Complete Subdivision Algorithms, I: Intersection of Bezier Curves

Chee K. Yap[*]
Department of Computer Science
Courant Institute, New York University
New York, NY 10012, USA
and
School of Computer Science and Engineering
Seoul National University
yap@cs.nyu.edu

ABSTRACT

We give the first complete subdivision algorithm for the intersection of two Bezier curves F, G, possibly with tangential intersections. Our approach to robust subdivision algorithms is based on geometric separation bounds, and using a criterion for detecting non-crossing intersection of curves. Our algorithm is adaptive, being based only on exact bigfloat computations. In particular, we avoid manipulation of algebraic numbers and resultant computations. It is designed to be competitive with current algorithms on "nice" inputs. All standard algorithms assume F, G to be relatively prime — our algorithm needs a generalization of this.

Categories and Subject Descriptors

I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling—*geometric algorithms, curves*; G.1.5 [**Numerical Analysis**]: Roots of Nonlinear Equations—*methods for polynomials*; D.m [**Software**]: Miscellaneous—*robust geometric computation*

General Terms

Algorithms, Performance, Reliability

Keywords

computational geometry, curve intersection, Bezier curves, subdivision method, robust numerical algorithms, exact geometric computation

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1. INTRODUCTION

The intersection and analysis of algebraic curves and surfaces is a fundamental problem in many areas of geometric modeling [16]. Most practical algorithms are based on free-form curves and surfaces [8, 6]. In this paper, we consider one class of free-form curves, Bezier curves. All current algorithms for intersecting Bezier curves are inexact, leading to well-known nonrobustness issues. Let us look at a fundamental reason for this.

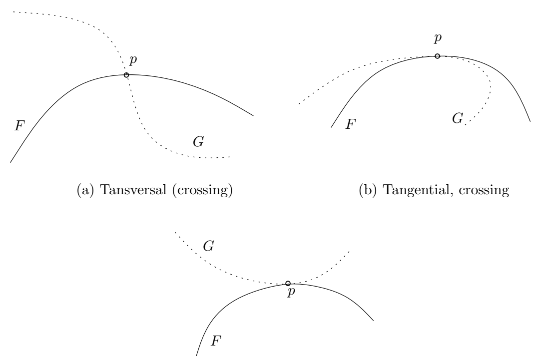
A Bezier curve F is a finite curve segment, represented by a sequence $P(F) = (p_0, \ldots, p_m)$ of control points [8, 6]. Let $CH(F)$ denote the convex hull of $P(F)$, viewed as a closed region. A pair (F, G) of Bezier curves is called a *candidate pair* if $CH(F) \cap CH(G)$ is non-empty. Standard algorithms for intersecting Bezier curves are based on two ideas. First, using the property that a Bezier curve F is contained in $CH(F)$, the algorithm can discard non-candidate pairs. Second, the principal algorithmic operation is *subdivision*, which divides a curve F into two subcurves F_0, F_1 using De Casteljau's algorithm. The *generic intersection algorithm* maintains a queue Q of candidate pairs. As long as Q is non-empty, it extracts a candidate pair (F, G) from Q. If the diameter[1] of $F \cup G$ is less than ε, it outputs this pair; otherwise it subdivides the curve with the larger diameter, say F, into subcurves F_0, F_1, and appends (F_0, G) and (F_1, G) to Q.

This algorithm depends on a constant $\varepsilon > 0$: pairs (F, G) with diameter less than ε are treated as intersecting. For display purposes, such constants are justifiable. But for topological analysis of curve arrangements, we want output pairs (F, G) that represent *unique* intersections. But the generic algorithm might output a pair (F, G) that has *no* intersection, or has *multiple* intersections.

Let p be an intersection point of F and G. Under the *standard assumption* that F, G have no common component, then p is an isolated point of $F \cap G$. The intersection at p can be *tangential* or *transversal*, depending on whether the tangents of F and G at p are coincident or not. Alternatively, we can classify p as a *crossing* or *non-crossing* intersection, depending on whether the curves cross each other at p or not; this amounts to whether the intersection multiplicity at p is odd or even. Non-crossing intersections must be tan-

[1]The diameter $\mathrm{diam}(X)$ of a closed set $X \subseteq \mathbb{R}^2$ is the distance between a farthest pair of points in X.

gential, and transversal intersections must be crossing. Thus we have 3 possibilities, as illustrated in Figure 1.



(a) Tansversal (crossing)

(b) Tangential, crossing

(c) Non-crossing (tangential)

Figure 1: Intersections: (a) Transversal (b) Tangential, crossing (c) Tangential, noncrossing

Can the use of ε be avoided? It seems plausible that if F and G have only crossing intersections, then we can design an intersection algorithm based on subdivision that does not use any ε-cutoff. This is not obvious, but it is implied by the intersection algorithm of this paper. In any case, the key issue is how to detect non-crossing intersections. Recently, Wolpert [23, 22] addressed this issue for nonsingular algebraic curves. The class of curves she addresses and the ones addressed in this paper are not directly comparable – although Bezier curves are rather special algebraic curves, they may be singular (see Figure 5). She introduced the technique of generalized Jacobi curves to detect non-crossing intersection of non-singular curves. She also uses subdivision of space to avoid the manipulation of algebraic numbers (unlike the traditional approach based on cylindrical decomposition). But her approach still uses strong algebraic tools such as resultants and root isolation. Such algebraic techniques are[2] expensive and reduce the effectiveness of adaptivity in other parts of the algorithm. A more recent paper of Seidel and Wolpert [20] addresses computing the topological arrangement of plane algebraic curves; again, a combination of subdivision and algebraic methods are used. In contrast, the only algebraic information we use are algebraic zero bounds. Otherwise, we perform purely numerical computations using bigfloat numbers and primitive geometric operations such as computing convex hulls and intersecting curves with a line. It is not obvious *a priori* that we can achieve our exact curve intersection goals using only such operations. For instance, a purely adaptive/numerical version in the Wolpert-Seidel setting is not known.

Current subdivistion algorithms deploy a variety of criteria for detecting intersection points. These are typically *partial criteria*: either a *rejection criterion* that affirms non-intersection or an *acceptance criterion* that affirms an intersection. A *complete criterion* is one that is both an acceptance and rejection criterion. The generic algorithm above uses only the *convex hull criterion*. Since this is a rejection criterion, the generic algorithm could never affirm intersections. Sederberg and Meyers [19] gave an acceptance criterion based on hodographs which affirms the presence of a transversal intersection. However there is no known acceptance criterion for non-crossing intersections; we will provide one in Section 3. In general, partial criteria is the geometric analogue of the concept of numerical *filters* [5]. Partial criteria can be very useful as a heuristic for quick reject/accept. But ultimately, a correct algorithm must use some complete criteria or some other global guarantee of completeness.

Overview of Our Approach. 1. The fundamental motivation of this work is to design an exact subdivision algorithm for Bezier intersection. Subdivisioning implies adaptivity. In contrast to approaches that combines algebraic methods with numerical ones [22, 23, 20], ours is "fully adaptive".

2. The main analytical tool we introduce is *geometric separation bounds*. They answer such questions as: *What is the closest distance between two curve segments, if they do not intersect?* or *What is the closest distance between a point q and a curve, if q is not on the curve?* These bounds, expressed as functions of the degrees and heights of the underlying polynomials and algebraic numbers, are denoted by various Δ's in this paper. See Sections 2 and 7.

3. The Δ-bounds provide stopping criteria for our numerical and subdivision procedures. The bounds are easily computed at the start of the algorithm. The logic of the algorithm is oblivious to the values of these Δ's. Thus, if improved Δ-bounds are available in the future, they can be directly incorporated without changing the algorithmic logic.

