Almost Tight Recursion Tree Bounds for the Descartes Method

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The Descartes Method

What is the Descartes Method?

Real root isolation by recursive interval bisection
using Descartes' Rule of Signs to test for roots.

What makes the Descartes Method interesting?

- It performs very well in practice.
- It is simple to implement.
- It is used a lot.

The Descartes Test for roots in an interval

Descartes Test (classical form) [Jacobi, 1835]

Consider the real polynomial $A(X)$ and an interval (c,d).
Let $A^\star(X) = \sum_{i=0}^{n} a_i^\star X^i = A((cX+d)/(X+1)) \cdot (X+1)^n$
and define

$$DescartesTest(A,\,(c,d)) := \mathrm{var}(a_0^\star,\ldots,a_n^\star).$$

Descartes Test (Bernstein form) [Pólya/Schoenberg, 1958]

Let $A(X) = \sum_{i=0}^{n} b_i B_i^n(X)$, where $B_i^n(X) = \binom{n}{i}\frac{(X-c)^i(d-X)^{n-i}}{(d-c)^n}$.
Then

$$DescartesTest(A,\,(c,d)) = \mathrm{var}(b_0,\ldots,b_n).$$

The Descartes Test for roots in an interval

Properties

Let $v = \mathit{DescartesTest}(A, (c,d))$.

- If $v = 0$, then $A(X)$ has no roots in (c,d).
- If $v = 1$, then $A(X)$ has exactly one root in (c,d), which is simple.
- If $v \geq 2$, then $A(X)$ has two or more roots (or a multiple root) in or near (c,d) in the complex plane.

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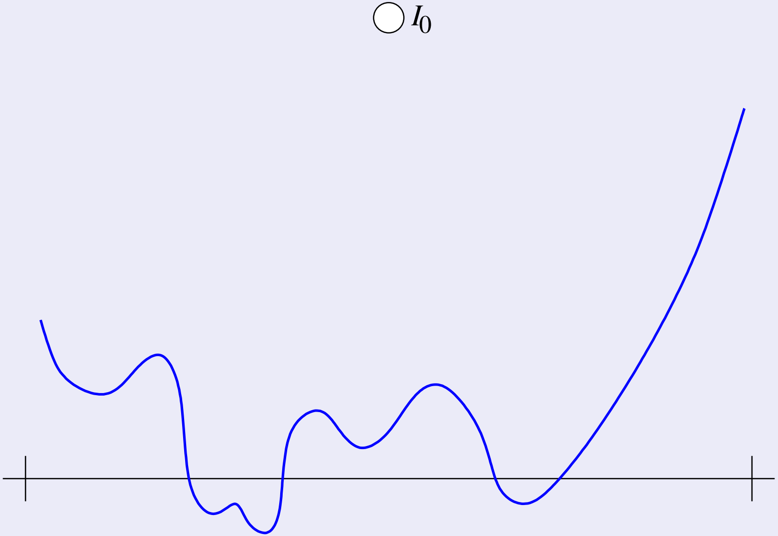
From now on, let $A(X)$ be square free.

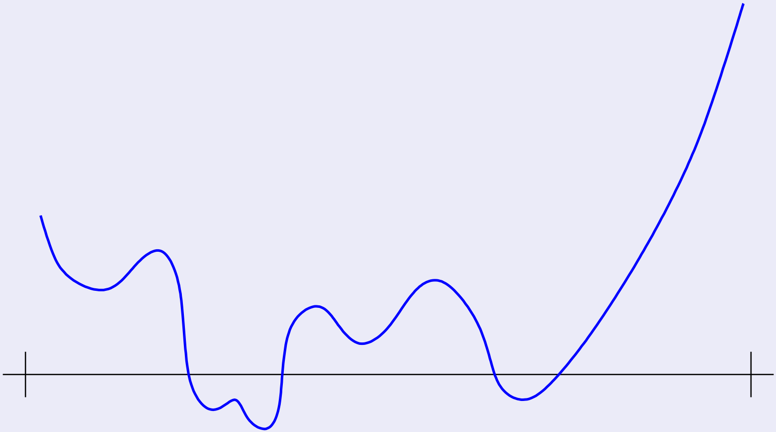
The Descartes Method



The initial interval I_0 is chosen to enclose all real roots.

