) \leq 2^{L+1}$ since all zeros α of ff' satisfies $|\alpha| \leq 2^L$ ([33]). Hence

$$\#(T'_1) \leq 2dL - \sum_{\alpha \in S \cup S'} \lg w(I_\alpha) = 2dL - \lg \prod_{\alpha \in S \cup S'} w(I_\alpha). \tag{22}$$

Note that $w(I_\alpha) \geq \frac{1}{2}\min\left\{B(\alpha), \frac{\rho(\alpha)}{4d(d-1)}\right\}$, by our stopping condition in Procedure H. If $\alpha \in S$, this reduces to $w(I_\alpha) \geq \frac{1}{2}\frac{\rho(\alpha)}{4d(d-1)}$, and hence we obtain (see [31]):

$$\begin{aligned} -\lg \prod_{\alpha \in S} w(I_\alpha) &\leq -\lg \prod_{\alpha \in S} \frac{\rho(\alpha)}{8d(d-1)} \\ &= O(d\log d + dL) = O(dL). \end{aligned} \tag{23}$$

Next, we consider the case $\alpha \in S'$. In this case, $B(\alpha) = \sqrt{\frac{|f(\alpha)|}{4(\log d)|f''(\alpha)|}}$. Thus $w(I_\alpha) \geq \min\{B(\alpha), \rho(\alpha)/8d(d-1)\}$. We can split S' into $S'_0 \cup S'_1$ where $\alpha \in S'_0$ iff $\min\{B(\alpha), \rho(\alpha)/8d(d-1)\} = B(\alpha)$. Thus

$$\prod_{\alpha \in S'} w(I_\alpha) \geq \prod_{\alpha \in S'_0} B(\alpha) \prod_{\alpha \in S'_1} \rho(\alpha)/8d(d-1).$$

We have

$$-\lg \prod_{\alpha \in S'_1} \rho(\alpha)/8d(d-1) = O(dL) \tag{24}$$

as in (23). Moreover,

$$-\lg \prod_{\alpha \in S'_0} \sqrt{\frac{|f(\alpha)|}{4(\log d)|f''(\alpha)|}} = O(dL) \tag{25}$$

using the evaluation bound of Theorem 26 in the appendix. From (22,23,24,25), we conclude that $\#(P_1) = O(dL)$.

6 Bounding the Integral $\int_{I'} \frac{dx}{\max\{G(x), G'(x)\}}$.

This section proves the following bound:

Theorem 15. $\int_{I'} \frac{dx}{\max\{G(x), G'(x)\}} = O(dL^2)$

The general strategy goes as follows. First, because of our choice of I', we can ignore one of the terms in the maximum:

$$\int_{I'} \frac{dx}{\max\{G(x), G'(x)\}} \quad \leq \quad \int_{I'} \frac{dx}{G(x)}.$$

We bound the remaining integral by a sum of two integrals:

$$\int_{I'} \frac{dx}{G(x)} = \int_{I'} \max\left\{\gamma(x), \frac{d|f'(x)|}{|f(x)|}\right\} dx \leq \int_{I'} \gamma(x)dx + d\int_{I'} \left|\frac{f'(x)}{f(x)}\right| dx = \Gamma + R$$

where $\Gamma := \int_{I'} \gamma(x)dx$ ("gamma integral") and $R := \int_{I'} \left|\frac{f'(x)}{f(x)}\right| dx$ ("logarithmic-derivative integral"). Note that R is closely related to $R_0 := \int_I \min\left\{\left|\frac{f'(x)}{f(x)}\right|, \left|\frac{f''(x)}{f'(x)}\right|\right\} dx$. Intuitively, R_0 is the "ideal integral" that captures the complexity of EVAL under ideal conditions. In the appendix, we prove that $R_0 = O(dL)$. A similar proof yields:

Lemma 16. $R = \int_{I'} \left|\frac{f'(x)}{f(x)}\right| dx = O(dL)$.

In the rest of this section, we outline the method to bound the gamma integral:

Lemma 17. $\Gamma = \int_{I'} \gamma(x)dx = O(d^2L)$.

Thus Theorem 15 follows from Lemma 16 and Lemma 17. The gamma function satisfies a key inequality:

Lemma 18. *Let $\beta_2, \ldots, \beta_d$ be all the critical points of $f(x)$ (i.e., zeros of f'). Then*

$$\gamma(x) \leq \sum_{j=2}^{d} \frac{1}{2|x - \beta_j|}$$

The proof exploits the relation $f^{(i)}(x)/f'(x) = \sum'_{(j_2,\ldots,j_i)} \prod_{\ell=2}^{i} \frac{1}{x-\beta_{j_\ell}}$, where j_ℓ's are taken from the set $\{1,\ldots,d-1\}$, and the prime in the summation indicates that the j_ℓ's are pairwise distinct.

Corollary 19.

$$\Gamma = \int_{I'} \gamma(x)dx \le \sum_{i=2}^{d} \int_{I'} \frac{dx}{2|x-\beta_i|}$$

Next, we write $\beta_i = r_i + \mathbf{i}s_i$ where $r_i = \mathtt{Re}(\beta_i)$ and $s_i = \mathtt{Im}(\beta_i)$ are the real and imaginary parts, and $\mathbf{i} = \sqrt{-1}$. Furthermore, assume $s_i = 0$ iff $i \le k$, so all the real roots of f' are given by $r_2, \ldots, r_k$. In the appendix, we construct two integer polynomials $R(X)$ and $S(X)$ of degrees $\le d^2$ whose zero set contains r_i and s_i (resp.). We split the summation from Corollary 19 into the real and complex parts:

Lemma 20 (Real Part).

$$\sum_{i=2}^{k} \int_{I'} \frac{dx}{2|x-r_i|} = O(d^2 L).$$

For the complex part, we obtain the better bound:

Lemma 21 (Complex Part).

$$\sum_{i=k+1}^{d} \int_{I'} \frac{dx}{2|x-\beta_i|} = O(dL).$$

These results exploits a "generalized evaluation bound" in the appendix. This completes the proof of Lemma 17.