4. Adaptivity means that these Δ-bounds are invoked only in the worst case scenario. For "nice inputs", an iteration may terminate long before the bound of that iteration is reached. Such early terminations rely on semi-criteria (i.e., filters) for determining intersection or non-intersection. For simplicity, we describe our algorithm using only the convex hull filter (Section 1). Although filters are not emphasized in this paper, they influence the basic design of our algorithms. We expect most filters to be easily incorporated into our algorithm with minor changes.

5. Section 3 provides the first complete criterion for detecting non-crossing intersection (NIC) of *elementary Bezier Curves*. By an "elementary curve" we mean the graph of a convex or concave function.

6. Our algorithm uses various numerical and geometric approximations as subroutines. Two key subroutines are for intersecting an elementary Bezier curve with a line (Section 4) and for evaluating signs of the "alpha function" (Section 5).

7. The Δ-bounds require careful application (it is not just a matter of substituting some Δ bound for ε in the generic intersection algorithm). Likewise, the application of the non-crossing intersection criterion (NIC) requires preparation: in Section 6, we describe a *coupling process* to create the necessary preconditions for applying NIC.

8. What about non-elementary Bezier curves? A general Bezier curve has *critical points*; elementary curves have no critical points. There are general methods to break up an algebraic curve at critical points (e.g., [2, 14]). These involve algebraic, non-adaptive methods which we wish to avoid. Instead, Section 7 shows how subdivisioning with separation

[2] Nicola Wolpert and Raimund Seidel (personal communication, and talk at Dagstuhl Workshop 2005) observed that their implementation does not necessarily run faster for easy examples. This is attributable to the non-adaptive aspects of their algorithm.

bounds can detect and isolate such critical points. Section 8 presents the overall intersection algorithm.

9. All our numerical computations are ultimately reduced to ring operations $(+, -, \times)$ on (binary) *bigfloats*, i.e., rational numbers of the form $n2^m$ where $n, m \in \mathbb{Z}$. These operations are[3] carried out *exactly*. For reasons of efficiency, we do not manipulate algebraic or even general rational numbers. We also do not manipulate polynomials or perform subresultant calculations, such as is found in the current exact intersection algorithms.

10. To emphasize the role of bigfloats in our representations, it is useful to introduce the following terminology. First, define the "standard parametrizations" of points, lines and Bezier curves as follows: a point p is given by its coordinates $p = (x, y)$, a line ℓ is given by the coefficients of its equation $\ell : aX + bY + c$, and a Bezier curve is given by its control points (which are in turn given by coordinates). When such standard parameters x, y, a, b, c, etc, are bigfloats, we called them *direct objects*; otherwise they are *indirect objects*. For instance, intersecting a "direct line" ℓ with a "direct Bezier curve" F yields a point p^* whose coordinates are generally algebraic numbers. So p^* is an indirect object. We must then provide alternate means of representing (and approximating) indirect objects by direct objects. We use "expressions" over direct objects. For instance, if p^* is the unique intersection of ℓ and F, we may use the expression "$Point[\ell, F]$" to represent p^*. Thus, ℓ, F are direct objects that serve as non-standard parameters for p^*. This representation can be *refined* as follows: subdivide F into the pair of subcurves (F_0, F_1) using De Casteljau's algorithm as in the generic algorithm. Check whether ℓ intersects F_0 (Section 4); if so, the refined representation is $Point[\ell, F_0]$, otherwise it is $Point[\ell, F_1]$. This process can be repeated as often as we wish, giving better and better approximation of p^*.

Computations with approximations of indirect objects are necessarily iterative, with stopping criteria given by appropriate Δ-bounds. To illustrate, suppose we wish to test whether $p^* = Point[\ell, F]$ lies on a standard Bezier curve G. Assume we could compute a bound $\Delta > 0$ such that if p^* does not lie on G, then its distance from G is at least Δ (Section 2). Then we refine $Point[\ell, F]$ as indicated above, until $\mathrm{diam}(F) < \Delta/2$. Next, we repeatedly subdivide G into subcurves G_j $(j = 0, 1, \ldots)$ using De Casteljau's algorithm. We discard G_j if $CH(G_j) \cap CH(F)$ is empty; we also stop the subdivision on G_j when $\mathrm{diam}(G_j) < \Delta/2$. Finally, we conclude that p lies on G iff the convex hulls $CH(F), CH(G_j)$ intersect for some j. The correctness of this procedure is not hard to see.

11. An appendix contains all the omitted proofs. Our full paper is available from
`http://cs.nyu.edu/cs/faculty/yap/papers/`.

Related Work. The computational literature on algebraic curves and surfaces is very large and diverse. We may roughly divide the computational approaches into two distinct viewpoints: (A) The *Algebraic Viewpoint* treats curves and surfaces as systems of algebraic equations to be solved, usually using symbolic or algebraic techniques. Such "algebraic algorithms" are exact and are (or can be made) complete. (B) The *Geometric Viewpoint* prefers curves and surfaces in parametric form, usually solved using numerical techniques such as homotopy or subdivision. Such "geometric algorithms" are often incomplete but widely used in practice. The Algebraic Viewpoint has made impressive advances in the last 20 years [3]. Nevertheless, many algebraic algorithms are not considered practical. The curves and surfaces in applications are usually bounded subsets ("patches") of an algebraic set. The geometric algorithms directly manipulate such patches; the algebraic algorithms treat complete algebraic sets, often assumed to be irreducible. This fact reduces the applicability of algebraic algorithms. To specify patches of an algebraic set, one could use semi-algebraic formulas (i.e., introduce inequalities). But it is not easy, say, to specify a particular branch of a curve in the neighborhood of a self-intersection using this method.

The computation and topological analysis of real plane curves is a well studied problem [1, 18], but the worst case complexity is prohibitive; the best current theoretical bound is $O(n^{16} \log^5 n)$ time for a curve $F(X, Y) = 0$ of degree $\leq n$ with 2-norm $\leq n$ [11]. Such algorithms are not considered practical [9]. When algebraic algorithms are combined with numerical techniques, more practical algorithms can be achieved [13, 12]. Recently, computational geometers have begun to address curves and surfaces [4, 10, 21]. These papers provide efficient and complete algebraic algorithms for low degree surfaces and curves. More generally, their goal is to make algorithms under the algebraic viewpoint more efficient (by careful considerations of the primitives) and complete (by explicit treatment of degeneracies); both these issues tend to be glossed over by more theoretical papers. In contrast, *our goal is to make algorithms under the geometric viewpoint completely robust while preserving their adaptive efficiency*. More generally, our ostensible goal is to do exact algebraic computation without algebraic manipulations. We only do numerical approximations, but achieve exactness by exploiting algebraic zero bounds.

The work most directly comparable to ours is Wolpert and Seidel's [23, 22, 20], mentioned above. Recently, Plantinga and Vegter [17] gave a topologically-correct algorithm for isotopic approximation of implicit nonsingular surfaces. Their subdivision-based approach is geometric and fully adaptive; however it is currently unknown whether it can be extended to handle singularities. Subdivision methods for solving multivariate polynomial systems have recently been considered by Gershon and Kim [7] and Mourrain and Pavone [15].

2. GEOMETRIC SEPARATION BOUNDS FOR ALGEBRAIC CURVES

Our key mathematical tool is the idea of separation bounds.

Suppose $A(x, y), B(x, y) \in \mathbb{Z}[x, y]$. Let $\|A\|_k$ denote the k-norm of the vector of the coefficients of A (we use $k = 1, 2$ and $k = \infty$). If $p \in \mathbb{R}^n$ is a point in Euclidean space, its k-norm has a similar notation, $\|p\|_k$. We omit the subscript k when referring to the 2-norm, i.e., Euclidean distance. Consider the curves F and G defined by the equations $A = 0$ and $B = 0$, respectively. A pair (p, q) where $p \in F$ and $q \in G$ such that the line through p, q is normal to p at F and to q at G is called an *(F, G)-antipodal pair*. In particular, if F and G intersect tangentially at p, then (p, p) is an (F, G)-antipodal pair. In the following, we assume there are only finitely many pairs of (F, G)-antipodal pairs. This implies that A, B are relatively prime (i.e., they have no common

[3] There is another view of bigfloats that do not require exact ring operations, only approximations to any desired error bound.

component). The following results depend on a multivariate root bound of Yap [24] (Theorem 11.45).

