Effective Subdivision Algorithm for Isolating Zeros of Real Systems of Equations, with Complexity Analysis

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1 INTRODUCTION

Solving multivariate zero-dimensional systems of equations is a fundamental task with many applications. We focus on the problem of isolating simple real zeros of a real function

\[ f = (f_1, \ldots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

within a given bounded box \( B_0 \subseteq \mathbb{R}^n \). We do not require \( f \) to be polynomial, only each \( f_i \) and its partial derivatives have interval forms. We require that \( f \) has only isolated simple zeros in \( 2B_0 \).

We call \( B_0 \) the region-of-interest (ROI) of the input instance. This formulation of root isolation is called a local problem in [14], in contrast to the global problem of isolating all roots of \( f \). The local problem is very important in higher dimensions because the global problem has complexity that is exponential in \( n \). In geometric applications we typically can identify ROI’s and can solve the corresponding local problem much faster than the global problem. Moreover, if \( f \) is not polynomial, the global problem might not be solvable: E.g., \( f = \sin x, n = 1 \). But it is solvable as a local problem as in [28].

In their survey of root finding in polynomial systems, Sherbrooke and Patrikalakis [26] noted 3 main approaches: (1) algebraic techniques, (2) homotopy, (3) subdivision. They objected to the first two approaches on “philosophical grounds”, meaning that it is not easy in these methods to restrict its computation to some ROI \( B_0 \). Of course, one could solve the global problem and discard solutions that do not lie in \( B_0 \). But its complexity would not be a function of the roots in \( 2B_0 \). Such local complexity behavior are provable in the univariate case (e.g., [4]), and we will also show similar local complexity in the algorithm of this paper.

Focusing on the subdivision approach, we distinguish two types of subdivision: algebraic and analytic. In algebraic subdivision, \( f \) is polynomial and one exploits representations of polynomials such as Bernstein form or B-splines [7, 11, 12, 22, 26]. Analytic subdivision [15, 23, 27] supports a broader class of functions; this is formalized in [28] and includes all the functions obtained from composition of standard elementary functions or hypergeometric functions. Many algebraic algorithms comes with complexity analysis, while the analytic algorithms typically lack such analysis, unless one views convergence analysis as a weak form of complexity analysis. This lack is natural because many analytic algorithms are what theoreti
cal computer science call “heuristics” with no output guarantees. Any guarantees would be highly conditional (cf. [27]). To our knowledge, there has been no subdivision algorithm that solves the root isolation problem until the present paper. The subdivision algorithms [7, 11, 12, 22, 26] suffer from two gaps. (1) Non-termination: they require an input \( \varepsilon > 0 \) to serve as termination criterion. (2) Non-isolation: the output box is not guaranteed to be isolating, i.e., to contain a unique root. So an output box could err in one of two ways: it may contain no roots or may have more than one root.

1 Sometimes, an algorithm is called “local” if it works in small enough neighborhoods (like Newton iteration), and “global” if no such restriction is needed. Clearly, this is a different local/global distinction.

2 The issue of “unconditional algorithms” is a difficult one in analytic settings. Even the algorithm in this paper is conditional: we require the zeros of \( f \) to be simple within \( 2B_0 \). But one should certainly specify any conditions upfront and try to avoid conditions which are “algorithm-induced” (see [29]).
root. To avoid the first error, some root existence test is needed: so Garloff and Smith [11, 12] considered the use of Miranda test. To avoid the second error, Elber and Kim [7] introduced a cone test to ensure that there is at most one solution. The cone test generalizes the hodograph test of Sederberg and Meyers (1988); unfortunately this is a nontrivial test and details on how to compute the cones are missing.

1.1 Generic Root Isolation Algorithms

It is useful to formulate a “generic algorithm” for local root isolation (cf. [19]). We postulate 5 abstract modules: three box tests (exclusion $C_0$, existence $EC$, Jacobian $JC$) and two box operators (subdivision and contraction). Our tests (or predicates, which we use interchangeably) is best described using a notation: for any set $B \subseteq \mathbb{R}^n$, $\#(B)$ denotes the number of roots, counted with multiplicity, of $f$ in $B$. These tests are abstractly defined by these implications:

$$
\begin{align*}
C_0(B) &\implies \#(B) = 0, \\
EC(B) &\implies \#(B) \geq 1, \\
JC(B) &\implies \#(B) \leq 1.
\end{align*}
$$

The partial correctness of Simple Isolate is clear, i.e., if it terminates, the output is correct. But termination is a serious issue: clearly it depends on instantiations of the three tests. But independent of the tests, termination can arise in two other ways: (1) Success of contraction ensures a reduction in the width $w(B)$, but this alone may not suffice for termination. (2) Presence of roots on the boundary of a box (e.g., $B_0$). We next discuss the research issues around this framework.

1.2 How to derive effective algorithms

In this paper, we describe Miranda, a subdivision algorithm for root isolation, roughly along the above outline. We forgo the use of the contraction operator as it will not figure in our analysis. For simplicity, assume that all our boxes are hypercubes (equidimensional boxes); this means our subdivision splits each box into $2^n$ children. With a little more effort, our analysis can handle boxes with bounded aspect ratios and thus support the bisection-based algorithms. As noted, termination depends on instantiations of our 3 tests: our exclusion and Jacobian tests are standard in the interval literature. Our existence test, called MK test, is from Moore-Khoussaino (MK) [20]. Our algorithm is similar to one in the Appendix of [18, Appendix]. In the normal manner of theoretical algorithms, one would proceed to “prove that Miranda is correct and analyze its complexity” This will be done, but the way we proceed is aimed at some broader issues discussed next.

**Effectivity:** how could we convert a mathematically precise algorithm (like Miranda) into an “effective algorithm”, i.e., certified and implementable. One might be surprised that there is an issue.

The non-triviality of this question can be illustrated from the history of isolating univariate roots: for about 30 years, it is known that the “benchmark problem” of isolating all the roots of an integer polynomial with $L$-bit coefficients and degree $n$ has bit-complexity $O(n^2L)$, a bound informally described as “near-optimal”. This is achieved by the algorithm of Schönhage and Pan (1981-1992). But this algorithm has never been implemented. What is the barrier? Basically, it is the formidable problem of mapping algorithms in the Real RAM model [2] or BSS model [6] into a bit-based Turing-computable model – see [30].

In contrast, recent progress in subdivision algorithms for univariate roots finally succeeded in achieving comparable complexity bounds of $O(n^2(L+n))$, and such algorithms were implemented

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**Simple Isolate** $(f, B_0)$

Output: sequence of isolating boxes for roots in $B_0$

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3 In [18, Appendix], only termination was proved (up to the interval level). There was no complexity analysis and we will correct an error in a lemma.
shortly after! Thus, these subdivision algorithms were “effective”. For two parallel accounts of this development, see [17, 25] for the case of real roots, and to [4, 5, 14] for complex roots. What is the power conferred by subdivision? We suggest this: the subdivision framework provides a natural way to control the numerical precision necessary to ensure correct operations of the algorithm. Moreover, the typical one-sided tests of subdivision avoid the “Zero Problem” and can be effectively implemented using approximations with suitable rounding modes.