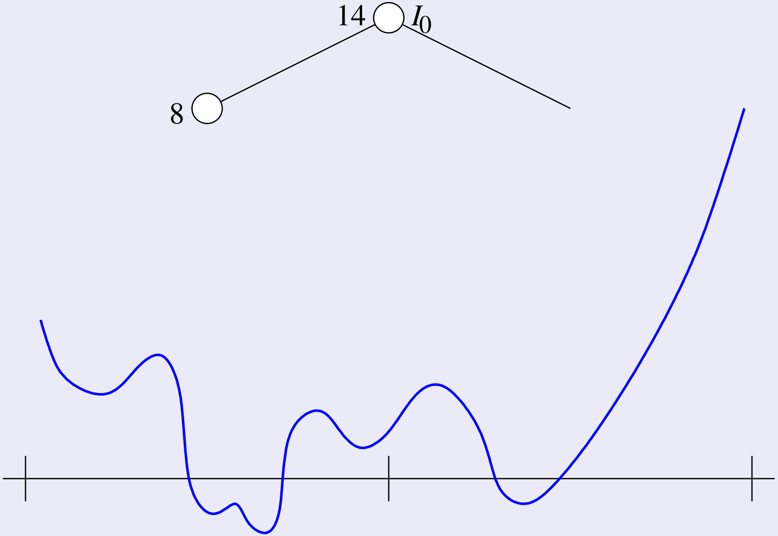
The Descartes Method

$14 \bigcirc I_0$



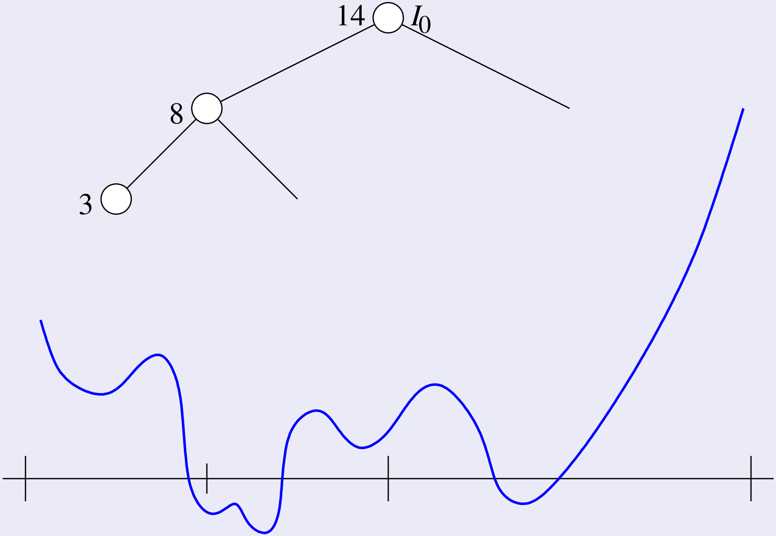
$DescartesTest(A, I_0) \geq 2 \implies$ subdivide I_0.

The Descartes Method



Continue recursively.