Our first result is a lower bound on the distance between distinct points in an antipodal pair.

Theorem 1. *Let F and G be defined by the equations $A = 0$ and $B = 0$, respectively. Assume F, G has finitely many antipodal pairs. Let $m = \deg(A)$, $n = \deg(B)$ and $\|A\|_2 = a$, $\|B\|_2 = b$. If (p, q) is a (F, G)-antipodal pair and $p \neq q$ then*
$\|p - q\| \geq \Delta_1(m, n, a, b) := (2^{\frac{3}{2}} NK)^{-D}\ 2^{-12m^2n^2}$ *where*

$$K = \max\{\sqrt{13}, 4ma, 4nb\}, \quad N = \binom{3 + 2m + 2n}{5},$$
$$D = m^2n^2(3 + \frac{4}{m} + \frac{4}{n}).$$

We next give separation bounds between distinct intersection points:

Theorem 2. *Let the curves $A = 0$ and $B = 0$ be relatively prime. If p, q are distinct points in their intersection then*
$\|p - q\| \geq \Delta_2(m, n, a, b) := (2^{\frac{3}{2}} NK)^{-D}\ 2^{-12m^2n^2}$ *where*

$$K = \max\{\sqrt{13}, m, n\}, \quad N = \binom{3 + 2m + 2n}{5},$$
$$D = m^2n^2(3 + \frac{4}{m} + \frac{4}{n}).$$

How close can a point p can get to an algebraic curve, without actually being on it? By *(L, ℓ)-bit floating point numbers* (floats, for short) we mean numbers of the form $x = m2^{k-L}$ where $m, k \in \mathbb{Z}$ with $|m| < 2^L$ and $0 \leq k \leq \ell$. Note that $-L \leq \lg|x| < \ell$ (where lg is $\log_2$). We simply say "L-bit floats" for (L, L)-bit floats.

Theorem 3. *Let $q = (u, v)$ be a point whose coordinates are (L, ℓ)-bit floats, $\ell \geq 2$, and $A(u, v) \neq 0$. If the curve $A = 0$ does not contain a circle centered at q, and p is a point on the curve $A = 0$ then*
$\|p - q\| \geq \Delta_3(m, a, L, \ell) := (2^{\frac{3}{2}} NK)^{-D}\ 2^{-8m^2}$ *where*

$$K = 2^{L+\ell+1} \max\{2^L, ma\}, \quad N = \binom{3 + 2m}{3},$$
$$D = m^2(3 + \frac{4}{m}).$$

The requirement that $A = 0$ does not contain a circle centered at q does not affect our application to Bezier curves, because circular arcs are non-Bezier.

The above bounds hold for general algebraic curves. To apply this to a Bezier curve F, we bound its 2-norm bound in terms of its control polygon, $(p_0, \ldots, p_m)$.

Theorem 4. *Let F be a Bezier curve of degree m, with control points which are L-bit floating point numbers. Then F satisfies an an integer polynomial equation $A(x, y) = 0$ of degree m where*

$$\|A\|_2 \leq (16^L 9^m)^m$$

The Δ-Separation Property. For any $\Delta > 0$, we say F, G have the *Δ-separation property* if for all $p \neq q$, if (p, q) is an antipodal pair of (F, G), or if $p, q \in F \cap G$, then $\|p - q\| > \Delta$. The above theorems give us an explicit bound for Δ. It is important to note that this Δ depends only on the initial input curves F, G; subsequent subdivisions of F, G do not change Δ.

3. ELEMENTARY CURVES

This section introduces the notion of elementary curves. The main result is a complete criterion for non-crossing intersection of elementary curves.

Let f be a bounded, continuously differentiable real function defined on the interval $[c, d]$ with $c < d$. Its *graph* is the parametrized curve $F = \{F(t) : t \in [c, d]\}$ where $F(t) = (t, f(t)) \in \mathbb{R}^2$. Let $\mathcal{G}[c, d]$ denote the set of all such graphs. Call $F(t)$ the *graph parametrization* of F, in contrast to the *Bezier parametrization* to be introduced later. Write "$F = F[c, d]$" to indicate that the domain of (graph or Bezier) parametrization is $[c, d]$. If $F = F[c, d]$, the line segment connecting $F(c)$ and $F(d)$ is called the *base segment* of F. Call F an *elementary curve* if it is the graph of a convex or concave function f. If F lies *above* (resp., *below*) its base segment, we call it *A-elementary* (resp., *B-elementary*).

Let $n_F(t)$ denote the *normal line* through $F(t)$, and let $\theta_F(t) \in [0, \pi)$ be the angle that $n_F(t)$ makes with the positive x-axis. Call $n_F(c)$ and $n_F(d)$ the *end normals* of $F = F[c, d]$. Divide the normal line $n_F(t)$ into two *half-normals*: an upper part $a_F(t)$ and a lower part $b_F(t)$ with $F(t)$ as the common end point of the half-lines. For an A-elementary F, define its *upper swept region* $U(F)$ to be bounded by $a_F(c)$ and $a_F(d)$ and F. See Figure 2. When we extend $a_F(c)$ and $a_F(d)$ until they meet, we obtain a cone $C(F)$ that contains $U(F)$. The important property is that $U(F)$ is "stratified" by the upper half-normals $a_F(t)$ ($t \in [c, d]$)

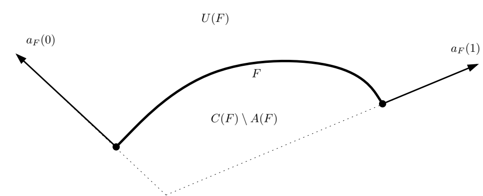


Figure 2: Upper swept region $U(F)$

Elementary Couples. Fix an A-elementary curve F; for simplicity, assume $F \in \mathcal{G}[0, 1]$. Suppose $G \in \mathcal{G}[c, d]$ is another elementary curve. We call (F, G) an *elementary couple* if (i) $G(c) \in a_F(0)$ and $G(d) \in a_F(1)$, and (ii) The entire curve G lies inside the cone $C(F)$. See Figure 3. Note that G can be an A- or B-elementary; accordingly, we call (F, G) an AA- or an AB-couple.

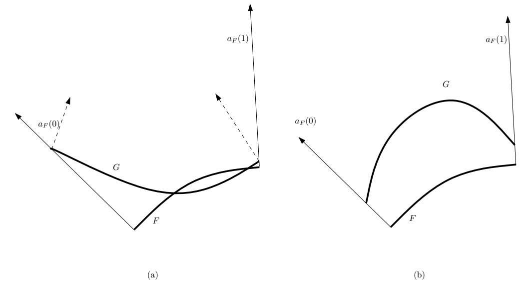


Figure 3: Elementary Couple (F, G): (a) AB-couple, (b) AA-couple

Lemma 5. *Let $(F, G) \in \mathcal{G}[0, 1] \times \mathcal{G}[c, d]$ be an elementary couple. If $G \subseteq U(F)$ then there is a unique continuous function $s : [0, 1] \to [c, d]$ such that for all $t \in [0, t]$, $G(s(t))$ lies on the upper half-normal $a_F(t)$.*

COROLLARY 6. *With (F,G) as in the previous lemma, if $G \subseteq U(F)$ then the angular function*

$$\alpha : [0,1] \to (-\pi, \pi) \quad (1)$$

given by $\alpha(t) = \theta_F(t) - \theta_G(s(t))$ is well-defined and continuous.