7 Conclusion

In this paper, we introduced novel techniques for analyzing the complexity of evaluation-based algorithms. Our bounds are based on an integral formula (10) and an amortized evaluation bound (Appendix). This can be viewed as a continuous amortization. We pose several open problems:
(a) Show that EVAL has subdivision complexity $O(dL)$ in the benchmark case.
(b) Show that the "ideal integral" (35) satisfies $R_0 = \Omega(dL)$.
(c) Extend integral analysis to higher dimensions, in particular the Plantinga-Vegter algorithms in 2 and 3-D.

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APPENDIX

8 Local Lipschitz Constants.

We would like to replace the global constants K_I used in Theorem 7 by more local constants. One suggestion, to use K_X in place of K_I, seems to lead to complicated conditions on our partitions and to require the integral to depend on the partition. Instead, we proceed as follows.

In this section, we fix the interval I, and throughout, X range over $\square I$. For any $a \in I$ and $\ell > 0$, define

$$K_{a,\ell} := \max_{\substack{X \subseteq I \\ a \in X \\ w(X) \leq \ell}} K_X. \tag{26}$$

.

If we replace K_X by K'_X in (26), the resulting constant will be denoted by $K'_{a,\ell}$.

Lemma 22. *Let $a \in I$ and $\ell > 0$.*
(i.a) $K_{a,\ell}$ is monotonically non-decreasing with ℓ.
(i.b) As $\ell \to \infty$, we have $K_{a,\ell} \to \infty$ and $\dfrac{|f(a)|}{K_{a,\ell}} \to \dfrac{|f(a)|}{K_I}$.
(i.c) As $\ell \to 0$, we have $K_{a,\ell} \to |f'(a)|$ and $\dfrac{|f(a)|}{K_{a,\ell}} \to \dfrac{|f(a)|}{|f'(a)|}$. (Hence, define $K_{a,0} = |f'(a)|$.)
(ii.a) The product $\ell \cdot K_{a,\ell}$ in strictly increasing with ℓ.
(ii.b) As $\ell \to \infty$, we have $\ell \cdot K_{a,\ell} \to \infty$.
(ii.c) As $\ell \to 0$, we have $\ell \cdot K_{a,\ell} \to 0$.

The proof is omitted. From (ii.a-c), we conclude that there is a unique $\ell = \ell_a$ such that $\ell_a \cdot K_{a,\ell} = |f(a)|$. Define $w(a) := \ell_a$ as the **local width** at a, and define $K_a := K_{a,w(a)}$ as the **local Lipschitz constant** at a. Note that $f(a) = 0$ implies $w(a) = 0$ and hence $K_a = |f'(a)|$.

We can also define the local width $w'(a)$ (resp., local Lipschitz constant K'_a) if we use $f', K'_{a,\ell}$ instead of $f, K_{a,\ell}$ in the above definitions of $w(a)$ (resp., K_a).

For all $a \in I$, we have

$$w(a) = \frac{|f(a)|}{K_a}. \tag{27}$$

From Lemma 22(ii), we immediately obtain:

Lemma 23. *Let $a \in I$ and $\ell > 0$. Then*

$$\ell \geq \frac{|f(a)|}{K_{a,\ell}} \quad \Leftrightarrow \quad \ell \geq w(a) \geq \frac{|f(a)|}{K_{a,\ell}}.$$

$$\ell \leq \frac{|f(a)|}{K_{a,\ell}} \quad \Leftrightarrow \quad \ell \leq w(a) \leq \frac{|f(a)|}{K_{a,\ell}}.$$

Moreover, equality is simultaneously achieved on both sides.

We define $f_\ell(a) = \max\left\{\dfrac{|f(a)|}{K_{a,l}}, \dfrac{|f'(a)|}{K'_{a,l}}\right\}$.

Using these facts, we define our candidate for a stopping functions:

$$f_*(a) := \max\left\{\frac{|f(a)|}{K_a}, \frac{|f'(a)|}{K'_a}\right\} \tag{28}$$

$$= \max\left\{w(a), w'(a)\right\}. \tag{29}$$

Using these definitions, Lemma 23 can be rephrased as follows:

$$\ell \geq f_\ell(a) \quad \Leftrightarrow \quad \ell \geq f_*(a) \geq f_\ell(a). \tag{30}$$

$$\ell \leq f_\ell(a) \quad \Leftrightarrow \quad \ell \leq f_*(a) \leq f_\ell(a). \tag{31}$$

Moreover, equality occurs simultaneously on both sides.

LEMMA 24. *f_* is a stopping function.*

Proof. Let $a \in X$. If $C_0(X)$ and $C_1(X)$ fail, as before, it means $w(X) \geq \max\{|f(a)|/K_X, |f'(a)|/K'_X\} = f_X(a)$. Thus $w(X) \geq f_{w(X)}(a)$. By (30), this is equivalent to $w(X) \geq f_*(a)$. Hence X is large.

Q.E.D.

THEOREM 25. *Let P_I be the partition of I at the end of Phase 1 of the Evaluation Algorithm. Then*

$$\#(P) \leq \max\left\{1, \int_I \frac{2da}{f_*(a)}\right\} = \max\left\{1, \int_I \min\left\{\frac{K_a}{|f(a)|}, \frac{K_a}{|f(a)|}\right\} 2da\right\}. \tag{32}$$

and this integral is finite.

We already know that f_* is a stopping function. This result follows from Theorem 5 if f_* is never 0. f_* is never 0 since f is square free and f and f' do not share any roots.

9 An Amortized Evaluation Bound

We prove an amortized evaluation bound that has independent interest. Unlike the usual Mahler-Davenport bounds, it does not involve root separation.