The Descartes Method



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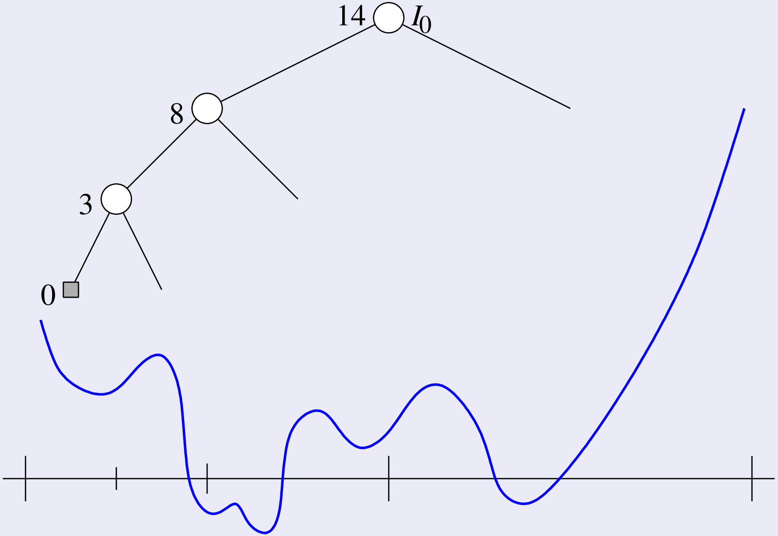
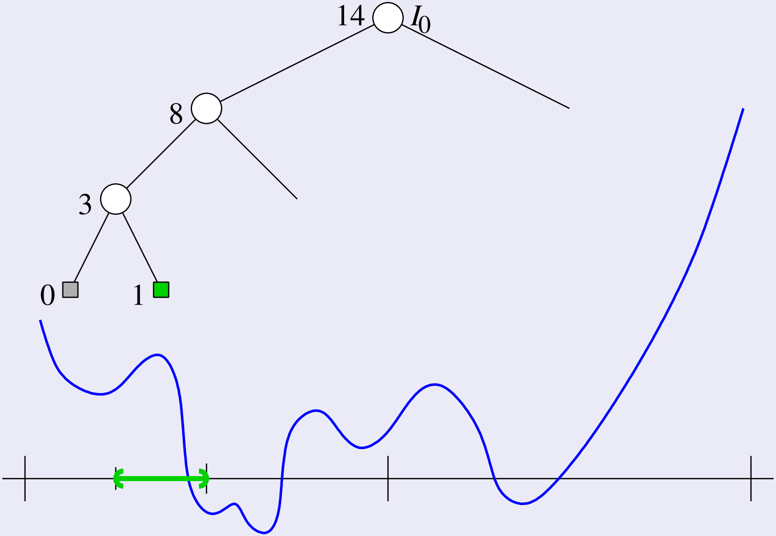


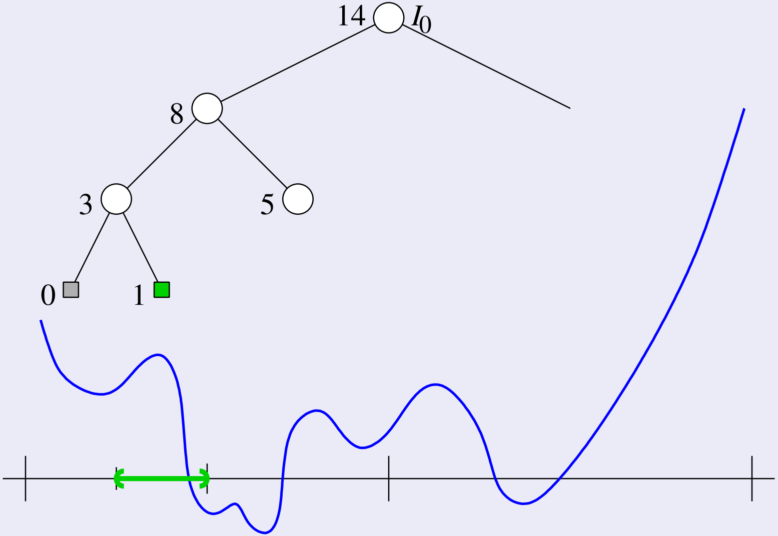
$DescartesTest(\ldots) = 0 \implies$ no roots found, return.

The Descartes Method



$DescartesTest(\ldots) = 1 \implies$ report isolating interval, return.

The Descartes Method



Continue recursion.

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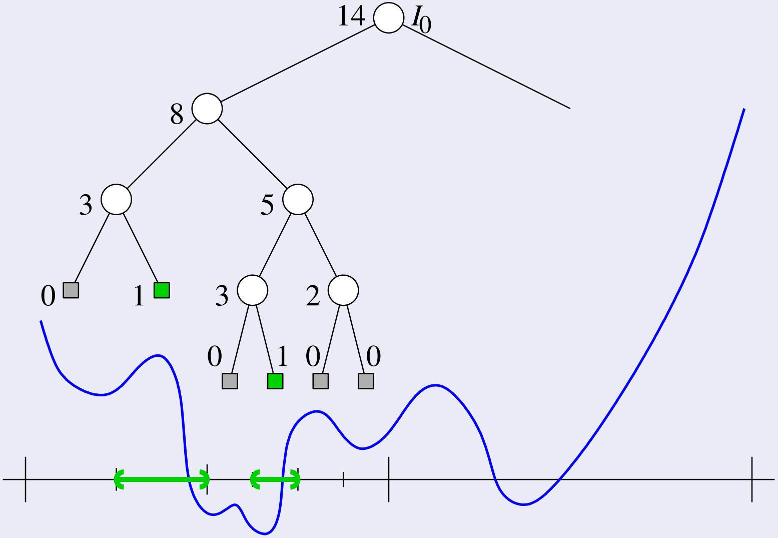