In this paper, we capture this pathway to effectivity by introducing 3 Levels of (algorithmic) Abstractions: (A) Abstract Level, (I) Interval Level, and (E) Effective Level. We normally identify Level (A) with the mathematical description of an algorithm or Real RAM algorithms. We assume our effective algorithms approximate real numbers by BigFloat or dyadic numbers, i.e., \( \mathbb{Z}[\frac{1}{2}] \). As illustration, consider the exclusion test \( C_0(B) \) (viewed as abstract) has correspondences in the next two levels:

(A): \( C_0(B) \Rightarrow 0 \not\in f(B) \)
(I): \( \square C_0(B) \Rightarrow 0 \not\in \square f(B) \)
(E): \( C_0(B) \Rightarrow 0 \not\in \square f(B) \)

where \( f(B) \) is the exact range of \( f \) on \( B \), \( \square f(B) \) is the interval form of \( f \), and \( \square f(B) \) the effective form. The 3 range functions here are related as follows:

\[
f(B) \subseteq \square f(B) \subseteq \mathcal{L} f(B).\tag{3}\]

In general, for any abstract test \( C(B) \), we derive its interval and effective forms to ensure the implications

\[
\square C(B) \Rightarrow \square C(B) \Rightarrow C(B).\tag{4}
\]

This means, the success of \( \square C(B) \) implies the success of \( \square C(B) \), and hence \( C(B) \). An abstract algorithm \( A \) is first mapped into an interval algorithm \( \square A \). But algorithms still involve real numbers. So we must map \( \square A \) to an effective algorithm \( \mathcal{L} A \). Correctness must ultimately be shown at the Effective Level; the standard missing link in numerical (even “certified”) algorithms is that one often stops at Abstract or Interval Levels.

**Complexity:** The complexity of analytic algorithms is often restricted to convergence analysis. But in this paper, we will provide explicit bounds on complexity as a function of the geometry of the roots in \( \mathbb{R}^n \). This complexity can be captured at each of our 3 levels, but we always begin by proving our theorems at the Abstract Level, subsequently transferred to the other levels. Although it is the Effective Level that really matters, it would be a mistake to directly attempt such an analysis at the Effective level: that would obscure the underlying mathematical ideas, incomprehensible and error prone. The 3-level description enforces an orderly introduction of new concerns appropriate to each level. Like structured programming, the design of effective algorithms needs some structure. Currently, outside of the subdivision framework, it is hard to see a similar path way to effectivity.

### 1.3 Literature Survey

There is considerable literature associated with each of our three tests: the exclusion test comes down to bounding range of functions, a central topic in Interval Analysis [24]. The Jacobian test is connected to the question of local injectivity of functions, the Bieberbach conjecture (or de Branges Theorem), Jacobian Conjecture, and theory of univalent functions. In our limited space, we focus on the “star” of our 3 tests, i.e., the existence test. It is the most sophisticated of the 3 tests in the sense that some nontrivial global/topological principle is always involved in existence proofs. In our case, the underlying principle is the fixed point theorem of Brouwer, in the form of Miranda’s Theorem (1940), and intimately related to degree theory.

We compare two box tests \( C \) and \( C’ \) in terms of their relative efficacy: say \( C \) is as efficacious as \( C’ \), written \( C \geq C’ \), if for all \( B \), \( C’(B) \) succeeds implies that \( C(B) \) succeeds. The relative efficacy of several existence tests have been studied [3, 9, 10, 13]. Goldsztejn considers four common existence tests, and argues that “in practice” there is an efficacy hierarchy

\[
(IN) \succeq (HS) \succeq (FLS) \succeq (K)\tag{5}
\]

where \( K \) refers to Krawczyk, (HS) to Hansen-Sengupta, (FLS) to Frommer-Lang-Schurr, and (IN) to Interval-Newton. Note that (K), (HS) and (IN) are all based on Newton-type operators (see (2)). Our Moore-Kioustelidis (MK) test is essentially (FLS). We say “essentially” because the details of defining the tests may vary to render the comparisons invalid. In our MK tests, we evaluate \( f \) on each box face using the Mean Value Form expansion at the center of the face. But the above analysis assumes an expansion is at the center of the box, which is less accurate. But we may also compare these tests in terms of their complexity (measured by the worst case number of arithmetic operations, or number of function evaluations); a complexity-efficacy tradeoff may be expected. Such complexity comparisons do not account for adaptive costs: Newton-type existence tests have non-adaptive costs while the Miranda-type tests are adaptive (we are testing \( n \) pairs of faces, and can break off as soon as one pair fails the test. Finally, evaluating these tests in isolation does not tell us how they might perform in the context of an algorithm. It is therefore premature to decide on the best existence test.

### 1.4 Overview

In section 2, we introduce some basic concepts of interval arithmetic and establish notations. Section 3 introduces the key existence test based on Miranda’s theorem. Section 4 proves conditions that ensure the success of these existence test. Section 5 introduces two Jacobian tests. Section 6 describes our main algorithm. Section 7 is the complexity analysis of our algorithm. We conclude in Section 8. All proofs are relegated to the Appendix.

### 2 INTERVAL FORMS

We first establish notations for standard concepts of interval arithmetic. Bold fonts indicate vector variables: e.g., \( f = (f_1, \ldots, f_n) \) or \( x = (x_1, \ldots, x_n) \).

Let \( \square \mathbb{R} \) denote the set of compact intervals in \( \mathbb{R} \). Extend this to \( \square \mathbb{R}^n \) for the set of compact \( n \)-boxes. In the remaining paper, we assume that all \( n \)-boxes are hypercubes (i.e., the width in each dimension is the same). For any box \( B \in \square \mathbb{R}^n \), let \( m_B = m(B) \) denote its center and \( w_B = w(B) \) be the width of any dimension. Besides boxes, we will also use ball geometry: let \( \Delta = \Delta(a,r) \subseteq \mathbb{R}^n \) denote the closed ball centered at \( a \in \mathbb{R}^n \) of radius \( r > 0 \). If \( r \leq 0 \), \( \Delta(a,r) \) is
just the point $a$. For any positive $k > 0$, let $kA$ and $kB$ denote the dilation of the ball $A$ and box $B$ relative to their centers. Let $A, B \subseteq \mathbb{R}^n$ be two sets. We will quantify their “distance apart” in two ways: their usual Hausdorff distance is denoted $d(A, B)$ and their separation, $\inf \{||a - b|| : a \in A, b \in B\}$ is denoted as $sep(A, B)$. Note that $q$ is a metric on closed subsets of $\mathbb{R}^n$ but $sep(A, B)$ is no metric.

Consider two kinds of extensions of a function $f : \mathbb{R}^n \to \mathbb{R}$. First, the set extension of $f$ refers to the function (still denoted by $f$) that maps $S \subseteq \mathbb{R}^n$ to $f(S) = \{f(x) : x \in S\}$. The second kind of extension is not unique: an interval form of $f$ is any function $\square f : \square \mathbb{R}^n \to \mathbb{R}$, satisfying two properties: (i) inclusion $f(B) \subseteq \square f(B)$; (ii) (convergence) if $p = \lim_{n \to \infty} B_n$ then $f(p) = \lim_{n \to \infty} \square f(B_n)$. For short, we call $\square f$ a box form of $f$.