Let $f = \sum_{i=t}^{d} c_i X^i \in \mathbb{C}[X]$ $(t = 0, \ldots, d)$ where $c_0 c_t \neq 0$. Recall that the height of f is $\|f\| = \max_{i=t}^{d} |c_i|$. Let $\mathrm{lc}(f) = |c_d|$ and $\mathrm{tc}(f) = |c_t|$ (resp.) denote the absolute values of the **leading coefficient** and **tail coefficient** (i.e., smallest non-zero coefficient) of f. We write $\mathtt{res}(f, g)$ for the resultant of two polynomials f, g. In addition to heights, we use the Mahler measure of polynomials, defined as $M(f) = \mathrm{lc}(f) M_1(f)$ where

$$M_1(f) := \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$$

where $\alpha_1, \ldots, \alpha_d$ are all the complex roots of f.

THEOREM 26. *Let $\phi(x), \eta(x) \in \mathbb{C}[x]$ be complex polynomials of degrees m and n. Let $\beta_1, \ldots, \beta_n$ be all the zeros of $\eta(x)$.*
(a)

$$\prod_{i=1}^{n} |\phi(\beta_i)| \leq ((m+1)\|\phi\|)^n \left(\frac{M(\eta)}{\mathrm{lc}(\eta)}\right)^m.$$

(b) Suppose there exists relatively prime $F, H \in \mathbb{Z}[x]$ such that $F = \phi\overline{\phi}$, $H = \eta\overline{\eta}$ for some $\overline{\phi}, \overline{\eta} \in \mathbb{C}[x]$. If the degrees of $\overline{\phi}$ and $\overline{\eta}$ are $\overline{m}$ and $\overline{n}$, then

$$\prod_{i=1}^{n} |\phi(\beta_i)| \geq \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot ((m+1)\|\phi\|)^{\overline{n}} M(\overline{\eta})^m \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n+\overline{n}} M(H)^{\overline{m}}}.$$

We also have

$$\prod_{i=1}^{n} |\phi(\beta_i)| \geq \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot ((m+\overline{m})\|F\|)^{\overline{n}} M(\overline{\eta})^{m+\overline{m}} \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n} M(\eta)^{\overline{m}}}.$$

Proof. (a) We may index the β_i's such that, for some $n' \in \{0, 1, \ldots, n\}$, we have $|\beta_i| \geq 1$ iff $i > n'$. Now for $i = 1, \ldots, n'$, we have $|\phi(\beta_i)| < \|\phi\|(m+1)$ and hence

$$\prod_{i=1}^{n'} |\phi(\beta_i)| \leq (\|\phi\|(m+1))^{n'}. \tag{33}$$

This inequality is strict iff $n' > 0$. For $i = n'+1, \ldots, n$, we have $|\phi(\beta_i)| \leq \|\phi\|(m+1)|\beta_i|^m$. So

$$\prod_{i=n'+1}^{n} |\phi(\beta_i)| \leq (\|\phi\|(m+1))^{n-n'} \left(\prod_{i=n'+1}^{n} |\beta_i|\right)^m = (\|\phi\|(m+1))^{n-n'} \left(\frac{M(\eta)}{\mathrm{lc}(\eta)}\right)^m \tag{34}$$

Part (a) follows from (33) and (34).

(b) We have $\mathtt{res}(F, H) = \mathrm{lc}(H)^{m+\overline{m}} \prod_{i=1}^{n+\overline{n}} F(\beta_i)$ where $\beta_1, \ldots, \beta_n, \beta_{n+1}, \ldots, \beta_{n+\overline{n}}$ are all the zeros of H (see [33, p. 167]). Thus,

$$\begin{aligned}
1 \leq |\mathtt{res}(F, H)| &= \mathrm{lc}(H)^{m+\overline{m}} \cdot \prod_{i=1}^{n} |\phi(\beta_i)| \left(\prod_{i=n+1}^{n+\overline{n}} |\phi(\beta_i)| \cdot \prod_{i=1}^{n+\overline{n}} |\overline{\phi}(\beta_i)|\right) \\
\prod_{i=1}^{n} |\phi(\beta_i)| &\geq \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot \prod_{i=n+1}^{n+\overline{n}} |\phi(\beta_i)| \cdot \prod_{i=1}^{n+\overline{n}} |\overline{\phi}(\beta_i)|} \\
&\geq \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot ((m+1)\|\phi\|)^{\overline{n}} M(\overline{\eta})^m \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n+\overline{n}} M(H)^{\overline{m}}},
\end{aligned}$$

where the last inequality is an application of the bound in part (a). Alternatively, we could proceed thus:

$$\begin{aligned}
\prod_{i=1}^{n} |\phi(\beta_i)| &\geq \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot \prod_{i=n+1}^{n+\overline{n}} |F(\beta_i)| \cdot \prod_{i=1}^{n} |\overline{\phi}(\beta_i)|} \\
&\geq \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot ((m+\overline{m})\|F\|)^{\overline{n}} M(\overline{\eta})^{m+\overline{m}} \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n} M(\eta)^{\overline{m}}}.
\end{aligned}$$

Q.E.D.

We also need the following bound:

LEMMA 27. *If $S \subseteq \{\alpha_1, \ldots, \alpha_d\}$ is a set of non-zero roots of f then*

$$\prod_{\alpha \in S} |\alpha| \geq \frac{\mathrm{tc}(f)}{M(f)}.$$

Proof.

$$\begin{aligned}
\prod_{\alpha \in S} |\alpha| &\geq \prod_{i=t+1}^{d} \min\{1, |\alpha_i|\} \\
&= \frac{\prod_{i=t+1}^{d} |\alpha_i|}{\prod_{i=t+1}^{d} \max\{1, |\alpha_i|\}} \\
&= \frac{\mathrm{lc}(f) \prod_{i=t+1}^{d} |\alpha_i|}{M(f)} \\
&= \frac{\mathrm{tc}(f)}{M(f)}.
\end{aligned}$$

Q.E.D.

So if f is an integer polynomial, $\prod_{\alpha\in S}|\alpha| \geq \frac{1}{M(f)}$.