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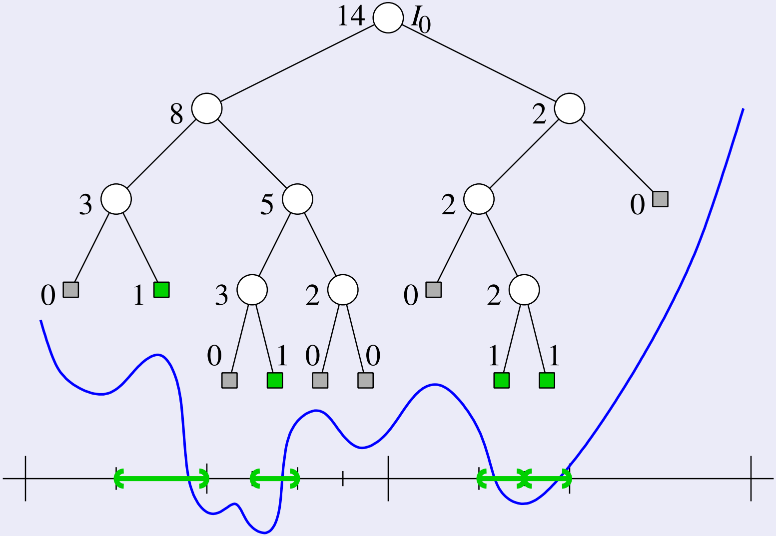

4 isolating intervals have been found.

Related Work (selection)

Description of the algorithm

- Classical / power basis variant: [Collins/Akritas, 1976]
- Bernstein basis variant: [Lane/Riesenfeld, 1981]
  (later: e.g., [Mourr./Vrah./Yakoubs., 2002] [Mourr./Rouillier/Roy, 2005])
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Tools from previous analyses

- [Krandick/Mehlhorn, 2006] used
  a Theorem of [Ostrowski, 1950] (also mentioned by [Batra, 1999]).
- [Johnson, 1991/98] [Krandick, 1995] applied
  a bound from [Davenport, 1985].

We use the same tools, but in a more direct way.

Tool #1: A partial converse of Descartes' Rule

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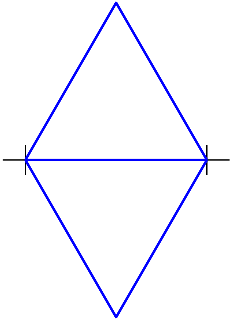

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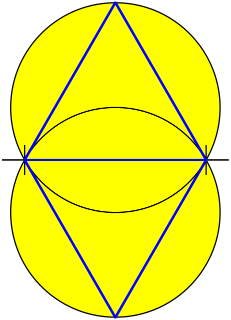

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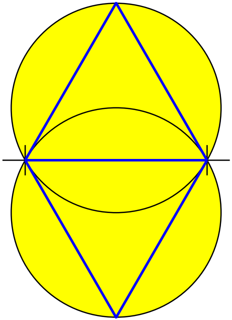

Two-circle Theorem (contrapositive)
([Ostrowski, 1950], see [Kra./Meh., 2006])

If $DescartesTest(A,(c,d)) \geq 2$*, then the two-circles figure in* $\mathbb{C}$ *around interval* (c,d) *contains two roots* α,β *of* $A(X)$*.*

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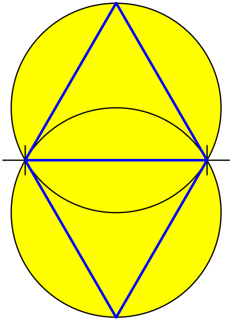

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Corollary

We can choose α, β *to be complex conjugate or adjacent real roots.*

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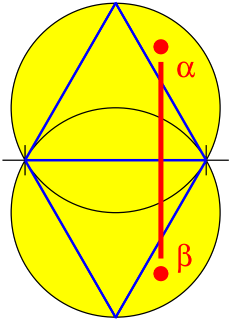