If $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$, we have corresponding set extension $f(S)$ and interval forms $\square f : \square \mathbb{R}^n \to \mathbb{R}^n$. For any set $S \subseteq \mathbb{R}^n$, let $Zero_f(S)$ denote the multiset of zeros of $f$ in $S$. We assume that $f$ is analytic and its zeros are counted with the proper multiplicity. Then $\#(S)$ is the size of the multiset $Zero_f(S)$. We may write $Zero(S)$ and $\#(S)$ when $f$ is understood.

The notation $\partial f$ is a generic box form; we use subscripts to indicate specific box forms. Thus, the mean value form of $f$ is

$$\partial_m f(B) = f(m(B)) + \nabla f(B)^T \cdot (B - m(B))$$

where $\nabla f$ is the gradient of $f$ (viewed as a column vector) and $\nabla f(B)^T$ is the transpose. The box $B - m(B)$ is now at the origin, i.e., $m(B - m(B)) = 0$. The appearance of the generic $\partial_m f(B)$ in the definition of $\partial_m f$ means that $\partial_m f$ is still not fully specified. In our complexity analysis, we assume that for any box form, if not fully specified, will have at least linear convergence. In this paper, all the box forms used in our predicates will be mean value forms. Next, we intend to convert the interval form $\partial_m f$ to some effective version $\square \partial_m f$. One reason that this is necessary may be seen in the fact that $\square \partial_m f$ assumes an exact value $f(m(B))$. Even if $m(B)$ is a dyadic number, we may need to approximate $f(m(B))$ (e.g., $f(x) = \sin(x)$).

## 3 MIRANDA AND MK TESTS

In the rest of this paper, we fix

$$f := (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n \quad (6)$$

to be a $C^2$-function (twice continuously differentiable), and $f$ and its partial derivatives have interval forms. We further postulate that $f$ has only finitely many simple zeros in any bounded region of interest (this means 2B0 in our algorithms). A zero $a$ of $f$ is simple if the Jacobian matrix $J_f(a)$ is non-singular. For any set $S \subseteq \mathbb{R}^n$, its magnitude is defined as $|S| := \max \{|x| : x \in S\}$.

We consider a classical test from Miranda (1940) to confirm that if a box $B \subseteq \square \mathbb{R}^n$ contains a zero of $f$. If the box $B$ is written as $B = \prod_{i=1}^n l_i$ with $l_i = [a_i^-, a_i^+]$, then it has two $i$-th faces, namely $B_i^- := l_1 \times \cdots \times l_{i-1} \times \{a_i^-\} \times l_{i+1} \times \cdots \times l_n$.

and $B_i^+$, defined similarly. Write $B_i^+$ to mean either $B_i^+$ or $B_i^-$. Consider the following box predicate called the simple Miranda test:

$$\text{MT}_f(B) \equiv \bigwedge_{i=1}^n (f_i(B_i^+) > 0) \land (f_i(B_i^-) < 0) \quad (7)$$

where $f$ is given in (6). The following result is classic:

**Proposition 1. [Miranda (1940)]**

If $\text{MT}_f(B)$ holds then $\#(B) \geq 1$.

For a box $B$ and $k > 0$, let $kB$ denote the box centered at $m(B)$ of width $k \cdot w(B)$, called the $k$-dilation of $B$. Next, we introduce the MK Test test $\text{MK}_k(B) = \text{MK}_f(B)$ that amounts an application of the simple Miranda test to the box $2B$ (instead of $B$), using a preconditioned form of $f$:

**Abstract MK Test**

**Input:** $f$ and box $B$  

**Output:** true iff $\text{MK}_k(B)$ succeeds

1. $C \leftarrow J_f(m(B))$, Jacobian matrix at $m(B)$  
2. If $C^{-1}$ does not exist, return false.  
3. Construct a "preconditioned version" $g$:  
4. Apply the Simple Miranda Test to $2B$ for $g$:  
5. For $i \leftarrow 1, \ldots, n$:  
6. If $g_i(2B_i^+) \leq 0$ or $g_i(2B_i^-) \geq 0$, return false  
7. Return true.

The notation $\text{MT}_f(B)$ in (7) refers to faces of the box $2B$, not the $2$-dilation of the faces of $B$. Here “MK” refers to Moore and Kuestelidaes [20]; the preconditioning idea first appearing in [16]. The MK Test was first introduced in [18].

Note that $\text{MK}_k(B)$ is mathematically exact and generally not implementable (even if it were possible, we may still prefer approximations). We first define its interval form, denoted $\square \text{MK}_k(B)$: simply by replacing $g_i(B_i^+)$ in line (4) by interval forms $\square g_i(B_i^+)$. Finally, we must define the effective form $\square \text{MK}_k(B)$ (Section 8). The key property is the relation (cf. (4)): $\square \text{MK}_k(B) \Rightarrow \square \text{MK}_k(B) \Rightarrow \text{MK}_k(B)$.

## 4 ON SUCCESS OF MK TEST

The success of the MK test implies the existence of roots. In this section, we prove some (quantitative) converses.

We need preliminary facts about mean value forms. Given $x, y \in \mathbb{R}$, the notation $x \pm y$ denotes a number of the form $x + by$, where $0 \leq |b| \leq 1$; thus “$\pm$” hides the implicit 0 in the definition. This notation is not symmetric: $x \pm y$ and $y \pm x$ are generally different. This notation extends to matrices: let $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ be two matrices. Then $A \pm B := (a_{ij} \pm b_{ij})_{i,j=1}^n$. Similarly, for a scalar $\lambda$, we have $A \pm \lambda := (a_{ij} \pm \lambda)_{i,j=1}^n$. Also, let $|x|$ denote the vector $(|x_1|, \ldots, |x_n|)$, for $x, y \in \mathbb{R}^n$, we write $|x, y|$ to denote the line segment connecting $x$ and $y$. We write $|x|$ for $\text{MT}_f(B)$.
and $\|A\|$ for the infinity norms of vector $x$ and matrix $A$. For convex set $C \subseteq \mathbb{R}^n$, define the matrix $K(C)$ with entries $(K(C))_{ij}^{n_{f-1}}$ where

$$K(C)_{ij} := \sum_{k=1}^{n_{f-1}} \left| \frac{\partial^2 f_i}{\partial x_j \partial x_k} (C) \right|.$$  

(8)

Below, $C$ may be a disc $D$ or a line $[x, y]$. Denote by $J_f(x)$ the Jacobian

matrix of $f$ at $x$. We write $J_f(x)$ as $J(x)$ when $f$ is understood.

The following is a simple application of the Mean Value Theorem (MVT):

**Lemma 2 (MVT).** Given two points $x, y \in \mathbb{R}^n$, we have:

(a) $J(x) = \frac{J(y) \pm K([x, y])]|x - y|}$

(b) $J(x) - f(y) = \left( J(y) \pm K([x, y]) \right) \cdot (x - y)$.

4.1 Sure Success of abstract MK Test

In this and the next subsection, we consider boxes that contain a root $\alpha$ of $f$. We prove conditions that ensure the success of the MK Test. We first prove this for the abstract test $\mathbf{MK}(B)$. The next section extends this result to the interval test $\mathbf{MK}(B)$. The key definition here is the bound $\Lambda_j(\alpha)$ which depends on $\alpha$ and $f$. We prove that if $w(B) \leq \Lambda_j(\alpha)$, then the abstract MK Test will succeed on $B$. By a critical point we mean $\alpha \in \mathbb{R}^n$ where the determinant of $J(\alpha)$ is zero. By definition, a root $\alpha$ of $f$ is simple if $\alpha$ is not a critical point.