To indicate the dependence on F, G, we may write $\alpha = \alpha_{F,G}$ and $s = s_{F,G}$ for these functions. Our main result for testing non-crossing intersection follows:

THEOREM 7 (Non-crossing Intersection Criterion (NIC)). *Let (F,G) be an elementary couple, $G \subseteq U(F)$ and α given by (1). Suppose F, G have the Δ-separation property and the diameter of $F \cup G$ is less than Δ.*
(i) If $\alpha(0)\alpha(1) \leq 0$ then F and G intersect tangentially, in a unique point.
(ii) If $\alpha(0)\alpha(1) > 0$ then F and G are disjoint.

In the following three sections, we show how to apply applying the NIC in an intersection algorithm.

4. INTERSECTION WITH A LINE

The special case of intersecting a Bezier curve F with a straight line ℓ is treated in standard textbooks [8]. We now give a complete algorithm for the case where F is elementary.

When F is represented by its control polygon $P(F) = (p_0, \ldots, p_m)$, then it has the *Bezier parametrization* $F(t) = \sum_{i=0}^{m} p_i B_m^i(t)$ and $B_m^i(t) = \binom{m}{i} t^i (1-t)^{m-i}$. Although the default curve is $F = F[0,1]$, we may also specify an arbitrary interval $I = [c,d]$ to define the curve $F[I] = F[c,d]$. Use the notation "$I \to (I_0, I_1)$" if an interval I is split into two subintervals I_0, I_1 (assume bisection of I unless otherwise noted). Similarly, we write "$F \to (F_0, F_1)$" to mean that $F = F[I]$ is subdivided into $F_0 = F[I_0], F_1 = F[I_1]$ where $I \to (I_0, I_1)$.

The output of our *line intersection algorithm* is a set of pairs (ℓ, F_i) where each F_i $(i \geq 0)$ is a subcurve of the original curve F and $\ell \cap F_i$ has a unique intersection. The relatively straightforward algorithm is omitted in this extended abstract. We note that, in this special case, we can detect tangential intersection using geometric separation bounds alone, without invoking NIC. This line intersection algorithm is used in several subroutines in the general algorithm.

Application to *Beneath-Beyond Test*: this is a form of "ray-shooting" that is used in the Coupling Process below. For points $q \neq v$, let $Ray(q,v)$ denote the ray originating at q and passing through v. On input (F, q, v), where F is elementary, this test produces one of three outputs: ON if $q \in F$; BEYOND if $Ray(q,v)$ intersects F but $q \notin F$; BENEATH otherwise. We first compute the intersection of F with the line through q, v. If there is no intersection, we return BENEATH. Else, we repeatedly subdivide $F \to (F_0, F_1)$ and replace F by F_i if $CH(F_i)$ contains q $(i = 0, 1)$. We terminate when $q \notin CH(F_0) \cup CH(F_1)$, or if $\mathrm{diam}(F) < \Delta_3/2$ (Theorem 3). On termination, if q lies outside $CH(F_0) \cup CH(F_1)$, we can easily determine whether to return BENEATH or BEYOND; otherwise we return ON since $\mathrm{diam}(F) < \Delta_3/2$.

5. ADAPTIVE SIGN OF ALPHA ANGLE

The NIC criterion (Theorem 7) requires sign of the angles $\alpha(0)$ and $\alpha(1)$. We develop an adaptive procedure for this sign determination. Let F be an elementary curve with the Bezier parametrization $F(t) = (F_x(t), F_x(t))$, and ℓ be a line with parametrization $L(t) = (ct + d, et + f)$ for constants c, d, e, f. We may assume $e > 0$. Let ℓ intersect F at the point $F(t^*)$ for some t^*. We want to determine the sign of the angle

$$\alpha(t^*) = \theta_F(t^*) - \theta_\ell$$

where $\theta_\ell \in (0, \pi)$ is slope angle of ℓ. We show that the sign of $\alpha(t)$ is equal to the sign of

$$S(t) := cF_x'(t) + eF_y'(t). \quad (2)$$

From (2), we can develop a lower bound for $S(t^*)$: if the control polygon of F has $m+1$ points with coordinates that are L-bit floats, and suppose c, d, e, f are L-bit floats, then $S(t^*) \neq 0$ implies

$$|S(t^*)| \geq (6m128^L 9^m)^{-m}.$$

We can now adaptively compute the sign of $S(t^*)$ by approximating t^* and estimating the error. See the full paper for details.

6. COUPLING PROCESS

In this section, assume that F, G are elementary Bezier curves satisfying the Δ-separation property, and such that $F \cup G$ has diameter less than Δ. This implies that $|F \cap G| \leq 1$. Our goal is to determine whether they intersect. If they intersect tangentially, we must ultimately reducing them to NIC, at least implicitly. This procedure is called the *coupling process*.

By way of motivation, note that to apply the NIC, we must subdivide F, G until we "see" an elementary couple (F', G') where $F' \subseteq F, G' \subseteq G$. This seems difficult to achieve in general. Instead, we propose to extend NIC to work with "half-couples". Let $F = F[0,1]$ be A-elementary and G lies *above* G, i.e., within the vertical strip $S(F)$ bounded by the vertical lines through $F(0)$ and $F(1)$, G lies above F. Suppose we found a half-normal $a_F(t)$ that intersects G uniquely. If G_0 is the restriction of G to $U(F[0,t])$, then we call $(F[0,t], G_0)$ a *half-couple*; we can similarly define another half-couple $(F[t,1], G_1)$. Note that G_0, G_1 are indirect Bezier curves. It turns out that we can adapte the NIC (Theorem 7) to half-couples.

In outline, the coupling process has three steps: *1. We detect if F and G have crossing intersection.* This is easily reduced to at most four beneath-beyond tests. If there is intersection, we are done. So assume otherwise. Up to symmetry, we may assume that $F = F[0,1]$ is A-elementary and G is *above* F. *2. Use binary search to find a $t \in [0,1]$ such that $a_F(t)$ intersects G.* It is not hard to fix the slight complication in case $a_F(t)$ intersects G in two points. In any case, we have reduced the search to two half-couples. *3. Apply the half-couple NIC below.* The three cases for the half-couples are seen in Figure 4.

LEMMA 8 (Half-Couple NIC). *Suppose $(F[0,1], G[c,d])$ is a half-couple in that $a_F(0)$ passes through $G(c)$ and $a_F(1)$ does not intersect G. Moreover, the end point $G(d)$ lies in $U(F)$, hence above F. Let $b_G(d)$ denote the lower half-normal at $G(d)$. Define $\theta_G(d)$ to be the angle that the upward normal at $G(d)$ makes with the positive x-axis (as before). We have three possibilities:*
(i) $b_G(d)$ intersects $a_F(0)$.

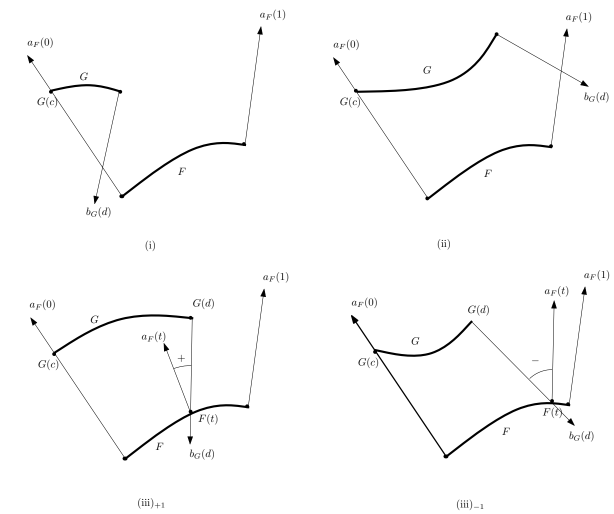


Figure 4: Half-couples: (i) $b_G(d)$ intersects $a_F(a)$, (ii) $b_G(d)$ intersects $a_F(b)$, (iii) $b_G(d)$ intersects F at $F(t)$.