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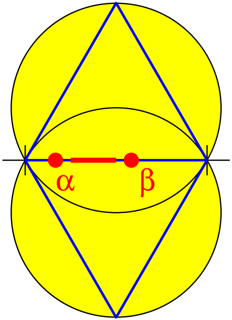

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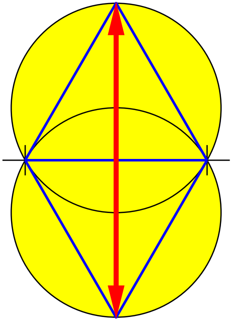

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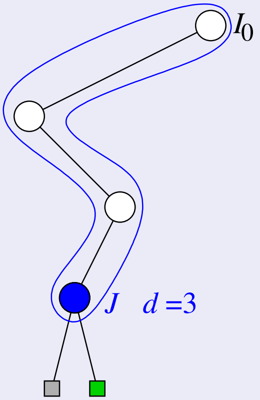
Corollary

We can choose α,β *to be complex conjugate or adjacent real roots. It holds that* $|\beta-\alpha| < \sqrt{3}(d-c)$*; i.e.,* $(d-c) > |\beta-\alpha|/\sqrt{3}$*.*

A tree bound in terms of roots (1)



A bound on path length

1. Consider any path in the recursion tree from I_0 to a parent J of two leaves.

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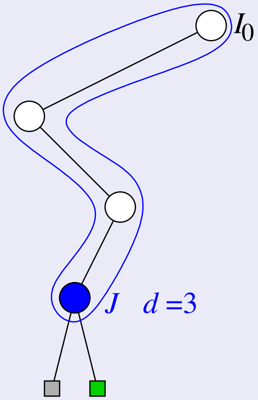

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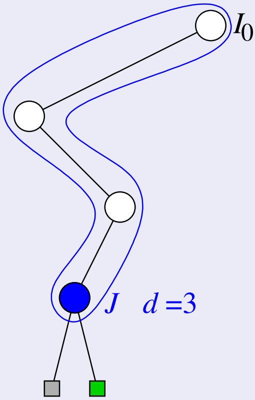

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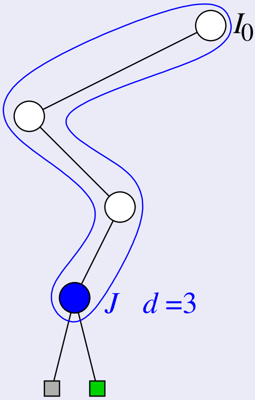

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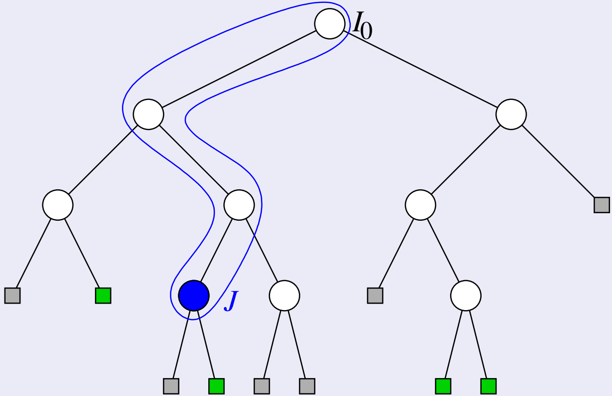
3. The whole path consists of $d+1$ internal nodes.
4. There is a pair of roots (α_J, β_J) such that $|J| > |\beta_J - \alpha_J|/\sqrt{3}$; hence $d+1 < \log|I_0| - \log|\beta_J - \alpha_J| + 2$.

A tree bound in terms of roots (2)



#(internal nodes on path) $<$ $\log|I_0| - \log|\beta_J - \alpha_J| + 2$

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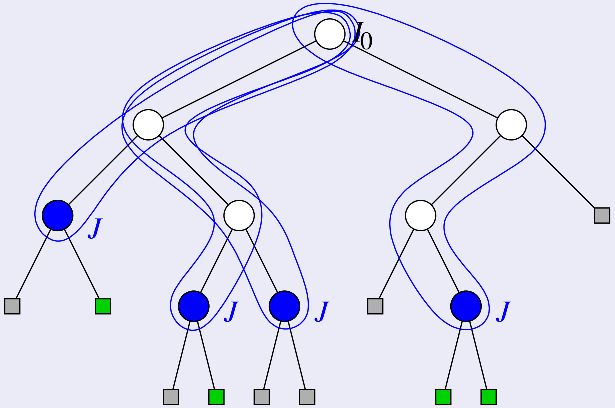