10 Bound on Integral of Logarithmic Derivatives

Our goal is to show Lemma 16. Of independent interest, we also bound a related integral,

$$R_0 = \int_a^b \min\left\{\left|\frac{f'(a)}{f(a)}\right|, \left|\frac{f''(a)}{f'(a)}\right|\right\} da. \tag{35}$$

Note that R_0 is a lower bound on the integral (32). The interest in R_0 comes from the hope that the integral (32) is asymptotically equal to R_0 "under ideal conditions".

¶9. Bounding R (Lemma 16). We may write $I' = \bigcup_{i=1}^k [a_i, b_i]$ where the $[a_i, b_i]$'s are pairwise disjoint. Moreover, $f'(x)/f(x)$ has constant non-zero sign over each $[a_i, b_i]$, and so the integral $\int_{a_i}^{b_i} |f'(x)/f(x)|dx = [\log|f(x)|]_{a_i}^{b_i} = \log|f(b_i)|/|f(a_i)|$. Therefore, $R = \sum_{i=1}^k \log|f(b_i)|/|f(a_i)|$. We can now directly apply the Evaluation Bound Theorem, Theorem 26: the polynomials $\phi(x), F(x)$ in Theorem 26 are just $f(x) = \phi(x) = F(x)$, and $\eta(x) = \prod_{i=1}^k (x - a_i)(x - b_i)$ with $H(x) = 2^N \eta(x)$ for a suitable integer N. Also, $\log M(H) = O(dL)$ and $\log\|f\| = O(dL)$. The overall bound of $O(dL)$ follows from a routine calculation.

¶10. Bounding R_0. Let $h = (f')^4 - (ff'')^2$. Note that f, f', h are pairwise relatively prime. Clearly, we have

$$R_0 = R_D + R_E = \int_D \left|\frac{f'(a)}{f(a)}\right| da + \int_E \left|\frac{f''(a)}{f'(a)}\right| da.$$

where D, E are unions of intervals with endpoints in $Zero(h)$, and R_D, R_E are defined by that $\arg\min$ of the integrand of R.

Let us focus on R_D since R_E is similar. Again,

$$R_D = R_D^+ - R_D^- = \int_{D^+} \frac{f'(a)}{f(a)} da - \int_{D^-} \frac{f'(a)}{f(a)} da$$

where $D^+ := \{x \in D : f'(x)f(x) \geq 0\}$ and $D^- := \{x \in D : f'(x)f(x) \leq 0\}$ Thus D^+, D^- are union of intervals whose endpoints belong to $Zero(hff')$. Note that R_D^+ and R_D^- are non-negative.

Let us focus on R_D^+ since R_D^- is similar. Suppose $D^+ = \bigcup_{i=1}^k [a_i, b_i]$. Then we have

$$\begin{aligned} R_D^+ &= \sum_i^k \int_{a_i}^{b_i} \frac{f'(x)}{f(x)} dx \\ &= \sum_i^k \log f(x)|_{a_i}^{b_i} \\ &= \log \prod_i^k \frac{f(b_i)}{f(a_i)}. \end{aligned} \tag{36}$$

We next bound R_D^+ in terms of the degree and height of $f \in \mathbb{Z}[x]$.

THEOREM 28. *Let* $\deg f = d$ *and* $\|f\| < 2^L$. *Then* $R_0 = O(dL)$.

Proof. It is enough to show that $R_D = O(dL)$ since R_E has the same bound. It is sufficient to upper bound R_D^+ since R_D^- will be bounded in exactly the same way.

The polynomial hff' has degree $< 6d$ and height $O(1)^L$, again using our assumption that $\log d \le L$. From (36), we see that

$$R_D^+ = \log(\prod_i^k |f(b_i)|) - \log(\prod_i^k |f(a_i)|.$$

Since the b_i's are distinct zeros of hff', we can apply the upper bound in Theorem 26:

$$\begin{aligned}\log(\prod_i^k |f(b_i)|)) &\le \log\left[((d+1)\|f\|)^{6d} M(hff')^d\right] \\ &= O(dL).\end{aligned}$$

Here, we have used the fact that $M(f) = O(\|f\|)$ since $\log d \le L$. Again, since the a_i's are distinct zeros of hff', we can apply the lower bound in Theorem 26:

$$\begin{aligned}-\log(\prod_i^k |f(a_i)|)) &\le \log\left[((d+1)\|f\|)^{6d} M(hff')^d\right] \\ &= O(dL).\end{aligned}$$

Q.E.D.

11 On the Real and Imaginary Part of Zeros.

Let $f \in \mathbb{R}[X]$ be a real polynomial of degree $d \ge 1$. Suppose its complex zeros are $\alpha_1, \ldots, \alpha_d$ and let $r_i = \mathtt{Re}(\alpha_i)$ and $s_i = \mathtt{Im}(\alpha_i)$ for each i. Our goal is to construct two integer polynomials $R(X), S(X)$ whose roots contains the r_i's and s_i's respectively. We also want to bound the heights of $R(X)$ and $S(X)$. CAVEAT: In this section, r_i, s_i here refer to real/complex parts of roots of f; elsewhere, they refer to real/complex parts of roots of f'.

¶11. REAL PART. We first construct a polynomial $R(X)$ whose roots include all the r_i's (cf. [33, p. 202]).