(ii) $b_G(d)$ intersects $a_F(1)$.
(iii)$_s$ $b_G(d)$ intersects F at $F(t)$, $t \in [0,1]$. In this case, let $s = \mathtt{sign}(\theta_F(t) - \theta_G(d)) \in \{0, \pm 1\}$.
Choose t_0 such that $a_F(t_0)$ intersects $G(d)$. Then we have

$$\alpha_{F,G}(t_0) \begin{cases} > 0 & \textit{in cases (i) or (iii)}_{+1}, \\ < 0 & \textit{in cases (ii) or (iii)}_{-1}, \\ = 0 & \textit{in case (iii)}_0 \end{cases}$$

7. CRITICAL POINTS

We now address the issue of non-elementary Bezier curves. A Bezier curve $F[0,1]$ can be subdivided into finitely many elementary subcurves. The points $F(t)$ at which such subdivisions occur are the "critical points". Figure 5(i) shows a cubic Bezier with a critical point.

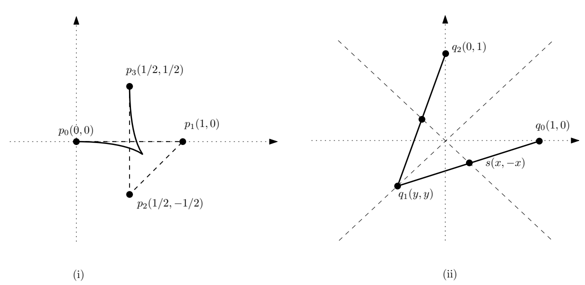


Figure 5: Singular cubic Bezier curve: (i) control polygon, (ii) hodograph

Let $F_x(t), F_y(t)$ be the coordinate functions of $F = F[0,1]$, i.e., $F(t) = (F_x(t), F_y(t))$, and let $F' = (F'_x, F'_y)$ denote derivatives with respect to t. Call $t \in (0,1)$ a *critical value*, and $F(t)$ a *critical point*, if one of the following conditions hold (cf. Kim and Lee [14]):

- (i) $F(t)$ is *stationary*, i.e., $F'_x(t) = F'_y(t) = 0$;
- (ii) $F(t)$ is *x-extreme*, i.e., $F'_x(t) = 0$, $F'_x(t^-)F'_x(t^+) < 0$ and $F'_y(t) \neq 0$;
- (iii) $F(t)$ is an *inflexion point*, i.e., $H_F(t) = 0$ and $H_F(t^+)H_F(t^-) < 0$, where
$$H_F(t) := F'_x(t)F''_y(t) - F'_y(t)F''_x(t).$$

Lemma 9. *If F has no critical points in its relative interior then F is elementary. Conversely, if F is elementary, then it has no x-extreme or inflexion points in its relative interior.*

Critical values are algebraic numbers. Hence we do not propose to subdivide Bezier curves into elementary parts by subdividing at critical points. Instead, we will subdivide a Bezier curve at bigfloat values so that the resulting subcurves are either elementary or contain at most one critical point. Call a curve containing exactly one critical point an *elementary critical curve*. Now, we appeal to additional separation bounds which are summarized by a single number $\Delta^* > 0$ to be explained.

In the rest of this section, assume that the control polygons of F and G are $P(F) = (p_0, \ldots, p_m)$ and $P(G) = (q_0, \ldots, q_n)$. Moreover, the coordinates of the p_i's and q_j's are L-bit bigfloats. First, we bound the degree and norms of critical points:

Lemma 10.
(i) If $F(t) = (F_x(t), F_y(t))$ is a stationary or x-extreme point, then $F_x(t)$ and $F_y(t)$ have degrees $\leq m-1$ and 2-norms at most $(4^L 9^m m)^m$.
(ii) If $F(t)$ is an inflexion point, then $F(t)$ has degree $\leq 2m-3$ and 2-norm at most $(2m^4 16^L 81^m)^m$.

This gives us:

Corollary 11. *Let p, q be two distinct critical points of F. If $2 \leq m \leq n$ then either $|p.x - q.x|$ or $|p.y - q.y|$ is larger than*

$$\Delta_4 := (16^{m+2} 256^L 81^{2m} m^5)^{-m}.$$

Next, we generalize Theorem 3 to the case where q is any algebraic point:

Theorem 12. *Let $q = (q.x, q.y)$ where $q.x, q.y$ are algebraic numbers with 2-norm $\leq c$ and degree $\leq d$. Let the polynomial $A(X,Y)$ have 2-norm a and degree m. If the curve $A(X,Y) = 0$ does not contain a circle centered at q then the distance from q to $A = 0$ is at least $\Delta_5(m, a, L, c, d) = (2^{3/2}NK)^{-D} 2^{-12m^2 d^2}$ where*

$$K = \max\{\sqrt{13}, 4ma, c\}, \quad N = \binom{5+2m+2d}{5},$$
$$D = m^2 d^2 (3 + \frac{4}{m} + \frac{4}{d}).$$

Let $\Delta_6 = \Delta(m, n, L)$ be the minimum separation between the curve G and a critical point p of F, assuming $p \notin G$. This bound comes from plugging Lemma 10 into Theorem 12. Finally, choose Δ^* to be the minimum of the bounds Δ_1 (Theorem 1), Δ_2 (Theorem 2), Δ_4 and Δ_6. If $\mathrm{diam}(F) < \Delta^*$, we may call F a *micro curve*.

It is well-known that the derivative with respect to t is given by

$$F'(t) = m \sum_{i=1}^{m} (p_i - p_{i-1}) B_{m-1}^{i-1}(t) = m \sum_{i=1}^{m} \nabla p_i B_{m-1}^{i-1}(t)$$

where we define $\nabla p_i := p_{i+1} - p_i$. Also, let $\nabla P(F)$ denote $(\nabla p_1, \ldots, \nabla p_m)$. Thus, we see that $m \nabla P(F)$ is the control polygon for the curve $F'(t)$, known as the *hodograph* of F. Hence F contains a stationary point iff $F'(t)$ passes through the origin. E.g., Figure 5(ii) shows $\nabla P(F)$ for the cubic Bezier of Figure 5(i); we may check that the hodograph passes through the origin in this case.

THEOREM 13 (Stationary Points). *Let $diam(F) < \Delta_3(m-1, ma, 2)/m$. Then F contains a stationary point iff the convex hull of $\nabla P(F)$ contains the origin $(0,0)$.*

THEOREM 14 (x-Extreme Points). *Let $diam(F) < \Delta^*$. Then F contains an x-extreme point iff* $(\nabla p_1).x(\nabla p_m).x \leq 0$ *and* $(\nabla p_1).y(\nabla p_m).y > 0$.

If p, q, r are planar points, let $\det(p, q, r)$ be the determinant of the 3×3 matrix whose rows are p^h, q^h, r^h, where $p^h = (p.x, p.y, 1)$. Also let $\det(p, q)$ denote $p.xq.y - p.yq.x$.

THEOREM 15 (Inflexion Points). *Let $diam(F) < \Delta^*$. Then F contains an inflexion point iff* $\det(p_0, p_1, p_2)\det(p_{m-2}, p_{m-1}, p_m) < 0$.

The preceding three theorems give us complete criteria for checking if a micro curve is an elementary critical curve.

8. INTERSECTION ALGORITHM

We now present the global algorithm for intersecting two arbitrary Bezier curves. We design the algorithm so that the generic intersection algorithm (see Introduction) is naturally embedded as the first phase of our algorithm. Simplicity rather than practical efficiency is the aim of the following description.