$$
\begin{array}{lcr}
\text{\#(internal nodes on path)} & < & \log|I_0| - \log|\beta_J - \alpha_J| + 2 \\
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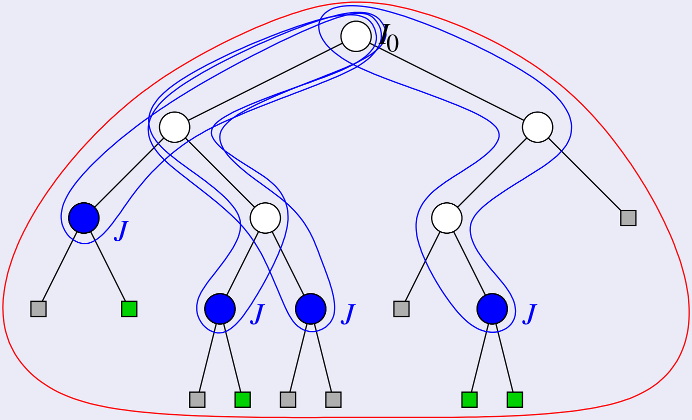

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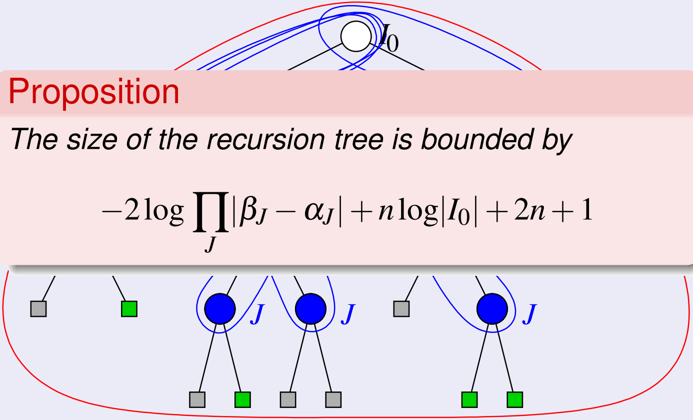

Proposition

The size of the recursion tree is bounded by

$$-2\log\prod_J|\beta_J-\alpha_J| + n\log|I_0| + 2n + 1$$

#(internal nodes on path) $<$ $\log|I_0| - \log|\beta_J-\alpha_J| + 2$

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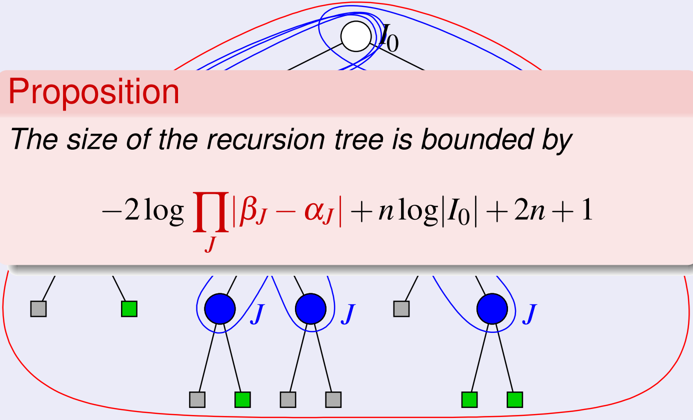

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Tool #2: The Davenport–Mahler bound

Theorem (Davenport–Mahler [Dav., 1985] [Johnson, 1991/98])

Consider a polynomial $A(X) \in \mathbb{C}[X]$ of degree n. Let $G = (V, E)$ be a digraph whose node set V consists of the roots $\vartheta_1, \ldots, \vartheta_n$ of $A(X)$. If

(i) $(\alpha, \beta) \in E \implies |\alpha| \leq |\beta|$,

(ii) $\beta \in V \implies \mathrm{indeg}(\beta) \leq 1$, *and*

(iii) G *is acyclic,*

then

$$\prod_{(\alpha,\beta)\in E} |\beta - \alpha| \geq \frac{\sqrt{|\mathrm{discr}(A)|}}{\mathrm{M}(A)^{n-1}} \cdot 2^{-O(n \log n)},$$

where

$$\mathrm{discr}(A) := a_n^{2n-2} \prod_{i>j} (\vartheta_i - \vartheta_j)^2 \quad \textit{and} \quad \mathrm{M}(A) := |a_n| \prod_i \max\{1, |\vartheta_i|\}.$$