Use the Taylor expansion of $f(X + \mathbf{i}Y)$ at the point X:

$$\begin{aligned}f(X+\mathbf{i}Y) &= f(X) + f'(X)(\mathbf{i}Y) + \frac{f''(X)}{2}(\mathbf{i}Y)^2 + \cdots + \frac{f^{(d)}(X)}{d!}(\mathbf{i}Y)^d \\ &= P(X,Y) + (\mathbf{i}Y)Q(X,Y)\end{aligned}$$

where

$$\begin{aligned}P = P(X,Y) &:= \sum_{j=0}^{\lfloor d/2 \rfloor} f_{2j}(X)(-Y^2)^j \\ Q = Q(X,Y) &:= \sum_{j=0}^{\lceil d/2 \rceil - 1} f_{2j+1}(X)(-Y^2)^j\end{aligned}$$

and $f_i(X) := (-1)^{\lfloor i/2 \rfloor} \frac{f^{(i)}(X)}{i!}$ is the "normalized" ith derivative (with sign). Note that $f_0(X) = f(X)$ and $\deg_Y(P) \geq \deg_Y(Q)$. It follows that r_i are real zeros of the resultant $R(X) := \mathtt{res}_Y(P(X,Y), Y \cdot Q(X,Y))$. It is easy to verify that

$$\mathtt{res}_Y(P, Y \cdot Q) = f_0(X)\mathtt{res}_Y(P, Q).$$

To further factor $R(X)$, let us assume $d \geq 3$, so that $\deg_Y(P) \geq \deg_Y(Q) \geq 2$. Then we can write

$$P(X,Y) = \overline{P}(X, Y^2), \qquad Q(X,Y) = \overline{Q}(X, Y^2)$$

where $\deg_Y \overline{P} = \lfloor d/2 \rfloor \geq \lceil d/2 \rceil - 1 = \deg_Y \overline{Q}$. Then we may verify

$$R(X) = f(X) \cdot \overline{R}(X)^2$$

where $\overline{R}(X) = \mathtt{res}_Y(\overline{P}, \overline{Q})$.

For the next bound, we use the 1-norm $\|f\|_1$ and 2-norm $\|f\|_2$ of f.

Lemma 29. *The degree of* $\overline{R} = \mathtt{res}_Y(\overline{P}, \overline{Q})$ *is*

$$\binom{d}{2} = \frac{d(d-1)}{2}.$$

Also, $\|\overline{R}\|_2 \leq (2^d \|f\|_1)^{d-1}$.

Proof. The degree of $\overline{R}$ comes from looking at the main diagonal of the Sylvester matrix defining the resultant. There are two cases: Case d is odd: here $\deg_Y \overline{P} = \deg_Y \overline{Q} = (d-1)/2$. The product of the diagonal elements is $(f_0)^{(d-1)/2}(f_d)^{(d-1)/2}$. Since $\deg f_0 = d$ and $\deg f_d = 0$, the degree of this product is $d(d-1)/2$. Case d is even: here $\deg_Y \overline{P} = d/2$ and $\deg_Y \overline{Q} = (d-2)/2$. The product of the diagonal elements is $(f_0)^{(d-2)/2}(f_{d-1})^{d/2}$. Since $\deg f_0 = d$ and $\deg f_{d-1} = 1$, the degree of this product is again $\frac{d(d-2)}{2} + d/2 = d(d-1)/2$.

For the height of $\overline{R}$, we use the Goldstein-Graham bound (see [33, p. 173]). Let $\mathtt{res}_Y(\overline{P}, \overline{Q}) = \det(T)$ where $T = [t_{ij}]_{i,j}$ is the $(d-1) \times (d-1)$ Sylvester matrix constructed from $\overline{P}, \overline{Q}$. For instance the first and last rows of T are (respectively) given by

$$(f_0, f_2, f_4, \ldots, f_{\lfloor d/2 \rfloor}, 0, \ldots, 0),$$
$$(0, \ldots, 0, f_1, f_3, , \ldots, f_{\lceil d/2 \rceil - 1}).$$

Let $W = [w_{ij}]_{i,j}$ be the $(d-1) \times (d-1)$ matrix whose (i,j)th entry is given by $w_{ij} = \|t_{ij}\|_1$. Each of the t_{ij} is of the form f_k for some $k = k(i,j)$. We use the simple estimate $\|f_k\|_1 \leq \binom{d}{k}\|f\|_1$ and hence the 2-norm of the first row of W is

$$\left(\|f_0\|_1^2 + \|f_2\|_1^2 + \|f_4\|_1^2 + \cdots + \|f_{\lfloor d/2 \rfloor}\|_1^2\right)^{1/2} < \left(\sum_{i \geq 0} \binom{d}{i}^2 \|f\|_1^2)\right)^{1/2} \leq 2^d \|f\|_1.$$

In fact, the 2-norm of every row of W is bounded by $2^d\|f\|_1$. The Graham-Goldstein bound says $\|\overline{R}\|_2$ is upper bounded by the product of these 2-norms, i.e., $\|\overline{R}\|_2 \leq (2^d \|f\|_1)^{d-1}$. **Q.E.D.**

Since $\lg \|f\|_1 \leq \lg d + L = O(L)$, we obtain

$$\lg \|\overline{R}\|_2 = O(d(d+L)). \tag{37}$$

¶12. COMPLEX PART. A similar procedure can be used to construct a polynomial $S(Y)$ whose roots include all the $s_i = \mathtt{Im}(\alpha_i)$. The details are somewhat different, which we proceed to derive. First, we write $f(X)$ as a sum of its even and odd parts:

$$f(X) = f_e(X) + f_o(X) \tag{38}$$
$$= \overline{f}_e(X^2) + X \cdot \overline{f}_o(X^2) \tag{39}$$

where $\overline{f}_e, \overline{f}_o \in \mathbb{R}[X]$ have degrees $\lceil (d-1)/2 \rceil$ and $\lfloor (d-1)/2 \rfloor$, respectively. For $i \geq 0$, we further write the i-th derivatives of f_e and f_e in the form:

$$f_e^{(i)}(X) = \begin{cases} \overline{f}_{e,i}(X^2) & \text{if } i = \textit{even} \\ X \cdot \overline{f}_{e,i}(X^2) & \text{if } i = \textit{odd}, \end{cases}$$
$$f_o^{(i)}(X) = \begin{cases} X \cdot \overline{f}_{o,i}(X^2) & \text{if } i = \textit{even} \\ \overline{f}_{o,i}(X^2) & \text{if } i = \textit{odd}. \end{cases}$$

The polynomials $\overline{f}_{e,i}$ and $\overline{f}_{o,i}$ are implicitly defined by these equations.