We have two work queues Q_0 and Q_1. Each queue is just a list of candidate pairs. A candidate pair (F, G) is called a *micro pair* if the diameter of $CH(F) \cup CH(G)$ is less than Δ^*; it is a *macro pair* otherwise. Call Q_0 and Q_1 the *macro queue* and *micro queue*, as they contain macro pairs and micro pairs, respectively. After the obvious initialization of Q_0, Q_1, the algorithm operates in two phases:

Macro Phase: This is basically the generic intersection algorithm of Section 1, operating on Q_0. The key difference is that after splitting a pair (F, G) into (F_i, G) $(i = 0, 1)$, if (F_i, G) is a candidate pair, we place it into Q_0 or Q_1, depending on whether it is a macro or micro pair. If we did nothing else in the macro phase, we would still be correct. But in practice, we would perform a variety of efficient partial criteria (e.g., a test for transversal intersection). This phase ends when Q_0 is empty.

Micro Phase: We now operate on Q_1: for each pair (F, G) extracted from Q_1, we check if F contains a critical point (Section 7). If so, we can output the pair. Otherwise, we apply the coupling process (Section 6).

Remarks. The macro phase is "standard" and easy to implement. The micro phase treats degenerate and unlikely situations. At the end of the macro phase, the micro queue is expected to be small or empty for "nice" inputs.

9. OPEN PROBLEMS

This paper opens up the possibility of achieving fully adaptive algorithms with exactness guarantees for other computational problems involving curves and surfaces. This new direction is intrinsically tied to the use of geometric separation bounds. Most obvious problems are open, so we only list a few questions related to the current paper:

- Our algorithm for intersecting F, G has this *Antipodal Assumption*: *F, G have finitely many antipodal pairs.* As noted, it generalizes the standard assumption that F, G are relatively prime. Clearly, if F is an offset of G, or vice-versa, then the Antipodal Assumption fails. We conjectured that the converse holds. Sungwoo Choi[4] has proved this conjecture. Choi's theorem implies that a Δ-separation property holds even when the Antipodal Assumption fails. Thus, the Antipodal Assumption is not essential. Unfortunately, we do not know how to bound Δ without the Antipodal Assumption.
- Give a complexity analysis of our algorithm, or some variant thereof. In general, the complexity of subdivision algorithms seems little understood.
- Implement an adaptive algorithm such as ours, and compare to non-robust subdivision algorithms, or to algebraic algorithms.
- Improve our separation bounds. We have somewhat improved bounds that exploit the fact that the curves are Bezier; these have been omitted for brevity. Developing bounding techniques that exploit the geometry would be of great interest.
- Provide a simpler algorithm in case there are only transversal intersections.

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APPENDIX: Proofs

We provide all the missing proofs of the paper. The results are re-stated for the readers' convenience.

THEOREM 1. *Let F and G be defined by the equations $A=0$ and $B=0$, respectively. Assume F,G has finitely many antipodal pairs. Let $m=\deg(A)$, $n=\deg(B)$ and $\|A\|_2=a$, $\|B\|_2=b$. If (p,q) is a (F,G)-antipodal pair and $p\neq q$ then*
$\|p-q\| \geq \Delta_1(m,n,a,b) :=(2^{\frac{3}{2}}NK)^{-D}\ 2^{-12m^2n^2}$ *where*

$$\begin{aligned} K &= \max\{\sqrt{13}, 4ma, 4nb\}, \quad N=\binom{3+2m+2n}{5}, \\ D &= m^2n^2(3+\frac{4}{m}+\frac{4}{n}). \end{aligned}$$

Proof. By definition of antipodal pairs, $p=(p.x,p.y)$ and $q=(q.x,q.y)$ and the distance $h=\|p-q\|$ satisfy the following system of five-variate polynomial equations:

$$\begin{aligned} \mathcal{A}_1 &: \quad A(p)=0, \\ \mathcal{A}_2 &: \quad B(q)=0, \\ \mathcal{A}_3 &: \quad h^2-\|p-q\|^2=0 \\ \mathcal{A}_4 &: \quad \left\langle (\frac{dA}{dy}(p),\ -\frac{dA}{dx}(p)),(p-q)\right\rangle=0, \\ \mathcal{A}_5 &: \quad \left\langle (\frac{dB}{dy}(q),\ -\frac{dB}{dx}(q)),(p-q)\right\rangle=0, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes inner product. Since A,B has finitely many antipodal pairs, this system is zero dimensional. To apply [24, Theorem 11.45], note that $\|\mathcal{A}_1\|_2=a$, $\|\mathcal{A}_2\|_2=b$ and $\|\mathcal{A}_3\|_2=\sqrt{13}$. Next, $\|A_y\|_2\leq ma$ where $A_y=dA/dy$. Since $\|A+B\|_2\leq\|A\|_2+\|B\|_2$, we get $\|A_y(p)\cdot(p.x-q.x)\|_2\leq 2ma$. Hence

$$\|\mathcal{A}_4\|_2=\|A_y(p)\cdot(p.x-q.x)-A_x(q)\cdot(p.y-q.y)\|_2\leq 4ma. \tag{3}$$

Similarly, $\|\mathcal{A}_5\|_2\leq 4nb$. Also, $d_1=d_3=m, d_2=d_4=n, d_5=2$. The result follows by plugging into [24, Theorem 11.45]. **Q.E.D.**

THEOREM 2. *Let the curves $A=0$ and $B=0$ be relatively prime. If p,q are distinct points in the intersection of these two curves then*

$$\|p-q\|\geq\Delta_2(m,n,a,b) :=(2^{\frac{3}{2}}NK)^{-D}\ 2^{-12m^2n^2}$$

where

$$\begin{aligned} K &= \max\{\sqrt{13}, m, n\}, \quad N=\binom{3+2m+2n}{5}, \\ D &= m^2n^2(3+\frac{4}{m}+\frac{4}{n}). \end{aligned}$$

Proof. Now p and q satisfy the following system of equations:

$$\begin{aligned} \mathcal{A}_1 &: \quad A(p)=0, \\ \mathcal{A}_2 &: \quad B(q)=0, \\ \mathcal{A}_3 &: \quad h^2-\|p-q\|^2=0, \\ \mathcal{A}_6 &: \quad A(q)=0, \\ \mathcal{A}_7 &: \quad B(p)=0. \end{aligned}$$

This system is zero-dimensional because A, B are relatively prime. The application of [24, Theorem 11.45] is direct in this case. **Q.E.D.**

THEOREM 3. *Let $q = (u, v)$ be a point whose coordinates are (L, ℓ)-bit floats, $\ell \geq 2$, and $A(u, v) \neq 0$. If the curve $A = 0$ does not contain a circle centered at q, and p is a point on the curve $A = 0$ then*

$$\|p - q\| \geq \Delta_3(m, a, L, \ell) := (2^{\frac{3}{2}} NK)^{-D}\ 2^{-8m^2}$$

where

$$\begin{aligned} K &= 2^{L+\ell+1} \max\{2^L, ma\}, \quad N = \binom{3+2m}{3}, \\ D &= m^2(3 + \frac{4}{m}). \end{aligned}$$

Proof. We may assume p is a closest point on A to q and $h = \|p - q\|$. Thus three variables $p.x, p.y, h$ satisfy the system,

$$\begin{aligned} \mathcal{A}_1 &: A(p) = 0, \\ \mathcal{A}_3 &: h^2 - \|p - q\|^2 = 0, \\ \mathcal{A}_4 &: \left\langle (\frac{dA}{dy}(p),\ -\frac{dA}{dx}(p)), (p - q) \right\rangle = 0. \end{aligned}$$