Turning our product into an admissible graph

We want to rewrite

$$\prod_{J} |\beta_J - \alpha_J| \text{ as } \prod_{(\alpha,\beta)\in E} |\beta - \alpha|.$$

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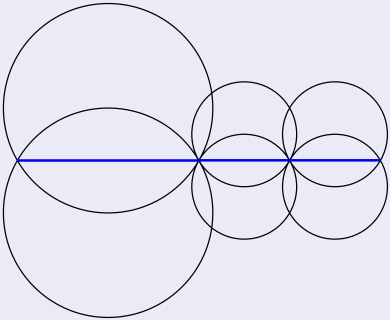
How often does $|\beta_J - \alpha_J|$ appear?

- adjacent real: ≤ 1



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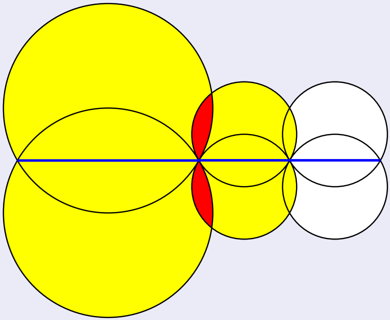

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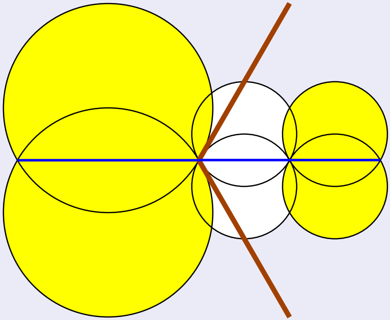

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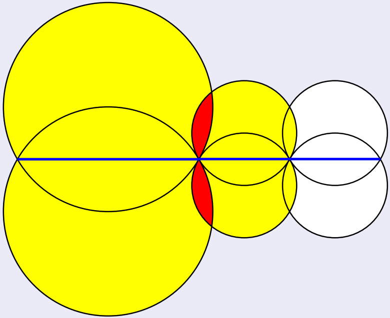
We need two graphs. (Paper: just 1.)



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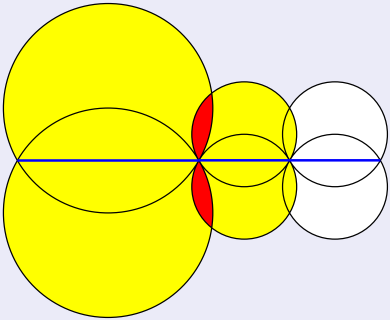

Conditions on $G = (V, E)$

(i) $(\alpha, \beta) \in E \implies |\alpha| \leq |\beta|$

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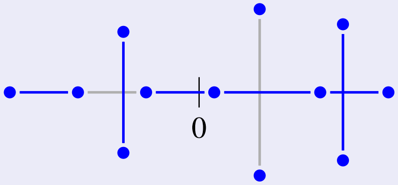

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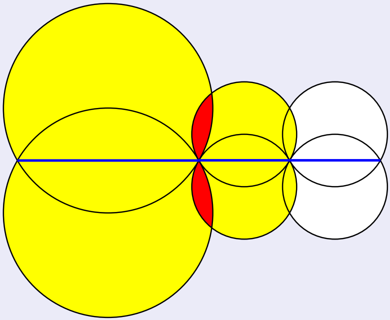

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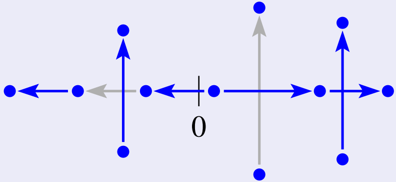

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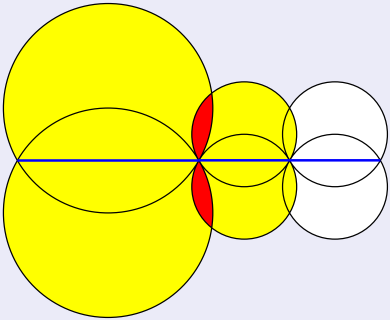