Use the Taylor expansion of $f(X + \mathbf{i}Y)$ at the point $\mathbf{i}Y$:

$$\begin{aligned} f(X+\mathbf{i}Y) &= \sum_{i\geq 0} f^{(i)}(\mathbf{i}Y)\frac{X^i}{i!} \\ &= \sum_{i\geq 0} \left[f_e^{(i)}(\mathbf{i}Y) + f_o^{(i)}(\mathbf{i}Y) \right] \frac{X^i}{i!} \\ &= \sum_{i\geq 0} \left[f_{e,2i}(-Y^2) + \mathbf{i}Y f_{o,2i}(-Y^2) \right] \frac{X^{2i}}{(2i)!} + \sum_{i\geq 0} \left[\mathbf{i}Y f_{e,2i+1}(-Y^2) + f_{o,2i+1}(-Y^2) \right] \frac{X^{2i+1}}{(2i+1)!} \\ &= \sum_{i\geq 0} \left[\frac{f_{e,2i}(-Y^2)}{(2i)!} + X\frac{f_{o,2i+1}(-Y^2)}{(2i+1)!} \right] X^{2i} + \mathbf{i}Y \sum_{i\geq 0} \left[X\frac{f_{e,2i+1}(-Y^2)}{(2i+1)!} + \frac{f_{o,2i}(-Y^2)}{(2i)!} \right] X^{2i} \\ &= P(X,Y) + \mathbf{i}YQ(X,Y) \end{aligned}$$

where

$$\begin{aligned} P(X,Y) &= \textstyle\sum_{i=0}^{2\lfloor (d-1)/2 \rfloor} p_i(Y)X^i, \\ &\quad \text{with } p_{2i}(Y) = \tfrac{f_{e,2i}(-Y^2)}{(2i)!} \text{ and } p_{2i+1}(Y) = \tfrac{f_{o,2i+1}(-Y^2)}{(2i+1)!}, \\ Q(X,Y) &= \textstyle\sum_{i=0}^{2\lceil (d-1)/2 \rceil} q_i(Y)X^i, \\ &\quad \text{with } q_{2i}(Y) = \tfrac{f_{o,2i}(-Y^2)}{(2i)!} \text{ and } q_{2i+1}(Y) = \tfrac{f_{e,2i+1}(-Y^2)}{(2i+1)!}. \end{aligned}$$

NOTE: we are reusing the symbols P, Q, and they should not be confused with the polynomials P, Q used in the definition of $R(X)$ above.

Now the imaginary part of the zeros of $f(X)$ are zeros of the resultant $S(Y) := \mathtt{res}_X(P, Q)$ since

$$\mathtt{res}_X(P(X,Y), Y \cdot Q(X,Y)) = Y^{2\lfloor (d-1)/2 \rfloor}\mathtt{res}_X(P,Q). \tag{40}$$

Note that $S(Y)$ is the determinant of a Sylvester matrix T whose first and last rows are

$$(p_0, p_1, p_2, \ldots, p_{2\lfloor (d-1)/2 \rfloor}, 0, \ldots, 0),$$
$$(0, \ldots, 0, q_0, q_1, \ldots, q_{2\lceil (d-1)/2 \rceil}).$$

The dimension of T is $(d-1) \times (d-1)$, and $\deg(S(Y)) \leq d(d-1)$.

To bound the height of $S(Y)$, we proceed as before: $\|p_i\|_1 \leq \binom{d}{i}\|f\|_1$ and $\|q_i\|_1 \leq \binom{d}{i}\|f\|_1$. Then the Goldstein-Graham bound implies $\|S\|_2 \leq (2^d\|f\|_1)^{d-1}$, or $\lg \|S\|_2 = O(dL)$.

Lemma 30. *The degree of $S = \texttt{res}_X(P,Q)$ is*

$$\binom{d}{2} = \frac{d(d-1)}{2}.$$

Also, $\|S\|_2 \le (2^d\|f\|_1)^{d-1}$.

12 Bounding the Gamma Integral

We first prove the key inequality of Lemma 18, restated here:

Lemma 18. *Let $\beta_2, \ldots, \beta_d$ be all the critical points of $f(x)$ (i.e., zeros of f'). Then*

$$\gamma(x) \le \sum_{j=2}^{d} \frac{1}{2|x-\beta_j|}$$

Proof. We have

$$\frac{f^{(i)}(x)}{f'(x)} = \sum_{(j_2,\ldots,j_i)}{}' \prod_{\ell=2}^{i} \frac{1}{x-\beta_{j_\ell}}$$

where the summation ranges over all ordered $(i-1)$-tuples $(j_2, j_3, \ldots, j_i)$ taken from $\{1, \ldots, d-1\}$, $1 \le j_2 < j_3 < \cdots < j_i \le d-1$. The prime in the summation symbol, $\sum'$, indicates the strict inequality, $j_2 < \cdots < j_i$. When we omit the prime in the summation, it means that the tuples could have duplicated components, $1 \le j_2 \le j_3 \le \cdots \le j_i \le d-1$. Thus

$$\begin{aligned}
\left|\frac{f^{(i)}(x)}{f'(x)}\right|^{1/(i-1)} &= \left|\sum_{(j_2,\ldots,j_i)}{}' \prod_{\ell=2}^{i} \frac{1}{x-\beta_{j_\ell}}\right|^{1/(i-1)} \\
&\le \left(\sum_{(j_2,\ldots,j_i)}{}' \prod_{\ell=2}^{i} \frac{1}{|x-\beta_{j_\ell}|}\right)^{1/(i-1)} \\
&\le \left(\sum_{(j_2,\ldots,j_i)} \prod_{\ell=2}^{i} \frac{1}{|x-\beta_{j_\ell}|}\right)^{1/(i-1)} && \text{unprimed summation} \\
&= \left(\left(\sum_{j=2}^{d} \frac{1}{|x-\beta_j|}\right)^{i-1}\right)^{1/(i-1)} \\
&\le \sum_{j=2}^{d} \frac{1}{|x-\beta_j|}.
\end{aligned}$$

For $i \ge 2$, we have $i! \ge 2^{i-1}$, and hence

$$\left|\frac{f^{(i)}(x)}{i!f'(x)}\right|^{1/(i-1)} \le \frac{1}{2}\left|\frac{f^{(i)}(x)}{f'(x)}\right|^{1/(i-1)} \le \frac{1}{2}\sum_{j=2}^{d} \frac{1}{|x-\beta_j|}.$$

Q.E.D.