This system is zero-dimensional since the curve $A = 0$ does contain a circle about q. We have $\|\mathcal{A}_1\| = a$, $\|\mathcal{A}_3\| \leq \sqrt{1 + 2(2^{2\ell} + 2^{\ell+1} + 1)} < 2^{1+\ell}$ (assuming $\ell \geq 2$). For $\|\mathcal{A}_4\|$, we have $\|A_x(p) \cdot q.x\| < 2^\ell ma$ but $\|A_x(p)\| = \|A_x(p) \cdot p.x\| \leq ma$ (because $p.x$ is a variable). Hence $\|A_x(p) \cdot (p.x - q.x)\| \leq ma(1 + 2^\ell)$. But instead of $\|\mathcal{A}_3\|$ and $\|\mathcal{A}_4\|$, we use the integer polynomials $2^{2L}\mathcal{A}_3$ and $2^L\mathcal{A}_4$, with $\|2^{2L}\mathcal{A}_3\| < 2^{2L+\ell+1}$ and $\|2^L\mathcal{A}_4\| \leq ma(1 + 2^\ell)2^L < ma2^{1+\ell+L}$. The result follows by plugging into [24, Theorem 11.45]. **Q.E.D.**

THEOREM 4. *Let F be a Bezier curve of degree m, with control points which are L-bit floating point numbers. Then F satisfies an an integer polynomial equation $A(x, y) = 0$ of degree m where*

$$\|A\|_2 \leq (16^L 9^m)^m$$

Proof. The curve $F = (F_1, F_2)$ satisfies the equation $B(x, y) = 0$ where $B(x, y) = \det(M)$ where M is a suitable matrix (see the full paper). The numbers in M are are (L, ℓ)-bit floats, so the matrix $2^L M$ has integer values. Let $A(x, y) = \det(2^L M) = 2^{2mL} B(x, y)$ is an integer matrix. The multivariate version of the Graham-Goldstein lemma [24, Sect. 11.11] says that $\|A\|_2$ is bounded by the product of the 2-norms of each row of the matrix $2^L M_1$ where M_1 is another matrix (see full paper). We can then bound the 2-norm of each row of $2^L M_1$ by $2^{2L}3^m$, Since there are $2m$ rows, the product is $(2^{2L}3^m)^{2m}$, as stated in the theorem. **Q.E.D.**

Lemma 5. *Let $(F, G) \in \mathcal{G}[0, 1] \times \mathcal{G}[c, d]$ be an elementary couple. If $G \subseteq U(F)$ then there is a unique continuous function $s : [0, 1] \to [c, d]$ such that for all $t \in [0, t]$, $G(s(t))$ lies on the upper half-normal $a_F(t)$.*

Proof. Since G lies in the upper swept region $U(F)$, for all $t \in [0, 1]$, $a_F(t)$ intersects G. Our lemma amounts to saying that $a_F(t)$ intersects G exactly once. Clearly $a_F(t)$ intersects G an odd number of times. However, since the region bounded by G and its base segment is convex, no line can intersect G more than twice. Hence G is intersected exactly once by $a_F(t)$, say at the point $G(s(t))$. The continuity of the function $s(t)$ follows from the continuity of the parameterization of $a_F(t)$ and the continuity of the curve G. **Q.E.D.**

Theorem 7 (Non-crossing Intersection Criterion (NIC)).

Let (F, G) be an elementary couple, $G \subseteq U(F)$ and α given by (1). Suppose F, G have the Δ-separation property and the diameter of $F \cup G$ is less than Δ.
(i) If $\alpha(0)\alpha(1) \leq 0$ then F and G intersect tangentially, in a unique point.
(ii) If $\alpha(0)\alpha(1) > 0$ then F and G are disjoint.

Proof. If $F \cap G$ contains two distinct points p and q, then $\|p - q\| \leq \text{diam}(F \cup G) \leq \Delta$. But this contradicts the Δ-separation property. We conclude that $F \cap G$ is either empty or contains a unique point. Thus, if $F \cap G$ intersects, they intersect tangentially. This also shows that $G \subseteq U(F)$. Therefore, the functions $s : [0, 1] \to [c, d]$ and α in (1) are well-defined and continuous.
(i) Assume $\alpha(0)\alpha(1) \leq 0$. So there exists $0 < t < 1$ such that $\alpha(t) = 0$. The pair $(p, q) = (F(t), G(s(t)))$ is therefore antipodal. If $p \neq q$, the Δ-separation property implies $\|p - q\| > \Delta$. But this contradicts our assumption that $F \cup G$ has diameter $\leq \Delta$. Thus $p = q$, i.e., F and G intersect tangentially.
(ii) Suppose, by way of contradiction, that F and G intersect at some (necessarily tangential) point p. Write $p = F(t_0)$ for some t_0. Then $\alpha(t_0) = 0$. Since $\alpha(0)\alpha(1) > 0$, we may assume $\alpha(0) > 0$ and $\alpha(1) > 0$ (the other case, $\alpha(0) < 0$ and $\alpha(1) < 0$, is similar). Since p is a tangential intersection and F is assumed to be below G, we see that $\alpha(t_0^-) > 0$ and $\alpha(t_0^+) < 0$. Then, by continuity of the function $\alpha(t)$, there exists $t_1 \in (t_0, 1)$ such that $\alpha(t_1) = 0$. Thus $(F(t_1), G(s(t_1)))$ is an antipodal pair. Again, we argue that $F(t_1) = G(s(t_1))$ must be a tangential intersection of F and G. This contradicts the Δ-separation property. **Q.E.D.**

Lemma 8 (Half-Couple NIC). *Suppose $(F[0, 1], G[c, d])$ is a half-couple in that $a_F(0)$ passes through $G(c)$ but $a_F(1)$ does not intersect G. Moreover, the end point $G(d)$ lies in $U(F)$, hence above F. Let $b_G(d)$ denote the lower half-normal at $G(d)$. We define $\theta_G(d)$ to be the angle that the upward normal at $G(d)$ makes with the positive x-axis (as usual). We have three possibilities:*
(i) $b_G(d)$ intersects $a_F(0)$.
(ii) $b_G(d)$ intersects $a_F(1)$.
(iii)$_s$ $b_G(d)$ intersects F at $F(t)$, $t \in [0, 1]$. In this case, let $s = \mathtt{sign}(\theta_F(t) - \theta_G(d)) \in \{0, \pm 1\}$.
Choose t_0 such that $a_F(t_0)$ intersects $G(d)$. Then we have

$$\alpha_{F,G}(t_0) \begin{cases} > 0 & \textit{in cases (i) or (iii)}_{+1}, \\ < 0 & \textit{in cases (ii) or (iii)}_{-1}, \\ = 0 & \textit{in case (iii)}_0 \end{cases}$$

Proof. In case (i), as t increases from $t = 0$ to $t = t_0$, the half normal $a_F(t)$ intersects $b_G(d)$ at a unique point that moves continuously until the final intersection point at $G(d)$. The angle formed at the intersection point is $\theta_F(t) - \theta_G(d)$, and this angle maintains its sign because it is never 0. Since the angle is initially positive, the final angle must be positive; but the final angle is of $\alpha_{F,G}(t)$. This proves $\alpha_{F,G}(t) > 0$. The case (ii) is similar. Cases (iii)$_{+1}$ and (iii)$_{-1}$ are (resp.) analogous to (i) and (ii). Finally, in case (iii)$_0$ we clearly have an antipodal pair $(G(d), F(t_0))$. **Q.E.D.**

Lemma 10. *Let $P(F) = (p_0, \ldots, p_m)$ where coordinates of the p_i's are (L, ℓ)-bit floats.*
(i) If $F(t) = (F_x(t), F_y(t))$ is a stationary or x-extreme point, then $F_x(t)$ and $F_y(t)$ has degree $\le m-1$ and 2-norm at most $(4^L 9^m m)^m$.
(ii) If $F(t)$ is an inflexion point, then $F(t)$ has degree $\le 2m-3$ and 2-norm at most $(2m^4 16^L 81^m)^m$.

Proof. (i) If $F(t) = (X, Y)$ is stationary or x-extreme, then X satisfies the equation $R(X) = \mathtt{res}_t(F_x(t) - X, F_x'(t))$. If $F_x(t) = \sum_{i=0}^m a_i t^i$, then $R(X)$ is the determinant of the $(2m-1) \times (2m-1)$ Sylvester matrix M', whose first $m-1$ rows are formed from the coefficients of $F_x(t) - X$, and the last m rows are formed from the coefficients of $F_x'(t)$.