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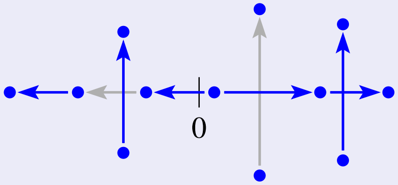

Main Result

Theorem

Let $A(X) \in \mathrm{R}[X]$ be a square-free polynomial of degree n.
The Descartes Method run on $A(X)$ starting from interval I_0 has a recursion tree $\mathcal{T}$ bounded in size by

$$|\mathcal{T}| = O(\log \frac{1}{|\mathrm{discr}(A)|} + n(\log \mathrm{M}(A) + \log n + \log |I_0|))$$

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Corollary

If $A(X) \in \mathbb{Z}[X]$ and $|a_i| < 2^L$, then easily $\log |I_0| = O(L)$, and one has

$$|\mathcal{T}| = O(n(L + \log n)).$$

Argument of [Krandick/Mehlhorn, 2006]: $|\mathcal{T}| = O(n \log n \, (L + \log n))$.

Almost tightness of the bound

Choose integers $n \geq 3$ and $a \geq 3$. Let $h = a^{-n/2-1}$. Consider

$$P(X) = X^n - 2(aX-1)^2 \quad \text{(irreducible)} \qquad \text{[Mignotte, 1981]}$$
$$P_2(X) = X^n - (aX-1)^2 \qquad \text{[Mignotte, 1995]}$$

The interval $(a^{-1}-h, a^{-1}+h)$ contains two roots of $P(X)$ and one root of $P_2(X)$ and thus three roots of $Q(X) = P(X) \cdot P_2(X)$.

Their median has an isolating interval of width less than $2h$, but $Q(X)$ has real roots outside $(0,1)$, so $|I_0| > 1$.

Hence recursion depth is more than $\log(1/(2h)) = \Omega(n \log a)$. $Q(X)$ has degree $2n = \Theta(n)$ and coefficient length $L = \Theta(\log a)$.

➥ Lower bound $\Omega(nL)$ matching $O(n(L + \log n))$ if $\log n = O(L)$.

Bit complexity for integer polynomials

Bit complexity depends on...

- the basis chosen to represent polynomials
  - Power basis $(x^i)_i = (1, x, x^2, \ldots, x^n)$
  - $[0,1]$-Bernstein basis $(\binom{n}{i} x^i (1-x)^{n-i})_i$
  - scaled $[0,1]$-Bernstein basis $(x^i (1-x)^{n-i})_i$

  (NB: Coefficient length L always refers to power basis.)
- the implementation of basic operations, esp. transformation of $A(X)$ to $A_L(X) = 2^n A(X/2)$ and $A_R(X) = 2^n A((X+1)/2)$.

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Classical subdivision

- Power basis + classical Taylor shift: $O(n^5 (L + \log n)^2)$.
  (Same bound as Johnson/Krandick/Mehlhorn, but simpler proof.)
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Asymptotically fast subdivision

- Power basis + fast Taylor shift [vzGathen/Gerhard, 1997]:
  $O(n\,(L+\log n)\,M(n^3(L+\log n))) = \widetilde{O}(n^4L^2)$.
  Same bound as [Du/Sharma/Yap, 2005] for Sturm's method.
- Bernstein basis: How to subdivide fast?
- A detour through the scaled Bernstein basis ("dual algorithm" of [Johnson, 1991]) makes it possible to apply a fast Taylor shift.
  Our tree bound $\rightsquigarrow \widetilde{O}(n^4L^2)$ [Emiris/Mourrain/Tsigaridas, 2006].

Summary

What have we done?

- Our paper gives a basis-free description of the Descartes Method for a uniform treatment of its power and Bernstein basis variants.
- We have recombined
  - tool #1: Ostrowski's partial converse of Descartes' rule
  - tool #2: the Davenport–Mahler bound

  in a new and simpler way.
- This gives a new and almost tight bound on the recursion tree.
- Bounds on bit complexity follow directly (some old, some new). Asymptotically fast variant attains $\widetilde{O}(n^4L^2)$ like Sturm's method.
- Replacing A by $A/\gcd(A,A')$ removes squarefreeness condition. Standard arguments show that our bounds remain valid.

Thank you!