Suppose $S_1$ and $S_2$ are two sets in $\mathbb{R}^n$. Define

$$\|J^{-1}(S_1)\| := \max_{x \in S_1} \|J^{-1}(x)\| \quad \text{and} \quad \|J^{-1}(S_1) \cdot K(S_2)\| := \max_{x, y \in S_1} \|J^{-1}(x) \cdot K(y)\|.$$

We see that both $\|J^{-1}(S_1)\|$ and $\|J^{-1}(S_1) \cdot K(S_2)\|$ are finite if $S_1$ does not contain a critical point of $f$. Consider the following function

$$s(r) := \frac{1}{18n} \|J^{-1}(\Delta(\alpha, 2\sqrt{n}r)) \cdot K(\Delta(\alpha, 2\sqrt{n}r))\|.$$

(9)

We then define $\lambda_j(\alpha)$ to be the smallest $r$ such that $s(r) = 0$, i.e.,

$$\lambda_j(\alpha) := \arg\min_r \{ s(r) = 0 \}.$$

**Lemma 3.** For any simple root $\alpha$ of $f$, $\lambda_j(\alpha)$ is well-defined.

From now on, let $\Lambda(\alpha)$ denote the dic

$$\Lambda(\alpha) := \Delta(\alpha, 2\sqrt{n}\lambda_j(\alpha)).$$

(10)

The following lemma corrects an gap in the appendix of [18].

**Lemma 4.** Let box $B$ contain a simple root $\alpha$ of $f$. If $w(B) \leq \lambda_1(\alpha)$, the preconditioned system $g_i(B) := J^{-1}(m(B)) f = (g_1, \ldots, g_n)$ is well-defined, and for all $i = 1, \ldots, n$,

$$g_i(2B_1^+) \geq \frac{w(B)}{4}, \quad g_i(2B^+) \leq -\frac{w(B)}{4}.$$

4.2 Sure Success of Interval MK Test

We now extend the previous subsection on the abstract MK Test $\mathbf{MK}(B)$ to the interval version $\mathbf{MK}(B)$. Again, assume $B$ is a box containing exactly one root $\alpha$ of $f$. We will give $\lambda_2(\alpha)$ which is analogous to $\lambda_1(\alpha)$ and prove that if $w(B) \leq \lambda_2(\alpha)$, then $\mathbf{MK}(B)$ will succeed.

To prove the existence of such a $\lambda_2(\alpha)$ as mentioned above, we need to make some assumptions on the property of the box functions. As in [21], a box function $\mathcal{D} f$ is called Lipschitz in a region $S \subseteq \mathbb{R}^n$ if there exists a constant $L$ such that

$$w(\mathcal{D} f(B)) \leq L \cdot w(B), \quad \forall B \subseteq S.$$  

(11)

We call any such $L$ a Lipschitz constant of $\mathcal{D} f$ on $S$. For our theorem, we need to know the specific box function in order to derive a Lipschitz constant. Consider the mean value form $\mathcal{D} f(B)$ on a region $S \subseteq \mathbb{R}^n$.

**Lemma 5.** Let $f$ be a continuously differentiable function defined on a convex region $S \subseteq \mathbb{R}^n$. Then a Lipschitz constant for $\mathcal{D} f(B)$ is

$$\sum_{k=1}^{n} \| \frac{\partial_f(B^+)}{\partial x_k} (S) \|.$$

Consider the sign tests of $\mathbf{MK}(B)$:

$$\lambda_i(2B_1^+) > 0 \quad \text{and} \quad \lambda_i(2B^+) < 0.$$

(12)

where $g_i$ is the $i$-th component of the system $f(m(B))$. We consider the mean value form $\mathcal{D} f(B)$.

**Theorem 7.** Let $B$ be a box containing a simple root $\alpha$ of $f$. If $w(B) \leq \lambda_1(\alpha)$, the preconditioned system $g_i(B) := J^{-1}(m(B)) f$ is well-defined and $\mathbf{MK}(B)$ will succeed.

5 TWO JACOBIAN CONDITIONS

We define the Jacobian test as follows:

$$\mathcal{JC}(B) \equiv \forall i \notin \det(J_f(3B)).$$

(14)

The order of operations in $\det(J_f(3B))$ should be clearly understood: first we compute the interval Jacobian matrix $J_f(3B)$, i.e., entries in this matrix are the intervals $\partial_x f_i(3B)$. Then we compute the determinant of the interval matrix. Also note that we use $3B$ instead of $B$. The following is well-known in interval computation (see [1, Corollary to Theorem 12.1]):

**Proposition 8.** [Jacobian test]

If $\mathcal{JC}(B)$ holds then $\# J(3B) \leq 1$.  

$\Box$
ABSTRACT Miranda($f, B_0$)
OUTPUT: Queue $P$ of non-overlapping isolating boxes of $f$ s.t.
$\bigcup_{B \in P} Z_f(B) \subseteq Z_f(2B_0)$
1. Initialize output queue $P \leftarrow \emptyset$ and priority queue $Q \leftarrow \{B_0\}$.
2. While $Q \neq \emptyset$ do:
3. Remove a biggest box $B$ from $Q$.
4. If $C_0(B)$ succeeds, continue;
5. If $JC(B)$ succeeds then
6. Initialize new queue $Q' \leftarrow \{B\}$.
7. While $Q' \neq \emptyset$ do:
8. $B' \leftarrow Q'.pop()$.
9. If $(B' = B) \land C_0(B')$ fails then
10. If $MK(B')$ succeeds then
11. $P.add(2B')$.
12. Discard from $Q$ the boxes contained in $3B$.
14. $Q.push(subdivide(B))$.
15. Else
16. $Q.push(subdivide(B))$.

Figure 1: Root Isolation Algorithm

We next introduce the following strict Jacobian test:

$$JC_0(B) \equiv 0 \neq (\det J_f)(3B)$$  \hspace{1cm} (15)

where $(\det J_f)(x)$ denotes expression obtained by evaluating the
determinant of the Jacobian matrix $J_f(x)$ with functional entries
$d_{ki}f_j(x)$. Finally, we evaluate $(\det J_f)(x)$ on $3B$. Note that $JC(B) \Rightarrow$ $JC_0(B)$ and so the strict test is more efficacious. Unfortunately
$JC_0(B)$ cannot be used by our algorithm since it is known that
$JC_0(B)$ does not imply $(3B) \neq 1$. Nevertheless, we now show that
it can serve as a uniqueness test in conjunction with the MK test:

Theorem 9.
If both $JC_0(B)$ and $MK(\frac{1}{2}B)$ succeed then $(3B) = 1$.

It follows that we could use $JC_0(B) \land MK(B)$ in our SIMPLE ISOLATE
algorithm in the introduction.