Recall that $\beta_i = r_i + \mathbf{i}s_i$ where $r_i = \texttt{Re}(\beta_i), s_i = \texttt{Im}(\beta_i)$. Also $s_i = 0$ iff $2 \le i \le k$. We next split the analysis into the real and nonreal parts.

¶13. REAL PART. Recall that Procedure H, we obtained disjoint special intervals $[a_i, b_i] \subseteq I$ containing the real roots of f'. Without loss of generality, assume that the real roots of f' in I are $r_1 < r_2 < \cdots < r_\ell$ for some $\ell \le k \le d$. Let $I'' := I \setminus \bigcup_{i=1}^{\ell} [a_i, b_i] \subseteq I'$. For consistency, we set $b_0 = a$ and $a_{\ell+1} = b$. Writing

$$\phi_R(X) = \prod_{i=2}^{k} (X - r_i), \tag{41}$$

we have:

$$\begin{aligned} \int_{I'} \sum_{i=2}^{k} \frac{dx}{|x - r_i|} &\le \int_{I''} \sum_{i=2}^{k} \frac{dx}{|x - r_i|} \\ &= \sum_{j=0}^{\ell} \int_{b_j}^{a_{j+1}} \sum_{i=2}^{k} \frac{dx}{|x - r_i|} \\ &= \sum_{j=0}^{\ell} \ln \left| \frac{\phi_R(a_{j+1})}{\phi_R(b_j)} \right|. \end{aligned}$$

Thus we have shown:

LEMMA 31.

$$\int_{I'} \sum_{i=1}^{k} \frac{dx}{|x - r_i|} \le \ln \prod_{j=0}^{\ell} \left| \frac{\phi_R(a_{j+1})}{\phi_R(b_j)} \right|.$$

It is interesting to note that that the real roots outside the interval I appears in this bound.

¶14. COMPLEX PART. Consider the case where β_i is nonreal, i.e., $i > k$. Initially, assume $a + |s_i| \le r_i \le b - |s_i|$ where $I = [a, b]$. Then

$$\begin{aligned} \int_{I'} \frac{dx}{|x - \beta_i|} &\le \int_a^b \frac{dx}{|x - \beta_i|} \\ &\le \int_a^b \frac{dx}{\max\{|x - r_i|, |s_i|\}} \\ &\overset{(*)}{=} \int_a^{r_i - |s_i|} \frac{dx}{r_i - x} + \int_{r_i - |s_i|}^{r_i + |s_i|} \frac{dx}{|s_i|} + \int_{r_i + |s_i|}^{b} \frac{dx}{x - r_i} \\ &= \ln\left(\frac{r_i - a}{|s_i|}\right) + 2 + \ln\left(\frac{b - r_i}{|s_i|}\right). \end{aligned}$$

where (*) is valid since $\max\{|x - r_i|, |s_i|\} = |s_i|$ iff $x \in [r_i - |s_i|, r_i + |s_i|]$. Next, suppose $r_i - |s_i| \le a$. Then the above bound holds, provided the term $\ln\left(\frac{r_i - a}{|s_i|}\right)$ be dropped. Similarly, if $r_i + |s_i| \ge b$ then the term $\ln\left(\frac{b - r_i}{|s_i|}\right)$ should be dropped. Combining all these cases, we obtain:

LEMMA 32.

$$\int_{I'} \frac{dx}{|x - \beta_i|} \le \ln \max\left\{1, \left(\frac{r_i - a}{|s_i|}\right)\right\} + 2 + \ln \max\left\{1, \left(\frac{b - r_i}{|s_i|}\right)\right\}.$$

We may assume that the roots β_i are indexed so that

$$r_{k+1} - |s_{k+1}| \le r_{k+2} - |s_{k+2}| \le \cdots \le r_d - |s_d|.$$

Then there exists $\ell \in \{k+1, \ldots, d+1\}$ such that $a < r_i - |s_i|$ iff $\ell \le i$. Note that $\ell = d+1$ means there is no such i.

Similarly, we have $\beta_{k+1}, \ldots, \beta_d$ such that

$$r_{k+1} + |s_{k+1}| \le r_{k+2} + |s_{k+2}| \le \cdots \le r_d + |s_d|.$$

There exists $\lambda \in \{k, k+1, \ldots, d\}$ such that $r_j + |s_j| < b$ iff $\lambda \le j$. Again, $\lambda = k$ means there is no such j. Thus Lemma 32 implies:

$$\begin{aligned}\int_{I'} \sum_{i=k+1}^{d} \frac{dx}{|x-\beta_i|} &\le \ln \prod_{i=k+1}^{d} \max\left\{1, \left(\frac{r_i - a}{|s_i|}\right)\right\} + 2(d-k) + \ln \prod_{i=k+1}^{d} \max\left\{1, \left(\frac{b - r_i}{|s_i|}\right)\right\} \\ &= \ln \prod_{i=\ell}^{d} \left(\frac{r_i - a}{|s_i|}\right) + 2(d-k) + \ln \prod_{i=k+1}^{\lambda} \left(\frac{b - r_i}{|s_i|}\right).\end{aligned}$$