Similar to the proof of Theorem 4, we derive from M' another matrix M_2 which is the Sylvester matrix of the following two sequences:

$$(2^{L+m}\binom{m}{0}, 2^{L+m-1}\binom{m}{1}, \ldots, 2^{L+1}m, 2^L + 1)$$

and

$$(m2^{L+m}\binom{m}{0}, (m-1)2^{L+m-1}\binom{m}{1}, \ldots, 2.2^{L+2}\binom{m}{2}, 2^{L+1}m).$$

The first $m-1$ rows of M_2 have 2-norms $2^L 3^m$ (see full paper). The last m rows of M_2 have 2-norms $\le m2^L 3^m$. Applying the generalized Hadamard bound, $\|R(X)\|$ is $\le (2^L 3^m)^{m-1}(m2^L 3^m)^m$, which is $2^{(2m-1)L} 3^{2m^2-m} m^m$ or $< (4^L 9^m m)^m$ as claimed.
(ii) We proceed as in (i), but the matrix M' now has $2m-3$ rows of the coefficients of $F_x(t) - X$, and has m rows of the coefficients of $F_x'(t)F_y''(t) - F_x''(t)F_y'(t)$. Thus the degree of $R(X) = \det(M')$ is $2m-3$. We have $\|F_x'(t)\| \le m2^L 3^m$, $\|F_y''(t)\| \le m^2 2^L 3^m$. Hence $\|F_x'(t)F_y''(t)\|$ is $\le m(m2^L 3^m)(m^2 2^L 3^m) = m^4 4^L 9^m$ using the fact that $\|AB\| \le m\|A\| \cdot \|B\|$ where A, B have degrees $\le m$. Hence $\|F_x'(t)F_y''(t) - F_x''(t)F_y'(t)\| \le 2m^4 4^L 9^m$. Thus we obtain $\|R(X)\|$ is $\le (2^L 3^m)^{2m-3}(2m^4 4^L 9^m)^m$ which is $\le (2m^4 16^L 81^m)^m$.
Q.E.D.

Corollary 11. *Let p, q be two distinct critical points of F. If $2 \le m \le n$ then either $|p.x - q.x|$ or $|p.y - q.y|$ is larger than*

$$\Delta_4 := (16^{m+2} 256^L 81^{2m} m^5)^{-m}.$$

Proof. By application of the previous lemma. See full paper. **Q.E.D.**

Theorem 12. *Let $q = (q.x, q.y)$ where $q.x, q.y$ are algebraic numbers with 2-norm $\le c$ and degree $\le d$. If $A = 0$ does not contain a circle centered at q then the distance from q to A is at least $\Delta_4(m, a, L, c, d) = (2^{3/2}NK)^{-D} 2^{-12m^2 d^2}$ where*

$$\begin{aligned} K &= \max\{\sqrt{13}, 4ma, c\}, \quad N = \binom{5 + 2m + 2d}{5}, \\ D &= m^2 d^2 (3 + \frac{4}{m} + \frac{4}{d}). \end{aligned}$$

Proof. We may assume p is a closest point on A to q and $h = \|p - q\|$. Thus q, p, h satisfy the system,

$$\begin{aligned} \mathcal{A}_1 &: A(p) = 0, \\ \mathcal{A}_3 &: h^2 - \|p - q\|^2 = 0, \\ \mathcal{A}_4 &: \left\langle (\frac{dA}{dy}(p), -\frac{dA}{dx}(p)), (p - q) \right\rangle = 0, \\ \mathcal{A}_9 &: P(q.x) = 0, \\ \mathcal{A}_{10} &: Q(q.y) = 0 \end{aligned}$$

where $P(X), Q(X)$ are polynomials of degree $\le d$ and 2-norm $\le c$ satisfied by $q.x$ and $q.y$, respectively. This system is zero-dimensional since $A = 0$ does contain a circle centered at q. We have $\|\mathcal{A}_1\|_2 = a$, $\|\mathcal{A}_3\|_2 \le \sqrt{13}$, $\|\mathcal{A}_4\|_2 \le 4ma$, $\|\mathcal{A}_9\|_2 \le c$, and $\|\mathcal{A}_{10}\|_2 \le c$. The result follows by plugging into [24, Theorem 11.45]. **Q.E.D.**

Theorem 13. *Let $diam(F) < \Delta_3(m-1, ma, 2)/m$. Then F contains a stationary point iff the convex hull of $\nabla P(F)$ contains the origin $(0, 0)$.*

Proof. If F contains a stationary point, then clearly its hodograph passes through $(0, 0)$. Hence the convex hull of $m\nabla P(F)$ contains $(0, 0)$. Equivalently, the convex hull of $\nabla P(F)$ contains $(0, 0)$. Conversely, suppose F does not contain a stationary point. Note that F' has degree $m-1$ and $\|F'\|_2 \le m\|F\|_2 = ma$. By Theorem 3, the distance from $(0, 0)$ to the curve F' is at least $\Delta_3(m-1, ma, 2)$ since $(0, 0)$ is a 2-bit float. Thus the distance from $(0, 0)$ to F'/m is at least $\Delta_3(m-1, ma, 2)/m$. On the other hand, the diameter of $\nabla P(F)$ is at most the diameter of F, i.e., less than $\Delta_3(m-1, ma, 2)/m$. This means $(0, 0)$ could not be inside the convex hull of $\nabla P(F)$. **Q.E.D.**

Theorem 14. *Suppose $diam(F) < \Delta^*$. Then F contains an x-extreme point iff $(\nabla p_1).x(\nabla p_m).x \le 0$ and $(\nabla p_1).y(\nabla p_m).y > 0$.*

Proof. If $(\nabla p_1).x(\nabla p_m).x \le 0$ and $(\nabla p_1).y(\nabla p_m).y > 0$ then clearly F contains an x-extreme point. Conversely, suppose $F(t)$ is an x-extreme point, then we know that both $F[0, t]$ and $F[t, 1]$ lies strictly to one side of the vertical line through $F_x(t)$. Since $\text{diam}(F)$ is small, there are no other critical points, not even y-extreme points. Thus, the two subcurves are elementary. Now it is clear that the first and last edges ∇p_1 and ∇p_m on the control polygon $\nabla P(F)$ has the claimed properties. **Q.E.D.**

Theorem 15. *Suppose $diam(F) < \Delta^*$. Then F contains an inflexion point iff*

$$\det(p_0, p_1, p_2)\det(p_{m-2}, p_{m-1}, p_m) < 0.$$

Proof. We make the connection between $\det(p_0, p_1, p_2)$ and inflexion points. Let $H(t) = F_x'(t)F_y''(t) - F_x''(t)F_y'(t)$. Note that $F'(0) = p_1 - p_0$ and $F''(0) = p_2 - 2p_1 + p_0$. Hence $H(0) = \det(p_1 - p_0, p_2 - 2p_1 + p_0) = \det(p_1 - p_0, p_2 - p_1)$. $H(0) = -\det(p_0 - p_1, p_2 - p_1) = \det(p_1, p_0, p_2) = -\det(p_0, p_1, p_2)$. Similarly, $H(1) = -\det(p_{m-2}, p_{m-1}, p_m)$. Therefore, if $\det(p_0, p_1, p_2)\det(p_{m-2}, p_{m-1}, p_m) < 0$, there must be some t such that $H(t) = 0$, and this would be an inflexion point. Conversely, if $F(t)$ is an inflexion point, then $H(t^-)H(t^+) < 0$. Since $\text{diam}(F)$ is small, there are no other inflexion points and therefore we conclude that $H(0)H(1) < 0$. **Q.E.D.**