6 THE MIRANDA ALGORITHM

Our main algorithm for root isolation is given in Figure 1. We use
$MK(B)$ and $JC(B)$ (respectively) for its existence and Jacobian tests.
It remains to specify the exclusion test $C_0(B)$:

$$C_0(B) \equiv (\exists i = 1, \ldots, n)[0 \neq f_i(B)]$$  \hspace{1cm} (16)

The algorithm in Figure 1 is abstract. To introduce the interval
version $\overline{\text{Miranda}}$, just replace the abstract tests by their interval
analogues: $\overline{\text{MK}}(B)$, $\overline{\text{C}}_0(B)$ and $\overline{\text{JC}}(B)$. In amounts to replacing
the set theoretic function in the abstract definition by their interval
analogues:

- $\overline{\text{C}}_0(B)$: $\exists i = 1, \ldots, n$ such that $0 \neq \overline{\text{f}}_i(B)$;
- $\overline{\text{JC}}(B)$: $0 \neq \overline{\text{det}}(f)(3B)$;
- In the definition of $\overline{\text{MK}}(B)$ (Section 3), replace each $g_i(2B^+)$
by $\overline{g_i}(2B^+)$.

Note that all these box forms are really mean value forms $\overline{\text{M}}$. For
the effective version, we use the tests $\overline{\text{MK}}(B)$, $\overline{\text{C}}_0(B)$ and $\overline{\text{JC}}(B)$,
which is discussed in Section 8.

Termination of each version of Miranda follows from the com-
plexity analysis below. Even if there are roots on the boundary of
$B_0$, we will terminate, although the isolated root might lie in
$2B_0 \setminus B_0$. But we first show that the output is correct when Miranda
halts:

Theorem 10 (Partial Correctness).
1. If $\text{Miranda}$ halts, the output queue $P$ is correct.
2. The same holds for $\overline{\text{Miranda}}$ and $\overline{\text{Miranda}}$.

7 COMPLEXITY UPPER BOUNDS

In this section, we derive a lower bound $\lambda > 0$ on the size of boxes
produced by Miranda. That is, any box $B$ with width $w(B) \leq \lambda$
would either be output or rejected. This implies that the subdivision
process is no deeper than $\log_2(w(B_0)/\lambda)$, yielding an upper bound on
computational complexity. This bound $\lambda$ will be expressed in terms
of quantities determined by the zeros in $2B_0$. We first prove this
for the abstract Miranda and $\overline{\text{Miranda}}$. From the algorithm, we see that a box $B$ is output if
$\neg C_0(B) \land \neg JC(B) \land \neg MK(B)$ holds in line 10; it is rejected if one of the 2
following cases is true: (1) $C_0(B)$ holds or (2) it is contained in $3B'$
where $JC(B')$ holds and a box in $B'$ is output, as indicated in line
12. The boxes that contain a root of $f$ will be finally verified by the
former predicate and the boxes that contain no root of $f$ will
eventually be rejected in one of the 2 cases.

To prove the existence of such a $\lambda$, we need to look into the tests
$C_0(B)$, $JC(B)$ and $MK(B)$. We will give bounds $\lambda_{JC}, \lambda_{MK}$ and $\lambda_{C_0}$
for the 3 tests respectively and show that for any box $B$ produced in
the algorithm

(1) if $\#(B) > 0$, it will pass $MK(B)$ when $w_B \leq \lambda_{MK}$,
(2) if $\#(B) > 0$, it will pass $JC(B)$ when $w_B \leq \lambda_{JC}$,
(3) if $\#(B) = 0$ and $B$ keeps a certain distance from the roots, it will
pass $C_0(B)$ when $w_B \leq \lambda_{C_0}$.

We have essentially proved item (1) in the Section 4. More precisely,
for each root $\alpha$, we had defined a constant $\lambda_2(\alpha)$. We now set

$$\lambda_{MK} := \min_{\alpha \in ZerO(2B_0)} \lambda_2(\alpha).$$  \hspace{1cm} (17)

7.1 Sure Success for $C_0(B)$ and $JC(B)$

We study conditions to ensure the success of the tests $JC$ and $C_0$.
We will introduce constants $\lambda_{JC}, \lambda_{C_0}$ in analogy to (17).

First consider $JC(B)$. Let box $B$ contain a simple root $\alpha$. By Mean
Value Theorem, $w(\frac{df_j}{dx_j}(3B)) \leq 3w_B \cdot K(3B)\{\{i\}$ (see (8) for defini-
tion). Since $\frac{df_i}{dx_j}(\alpha) \in \overline{\text{MK}}(3B)$, it holds $\frac{df_i}{dx_j}(3B) \leq \frac{3}{\lambda_{MK}}$ $(\alpha) -
3w_B \cdot K(3B)\{\{i\}$. (8) (i, j = 1, . . . , n). Denoting
$U(\alpha) = \max_{1 \leq i \leq n} |\frac{df_i}{dx_j}(\alpha)|$ and $V := \max_{1 \leq i \leq n} |\text{K}(3B)\{\{i\}|$, we get $|\frac{df_i}{dx_j}(3B)| \leq U(\alpha) + 3w_B$ and $w(\frac{df_i}{dx_j}(3B)) \leq 3Vw_B$. By
applying the rules $w(l_1 + l_2) = w(l_1) + w(l_2)$ and $w(l_1 \cdot l_2) \leq
w(l_1) \cdot |l_2| + w(l_2) \cdot |l_1|$ where $l_1, l_2$ are intervals, we may verify
by induction that $w(\prod_{i=1}^n (\frac{df_i}{dx_j}(3B)) \leq 3nV(U(\alpha) + 3w_BV)^{n-1}w_B$
for any permutation $\sigma$. Hence, it follows $w(\det(f_{\sigma}(3B))) \leq 3n \cdot n! \cdot V(U(\alpha) + 3Vw_B)^n - 1 \cdot w_B$.

Set $\lambda_{f}(\alpha)$ to be the smallest positive root of the equation

$$| \det(f_{\sigma}(\alpha)) | = 3n \cdot n! \cdot V(U(\alpha) + 3Vx)^n - 1 \cdot x = 0. \quad (18)$$

The following lemma implies the existence of $\lambda_{JC}$:

**Lemma 11.** If box $B$ contains a simple root $\alpha$ and $w_B < \lambda_{f}(\alpha)$ then $JC(B)$ succeeds.

Thus we may choose $\lambda_{JC} := \min_{\alpha \in \text{ZerO}(2B_0)} \lambda_{f}(\alpha)$ and set

$$\ell_1 := \min \{ \lambda_{JC}, \lambda_{MK} \}.$$  

**Lemma 12 (Lemma A).** If $\#(B) > 0$ and $w_B \leq \ell_1$ then $MK(B)$ and $JC(B)$ holds.

**Corollary 13.** Each root in $B_0$ will be output in a box of width $> 3\ell_1/2$.

Let $R_0 \subseteq 2B_0$ be a region that excludes discs around roots:

$$R_0 := 2B_0 \setminus \bigcup_{\alpha \in \text{ZerO}(2B_0)} \Delta_{\alpha, \ell_1^*},$$

where $\Delta$ is the interior of $\Lambda$. Denote the zero set of $f_i$ as $S_i$ for $i = 1, \ldots, n$ and define $d_0 := \inf_{p \in R_0} \max_{i=1}^n \text{sep}(p, S_i)$. Since all the roots in $2B_0$ are removed from the set $R_0$, we can verify that $\max_{i=1}^n \text{sep}(p, S_i) > 0$ for all $p \in R_0$. Combining with the compactness of $R_0$, we obtain $d_0 > 0$. Finally we set

$$\lambda_{C_{0}} := \frac{d_0}{2\sqrt{n}}.$$  

**Lemma 14 (Lemma B).** Suppose $\#(B) = 0$ with $\text{sep}(m_B, \text{ZerO}(2B_0)) \geq \ell_1$, if $w_B \leq \lambda_{C_{0}}$ then $C_0(B)$ holds.