In order to bound the integral in Lemma 33 in terms of d and L, we introduce the polynomials

$$\phi_A(X) := \prod_{i=\ell}^{d} (r_i - X) \tag{42}$$

$$\phi_B(X) := \prod_{i=k+1}^{\lambda} (X - r_i) \tag{43}$$

$$\phi_C(X) := \prod_{i=k+1}^{d} (X - s_i). \tag{44}$$

It follows from Lemma 27 that

$$\prod_{i=\ell}^{d} |s_i| \ge \frac{1}{M(\phi_C)} \ge \frac{1}{M(S)}, \qquad \prod_{i=k+1}^{\lambda} |s_i| \ge \frac{1}{M(\phi_C)} \ge \frac{1}{M(S)} \tag{45}$$

where $S(Y)$ is the polynomial of Lemma 30. This allows us to rephrase the preceding integral bound in a compact form:

Lemma 33.

$$\int_{I'} \sum_{i=k+1}^{d} \frac{dx}{|x-\beta_i|} \le \ln \frac{\phi_A(a)\phi_B(b)}{M(\phi_C)^2} + 2(d-k)$$

13 Applying the Evaluation Bounds.

In the previous section, we bounded the integrals for the real part (Lemma 31) and non-real parts (Lemma 33). These bounds were given in terms of the polynomials $\phi_R, \phi_A, \phi_B, \phi_C$ (see (41) and (42)) evaluated at suitable points. To convert these into explicit bounds in terms of d and L, we now use the Evaluation Bound in Theorem 26.

¶15. BOUND ON REAL PART. Consider the integral in Lemma 31. Recall that the Mahler measure of a rational number is $M(p/q) = \max\{|p|, |q|\}$ if p/q is a rational in lowest terms. It is not hard to see that the following amortized bound on the Mahler measures of the a_j's and b_j's:

$$\lg \prod_{j=0}^{\ell} M(a_{j+1})M(b_j) = O(dL). \tag{46}$$

We may set

$$\eta_A(X) = \prod_{j=0}^{\ell}(X - a_{j+1}), \qquad \eta_B(X) = \prod_{j=0}^{\ell}(X - b_j).$$

By multiplying η_s ($s = A, B$) with a suitable power of two, K_s, we obtain $H_s(X) = K_s\eta_i(X) \in \mathbb{Z}[X]$. It is clear that $\lg M(H_s) = O(dL)$ and $\lg \mathrm{lc}(H_s) = \lg K_s = O(dL)$.

We split the proof into two steps. The first step is upper bound

$$\lg \prod_{j=0}^{\ell} |\phi_R(a_{j+1})|.$$

We must exploit the fact that all the zeros of ϕ_R (see (41)) are also zeros of f' (this is in contrast to (47) below). Hence we have

$$\|\phi_R\| \le 2^d M(\phi_R) \le 2^d M(f').$$

We apply Theorem 26(a), with ϕ replaced $\phi_R(X)$, η replaced by $\eta_A(X)$, H by H_A. Hence $m \le d$ and $n \le d$, and

$$\begin{aligned}
\prod_{j=0}^{\ell} |\phi_R(a_{j+1})| &\le ((d+1)\|\phi_R\|)^d \, M(\eta_A)^d \\
&\le \Big((d+1)2^d M(\overline{R})\Big)^d \, M(\eta_A)^d. \\
\lg \prod_{j=0}^{\ell} |\phi_R(a_{j+1})| &= O(d^2 L).
\end{aligned}$$

The second step is to lower bound

$$\lg \prod_{j=0}^{\ell} |\phi_R(b_j)|.$$

We apply Theorem 26(b), with ϕ replaced $\phi_R(X)$ as before, but η replaced by $\eta_B(X)$, H by H_B, F by f'. We have $m \le d-1$ and $n \le d$ as before. Now $\overline{\phi}$ is given by f'/ϕ_R of degree $\overline{m} \le d-1$, and $\overline{\eta} = H_B/\eta_B = K_B$ of degree $\overline{n} = 0$.

$$\begin{aligned}
\prod_{j=0}^{\ell} |\phi_R(b_j)| &\ge \frac{1}{\mathrm{lc}(H_B)^{d-1} \cdot ((d\|\phi_R\|)^0 \, M(\overline{\eta})^{d-1} \cdot ((\overline{m}+1)\|\overline{\phi}\|)^d M(H_B)^{d-1}} \\
-\lg \prod_{j=0}^{\ell} |\phi_R(b_j)| &= O(d^2 L).
\end{aligned}$$

This concludes the proof of Lemma 20.

¶16. BOUND ON NONREAL PART. Consider the polynomial $\phi_A(X)$ in (42). We have ([33, p. 118])

$$\|\phi_A\| \le 2^d M(\phi_A) \le 2^d M(\overline{R}') \tag{47}$$

where $\overline{R}'$ is defined for f', analogous to the definition of $\overline{R}$ defined for f above. Recall that w.l.o.g. a, b are integers satisfying $|a|, |b| \le 2^L$. We now apply Theorem 26(a) where we take the polynomial $\phi(X)$ to be ϕ_A, and F to be $\overline{R}'$. The polynomial $\eta(X)$ is just $X - a$, and $H = \eta$. Hence $m = d$ and $n = 1$. Also $M(\eta) < 2^L$ and we have

$$|\phi_A(a)| \le ((d+1)\|\phi_A\|) \cdot M(\eta)^d \le (d+1)M(\overline{R}') \cdot 2^{Ld}$$

and taking logs,

$$\lg \phi_A(a) = O(dL). \tag{48}$$

Similarly, $\lg \phi_B(b) = O(dL)$.

From (45), we see that $-\lg \prod_i |s_i| \le \lg M(S)$. By Lemma 30, we get $\lg M(S) = O(dL)$. Plugging this and (48) into Lemma 33, we obtain a tight complexity bound for the integral over non-real roots β_i's:

$$\int_{I'} \sum_{i=k+1}^{d} \frac{dx}{|x - \beta_i|} = O(dL). \tag{49}$$

This concludes the proof of Lemma 21.