**Lemma 15 (Lemma C).** Every box produced by the Miranda has width $\geq \frac{1}{4} \min \{ \lambda_{C_{0}}, \lambda_{JC}, \lambda_{MK} \}$.

### 7.2 Sure Success for $C_0(B)$ and $JC(B)$

We now consider the interval tests $JC(B)$ and $C_0(B)$ under the assumption that the underlying interval forms are Lipschitz.

Let $\tilde{L}$ be a global Lipschitz constant for $f_{\ell}$ and $\frac{\partial f_{\ell}}{\partial x}$ for all $i, j = 1, \ldots, n$ in $3B_0$. Then we will develop corresponding bounds $\lambda_{JC}$, $\lambda_{MK}$, $\lambda_{C_{0}}$ and $\lambda_{C_{1}}$:

Observe that if we replace the bounds $\lambda_{MK}, \lambda_{JC}, \lambda_{C_{0}}$, in the abstract version by the bounds $\lambda_{JC}$, $\lambda_{MK}$, $\lambda_{C_{0}}$, all the statements and proofs in the previous section remain valid. So in this section, we do not repeat the statements, except to give the bounds $\lambda_{JC}$ and $\lambda_{C_{0}}$.

First look at the test $JC(B)$. With the same arguments as in abstract level, we obtain

$$\lambda_{JC} := \min_{\alpha \in \text{ZerO}(2B_0)} \lambda_{f}(\alpha)$$

where $\lambda_{f}(\alpha)$ is the smallest positive root of the equation

$$| \det(f_{\sigma}(\alpha)) | = 3n \cdot n! \cdot \tilde{L}(U(\alpha) + 3\tilde{L}x)^n - 1 \cdot x = 0. \quad (19)$$

With $\lambda_{JC}$ and $\lambda_{MK}$, we have an interval analogue of Lemma A:

**Lemma 16 (Lemma A).** If $\#(B) > 0$ and $w_B \leq \ell_{1}'$ with

$$\ell_{1}' := \min \{ \lambda_{JC}, \lambda_{MK} \},$$

then $MK(B)$ and $JC(B)$ succeeds.

Next look at the test $C_0(B)$. Arguing as in the abstract level, we only consider the boxes in the region $R_0' := 2B_0 \cup \{ \alpha \in \text{ZerO}(2B_0) \} \setminus \Delta(\alpha, \ell_{1}')$ with $\ell_{1}' := \min \{ \lambda_{JC}, \lambda_{MK} \}$. Define $u := \inf_{p \in R_0'} \max_{i=1}^n \frac{|f_i(p)|}{L}$. It is easy to see that $\max_{i=1}^n \frac{|f_i(p)|}{L} > 0$ for any $p \in R_0'$. Since the function $f_{\ell}(x)$ is continuous and the set $R_0'$ is compact, we obtain $u > 0$. Setting $\lambda_{C_{0}} := \frac{u}{2}$, we have the following lemma:

**Lemma 17 (Lemma B).** Let $\text{sep}(m_B, \text{ZerO}(2B_0)) > \ell_{1}'$ with $\ell_{1}' := \min \{ \lambda_{JC}, \lambda_{MK} \}$. If $\#(B) > 0$ and $w_B \leq \lambda_{C_{0}}$, then $C_0(B)$ succeeds.

Combining Lemma A and Lemma B, we obtain:

**Lemma 18 (Lemma C).** Every box produced by the Miranda has width $\geq \frac{1}{4} \min \{ \lambda_{C_{0}}, \lambda_{JC}, \lambda_{MK} \}$.

### 8 EFFECTIVE MIRANDA

We now extend our results from Miranda to Miranda by introducing the effective tests $\text{M}(B)$, $JC(B)$ and $C_0(B)$. Inside these tests are various box forms, say $\text{M}(B)$. Recall that they are actually mean value forms $\text{M}(B)$ (we write $\text{M}(B)$ for simplicity). We convert each $\text{M}(B)$ to its effective version $\text{M}(B)$, whose output interval has dyadic endpoints and which satisfies $\text{M}(B) \subseteq \text{M}(B)$. The main issue is the accuracy of the effective forms, which we express by upper bounds on the Hausdorff distance $g(\text{M}(B), \text{M}(B))$. It is always bounded as a linear function of the width $w_B$, i.e., $g(\text{M}(B), \text{M}(B)) = O(w_B)$. However, we cannot stop here – the implicit constant in the asymptotic notation must be made explicit for implementation purposes.

Specifically, in $C_0(B)$, we require $g(\text{M}(f_{i}(B)), \text{M}(f_{i}(B))) \leq \frac{1}{16} w_B$ for each $i = 1, \ldots, n$. In $\text{M}(B)$, we require $g(\text{M}(f_{i}(2B_B^+) \cup \text{M}(f_{i}(2B_B^-))) \leq \frac{1}{16} w_B$. In $JC(B)$ we require that $g(\text{M}(f_{i}(3B)), \text{M}(f_{i}(3B))) \leq \frac{1}{16} w_B$ for each entry $\text{M}(f_{i}(3B))$. We get effective versions of all our lemmas and theorems, with modified constants such as $\lambda_{JC}$ and $\lambda_{C_{0}}$.

### 9 CONCLUSION

We have provided the first effective subdivision algorithm Miranda for isolating simple real roots of a system of equations $f = 0$, provided $f$ and its derivatives have interval forms. Our result are novel for its completeness (previous algorithms need $\varepsilon$-termination and have no isolation guarantees), its generality (going beyond the polynomial case), and its complexity analysis (going beyond termination proofs). We also contributed to the theory of subdivision algorithms by formalizing a 3-level description to provide a pathway from abstract algorithms to effective ones. Given that many existing numerical algorithms still lack effective versions, this is a promising line of work. In the future, we plan to implement and develop our algorithm into a practical tool.

**REFERENCES**


A APPENDIX: ALL PROOFS

**Lemma 2 (MVT).** Given two points \( x, y \in \mathbb{R}^n \), we have:

\[
|f(x) - f(y)| = |K([x, y])||x - y|
\]

Proof. (a) We apply the Mean Value Theorem to each entry \( f_{ij} = \frac{\partial f}{\partial x_j} \).

\[
J_{ij}(x) = J_{ij}(y) + \nabla J_{ij}(\tilde{y}) \cdot (x - y)
\]

with \( \tilde{y} \in [x, y] \)

\[
= J_{ij}(y) + K([x, y])||x - y||
\]

(b) We apply the Mean Value Theorem twice. The first application gives:

\[
f_i(x) - f_i(y) = \nabla f_i(\tilde{y}) \cdot (x - y)
\]

\[
= J_i(y) = K([x, y])||x - y||
\]

where \( \tilde{y} \in [x, y] \) and \( J_i = \frac{\partial f_i}{\partial x_i} \). Applying the Mean Value Theorem again to each \( J_{ij}(\tilde{y}) \):

\[
J_{ij}(\tilde{y}) = J_{ij}(y) + \nabla J_{ij}(\tilde{y}) \cdot (y - \tilde{y})
\]

with \( \tilde{y} \in [y, y] \)

\[
= J_{ij}(y) + K([y, y])||x - y||
\]

Hence

\[
f_i(x) - f_i(y) = (J_i(1) + K([x, y]))||x - y||, \ldots ,
\]

\[
J_n(y) = K([x, y])||x - y|| \cdot (x - y)
\]

for \( i = 1, \ldots , n \). This proves (21).

Q.E.D.

**Lemma 3.** For any simple root \( \alpha \) of \( f \), \( \lambda_1(\alpha) \) is well-defined.

Proof. Note that \( s(0) \) is well-defined since \( \alpha \) is a simple root. We also deduce that \( s(0) < 0 \) and that \( s(r) = s(0) \) for all \( r < 0 \). Thus \( \lambda_1(\alpha) > 0 \) if it is well-defined. Let \( r^* \) be the smallest radius such that \( \lambda(\alpha, r^*) \) contains a critical point: if \( f \) has no critical point, then \( r^* \) is defined to be infinity. It follows that \( r^* = r^* + \frac{1}{2\pi} = r^* \). Thus \( s(0) < s(r^*) \). From the fact that \( \|f^{-1}(\lambda(\alpha, 2\pi r^*)))K(\lambda(\alpha, 2\pi r^*))\| \) is a continuous non-decreasing function of \( r \) in the range \([0, r^*) \), we conclude that there exists some \( r \in (0, r^*) \) such that \( s(r) = 0 \).

Q.E.D.

**Lemma 4.** Let box \( B \) contain a simple root \( \alpha \) of \( f \).

If \( |B| \leq \lambda_1(\alpha) \), the preconditioned system \( g_B := J^{-1}(m(B))f \) = \( (g_1, \ldots , g_n) \) is well-defined, and for all \( i = 1, \ldots , n \),

\[
g_i(B^*_B) \geq \frac{B^*_B}{4} \geq g_i(B^*_B) \leq -\frac{B^*_B}{4}
\]

Proof. For simplicity, we write \( m(B) \) as \( m \). From the definition of \( \lambda_1(\alpha) \) and the fact that \( B \) contains \( \alpha \) we know that \( J^{-1}(\alpha) \) is well-defined.
Let $x$ be a point on the boundary of the box $2B$. Then

$$g_j(x) = J^{-1}(m)(x) \quad \text{(by definition of } g_B).$$

$$= J^{-1}(m)f(x) + (J_x + K(x, \alpha)) \cdot (x - \alpha) \quad \text{(by MVT (21))}.$$

$$= J^{-1}(m)f(x) + K(x, \alpha) \cdot ((x - \alpha) - (x - \alpha)) \quad \text{(since } \alpha \text{ is a root)}.$$

$$= J^{-1}(m)f(x) + K(x, \alpha) \cdot ((x - \alpha) - (x - \alpha)) \quad \text{(by MVT (20))}.$$

$$= J^{-1}(m)f(x) + 2K(2B) |x - \alpha| \cdot (x - \alpha) \quad \text{(since } |x - \alpha| \leq |\alpha - x|).$$

$$= (1 \pm J^{-1}(m)(K)(2B) |x - \alpha|) \cdot (x - \alpha) \quad \text{(1 is the identity matrix).}$$

The $i$-th component in $g_B(x)$ is the $g_i$; thus

$$g_i(x) = (x_i - \alpha_i) \pm 2(J^{-1}(m)(K)(2B) |x - \alpha|) \cdot (x - \alpha).$$

In the following, we write $\lambda_1$ for $\lambda_1(\alpha)$ and note that $\alpha \in B$ and $\omega_B \leq \lambda_1$ implies

$$m \in \Delta_\alpha,$$

then

$$\left| g_i(x) - (x_i - \alpha_i) \right| \leq 2 \|J^{-1}(m)(K)(2B)\| \cdot \|x - \alpha\| \left( \sum_{j=1}^{n} |x_j - \alpha_j| \right),$$

$$\leq 3n \omega_B \|J^{-1}(m)(K)(2B)\| \cdot \|x - \alpha\| \left( \sum_{j=1}^{n} |x_j - \alpha_j| \right),$$

$$\leq \frac{3}{4}n \omega_B \left( 18n \|J^{-1}(m)(K)(2B)\| \right) \leq \frac{3}{4}n \omega_B \left( 18n \omega_B \right),$$

$$\leq \frac{3}{4}n \omega_B \left( 1 \right) \quad \text{(definition of } \lambda_1),$$

$$\leq \frac{3}{4}n \omega_B \left( 1 \right) \quad \text{(since } \omega_B \leq \lambda_1).$$

This last inequality gives

$$\left| g_i(x) - (x_i - \alpha_i) \right| \leq \omega_B \cdot 2.$$ 

It remains to show that $g_i(2B^+ \cap x \in B \text{ and } \omega_B \leq \lambda_1(\alpha)$). This amounts to proving $g_i(x) \geq \frac{\omega_B}{2}$ holds for all $x \in B^+$; first we note that

$$x_i - \alpha_i \geq \omega_B / 2 \quad \text{(44)}$$

since $x \in B^+$ and $\alpha \in B$. The inequalities (23) and (44) together imply $g_i(x)$ and $x_i - \alpha_i$ must have the same sign. Since $x_i - \alpha_i$ is positive, we conclude that $g_i(x)$ must be positive. Combined with (23) and (44), we conclude that $g_i(x) \geq \omega_B / 4$, as claimed. \textbf{Q.E.D.}

\textbf{THEOREM 7.} Let $B$ be a box containing a simple root $\alpha$ of width $\omega_B \leq \lambda_1(\alpha)$.

(a) If $\omega(B) \leq \frac{1}{2\pi n}$ for each $j = 1, \ldots, n$, then $g_B := J^{-1}(m)(B)$ is well-defined and $B^\alpha(\alpha)$ will succeed.

(b) If $\omega_B \leq \lambda_1(\alpha)$ with $\lambda(\alpha) := \min \{ \lambda_1(\alpha), \lambda_1(\alpha) \}$, then $B^\alpha(\alpha)$ will succeed.

\textbf{Proof.} (a) We show the first part of the theorem. In Lemma 4, it is proven that when $\omega_B \leq \lambda_1(\alpha)$, the system $g_B$ is well-defined and it holds that $g_i(2B^+) \geq \frac{\omega_B}{2}$ and $g_i(2B^-) \leq -\frac{\omega_B}{2}$. From Proposition 6, we have

$$q(B, g_i(2B^+), g_i(2B^-)) \leq 2 \omega(2B) \sum_{j=1}^{n} w_j \left( \frac{\omega(B)}{2} \right),$$

$$\leq 4n \omega_B \cdot \max_{1 \leq j \leq n} w_j \left( \frac{\omega(B)}{2} \right).$$

By the convergence property of box functions, $w(\omega(B))$ approaches 0 when $\omega_B$ approaches 0 for $j = 1, \ldots, n$. Thus when $\omega_B$ is small enough, we have $w(\omega(B)) \leq \left( \frac{1}{2\pi n} \right)^{18n}$. Then

$$\omega_B \leq \omega(B) \leq \frac{\omega_B}{2},$$

which gives the first part of the theorem. \textbf{Q.E.D.}

(b) Now we prove the second part of the theorem. From the proof of the first part, it suffices to prove that when $\omega_B \leq \lambda_1(\alpha)$, the inequality $w(\omega(B)) \leq \left( \frac{1}{2\pi n} \right)^{18n}$ holds for all $i, j = 1, \ldots, n$. To show this, we observe that

$$w(\omega(B)) \leq \left( \frac{1}{2\pi n} \right)^{18n} \leq \left( \frac{1}{2\pi n} \right)^{18n} \leq \left( \frac{1}{2\pi n} \right)^{18n},$$

which gives the second part of the theorem. \textbf{Q.E.D.}

\textbf{LEMMA 5.} Let $f$ be a continuously differentiable function defined on a convex region $S \subseteq \mathbb{R}^n$. Then a Lipschitz constant for $\nabla f$ on $S$ is

$$\sum_{k=1}^{n} \left| \frac{df}{dx_k}(S) \right|. \quad \textbf{Q.E.D.}$$

Proof. Recall that $\nabla f(B) = f(m(B)) + \nabla f(B)^T \cdot (B - m(B)) = f(m(B)) + \frac{1}{2} \omega_B \cdot \sum_{k=1}^{n} \frac{df}{dx_k}(B) \cdot (B - m(B)) = \frac{1}{2} \omega_B \cdot \sum_{k=1}^{n} \frac{df}{dx_k}(B) \leq \omega_B \cdot \sum_{k=1}^{n} \frac{df}{dx_k}(B) \leq \omega_B \cdot \sum_{k=1}^{n} \frac{df}{dx_k}(S).$$

The lemma follows. \textbf{Q.E.D.}

\textbf{LEMMA 6.} If both $J(C,B)$ and $B^\alpha(\alpha)$ succeed then $\nabla f(3B) = 1$.

\textbf{Proof.} From [8], the success of $\nabla f(3B)$ implies

$$\sum_{y \in \mathbb{R}^n} \text{sign}(\det J_f(y)) = \pm 1$$

where $\text{sign}(\det J_f(y))$ is the sign of $\det J_f(y)$. By the success of $J(C,B)$, we further know that $\text{sign}(\det J_f(y))$ is the same for all $y \in 3B$. Thus there is only one root in $3B$. \textbf{Q.E.D.}
The second containment (**) is immediate because all our output boxes have the form $B$ where $B$ is an aligned box. Such boxes are contained in $2B_0$. To show (**), it suffices to prove that if $B'$ is a discarded box, then either $B'$ has no roots, or any root in $B'$ is already output. From the algorithm, a box $B'$ is discarded in two lines: The first is Line 4, when $C_0(B')$ succeeds. But this implies $B$ has no roots. The second is Line 12 of the algorithm. Since $B'$ contains in $3B$ (where $2B$ is the output), we know that $JC(B)$ holds, and thus there is at most one root in $3B$. So if $B'$ contains any root, it must be the root already identified by $2B$. Thus all discarded boxes are justified.

Q.E.D.

**Lemma 11.** If box $B$ contains a simple root $\alpha$ and $w_B < \lambda_3(\alpha)$ then $JC(B)$ succeeds.

**Proof.** The fact $\alpha \in B$ implies $f(\alpha) \in f(3B)$. Since $\alpha$ is a simple root, we have $\det(f(\alpha)) \neq 0$, and thus $\lambda_3(\alpha) \neq 0$. From the definition of $\lambda_3(\alpha)$, we know that if $w_B < \lambda_3(\alpha)$, then $3n \cdot 1! \cdot V(U(\alpha) + 3Vw_B)^{n-1} \cdot w_B < |\det(f(\alpha))|$, and thus $\det(f(3B)) < |\det(f(\alpha))|$. It follows $0 \not\in \det(f(3B))$. The test $JC(B)$ succeeds. Q.E.D.

**Lemma 14 (Lemma B).** Suppose $\#(B) \equiv 0$ with $\text{sep}(m_B, \text{Zero}(2B_0)) \geq \ell_1$, if $w_B \leq \lambda_{c_0}$ then $C_0(B)$ holds.

**Proof.** We note that each output box in $\text{Miranda}$ follows from the partial correctness of $\text{Miranda}$ by the general observation\(^5\) that the predicates in $\text{Miranda}$ are one-sided, and (as can be verified below) none of our arguments are predicated upon the failure of the tests. We need to further note that for the effective version, we must assume that the ROI $B_0$ is a dyadic box, so that all subdivisions are done without approximation.

Hence it remains to prove the partial correctness of $\text{Miranda}$:

(1) We note that each output box in $P$ is isolating. A box $2B$ is output in line 11 upon passing $MK(B)$. This is inside the inner while loop for subboxes of some $B$ which passes $JC(B')$. But $MK(B)$ implies $\#(2B) \geq 1$ and $JC(2B')$ implies $\#(3B) \leq 1$. Thus $\#(2B) = 1$.

(2) Next we claim no root is output twice in $P$. This follows by showing that if $2B$ and $2B'$ are output, then their interiors are disjoint. It does not matter if the boundaries of $2B$ and $2B'$ intersect because there are no roots on their boundary – this is ensured by the success of the Simple Miranda test on these output boxes. The reason for our concern comes from the fact that, although the boxes in $Q$ have pairwise disjoint interiors, each $B$ in $Q$ can cause a larger box ($2B$) to be output.

CLAIM: Suppose $2B$ is output in line 11. Then immediately after line 12, every box $B'$ in $Q$, the interior of $2B'$ is disjoint from $2B$. Pf: Suppose the interior of $2B'$ intersects $2B$. By the priority queue property, we have $w(B') \leq w(B)$. It follows that $B'$ actually is contained in the annulus $3B' \setminus B$. This follows from two facts about aligned boxes: (a) any two aligned boxes have disjoint interiors or have a containment relationship, and (b) $3B' \setminus B$ is a union of 8 aligned boxes. If $B'$ is contained in this annulus, then line 12 would have removed it. This proves our claim.

(3) We must show that

$$Z_f(B_0) \cap \bigcup_{B \in P} Z_f(B) \subseteq Z_f(2B_0).$$

The second containment (**) is immediate because all our output boxes have the form $2B$ where $B$ is an aligned box. Such boxes are contained in $2B_0$. To show (**), it suffices to prove that if $B'$ is a discarded box, then either $B'$ has no roots, or any root in $B'$ is already output. From the algorithm, a box $B'$ is discarded in two lines: The first is Line 4, when $C_0(B')$ succeeds. But this implies $B$ has no roots. The second is Line 12 of the algorithm. Since $B'$ contains in $3B$ (where $2B$ is the output), we know that $JC(B)$ holds, and thus there is at most one root in $3B$. So if $B'$ contains any root, it must be the root already identified by $2B$. Thus all discarded boxes are justified.

**Q.E.D.**

\(^5\) If we were proving termination, the reverse implication hold: if $\text{Miranda}$ terminates than $\text{Miranda}$ terminates.

\(^6\) Here aligned boxes means those that can arise by repeated subdivision of $B_0$. Clearly $B$ and $B'$ are aligned, but $kB$ and $kB'$ are not aligned for any $k > 1$. 