Shortest Path amidst Disc Obstacles is Computable

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ABSTRACT

An open question in Exact Geometric Computation is whether there are transcendental computations that can be made "geometrically exact". Perhaps the simplest such problem in computational geometry is that of computing the shortest obstacle-avoiding path between two points p, q in the plane, where the obstacles are a collection of n discs.

This problem can be solved in $O(n^2 \log n)$ time in the Real RAM model, but nothing was known about its computability in the standard (Turing) model of computation. We first show the Turing-computability of this problem, provided the radii of the discs are rationally related. We make the usual assumption that the numerical input data are real algebraic numbers. By appealing to effective bounds from transcendental number theory, we further show a single-exponential time upper bound when the input numbers are rational.

Our result appears to be the first example of a non-algebraic combinatorial problem which is shown computable. It is also a rare example of transcendental number theory yielding *positive* computational results.

Categories and Subject Descriptors

F.2.2 [**Nonnumerical Algorithms and Problems**]: Computational geometry—*Computability*; F.2.1 [**Numerical Algorithms and Problems**]: Algebraic and transcendental computation—*Complexity*

General Terms

Algorithms, Theory

Keywords

Shortest path, disc obstacles, exact geometric computation, robust numerical algorithms, guaranteed precision computation, exponential complexity, real RAM model

1. INTRODUCTION

> "*It can be of no practical use to know that π is irrational, but if we can know, it surely would be intolerable not to know.*"
>
> E.C. Titchmarsh

Most problems in Computational Geometry are algebraic [21]. Algebraic problems can be solved "exactly, in the geometric sense". This means that the geometric relations required by the algorithm can be determined without error. Examples of geometric relations are "point p is inside triangle T", "line ℓ intersects disc D", etc. Exact Geometric Computation (EGC) is the basis of a highly successful approach to robust geometric algorithms. This approach is currently embodied in software libraries such as LEDA [3, 13], CGAL [7] and the Core Library [10].

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In contrast to algebraic problems, we do not know of any non-algebraic problem[1] that was known to be solvable in the EGC sense. This paper will furnish the first such example: *Given a collection S of discs in the plane, and given two points $p, q \in \mathbb{R}^2$, to compute a shortest S-avoiding path between p and q.* This **disc shortest path problem** is one of the simplest non-algebraic problem in computational geometry. We assume, as is the standard practice, that the input numbers in this problem are algebraic. Nevertheless the problem is non-algebraic since the lengths of shortest paths may involve transcendental quantities. For instance, consider the situation in Figure 1 involving two discs A, B. There are two feasible paths which might be shortest, and we must compare their path lengths: these lengths are numbers of the form $\alpha + \theta$ where α is algebraic and θ is provably transcendental when the input numbers are rational.

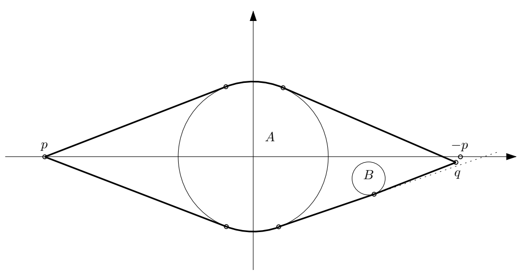

Figure 1: Two feasible paths from p to q.

For brevity, we say "EGC-solvable" when a real computational problem is Turing-computable in the EGC sense. This is precisely defined below. Using results from transcendental number theory, we show that the disc shortest path problem is EGC-solvable, provided the radii of the input discs are **commensurable** (i.e., rationally related). Usually, transcendental number theory yields negative results[2] about computability. For instance, Lindemann's proof (1882) that π is transcendental implies the impossibility of the ancient Greek quest to "square the circle", i.e., to construct a square whose area is equal to the area of a given circle, using only ruler and compass operations. Our paper is one of few examples where transcendental number theory yields a positive computational result. Another example is Lyapunov's work in theoretical mechanics, cited by Fel'dman and Nesterenko [8, p.88]. Besides [8], the books of Baker [1] and Lang [12] may serve as general references.

More generally, we are interested in general classes of transcendental problems that are EGC-solvable. EGC-solvability is invariably hinged on the decidability of the **zero problem** for numerical expressions [14, 15, 21]. For non-algebraic expressions, the only positive result is a conditional one, from Richardson [14]. This result says that the zero problem for complex expressions involving the (complex) functions $\exp(x)$, $\log(x)$ and algebraic functions is decidable, provided Schanuel's conjecture [1, p. 120] is true. Richardson's result essentially implies that our disc shortest path problem is *conditionally* EGC-solvable.

1.1 Standard Disc Shortest Path Solution

A path from p to q is a continuous rectifiable function $\mu : [0,1] \to \mathbb{R}^2$ with $\mu(0) = p$, $\mu(1) = q$ and $\mu(t) \notin \cup\{D : D \in S\}$ (we regard the D's to be open discs). Its length is denoted $d(\mu) \geq 0$. If this length is minimum, then we denote $d(\mu)$ by $d(p,q)$ or $d_S(p,q)$. Clearly, (the range of) μ is comprised of an alternating sequence of straight-line segments and arcs. The arcs are portions of the boundaries of discs in C. We may write

$$\mu = \mu_1; \mu_2; \cdots ; \mu_k \qquad (1)$$

as a concatenation of subpaths where μ_i is a straight line segment iff μ_{i+1} is an arc. Let p_i be the common end point of μ_i and μ_{i+1}. Also, let $p = p_0$ and $q = p_k$. Call $p_0, p_1, \ldots, p_k$ the **nodes** of the path. The straight line segments are tangent to the discs at the points of contact. In general, any such path μ is said to be **feasible**.

The underlying algorithm here is Dijkstra's shortest path algorithm. First we compute a combinatorial graph $G = (V, E)$ whose nodes $v \in V$ are the points on the boundary of some disc D_v in S. These points are the nodes of feasible paths. Each node v has a partner $u \in V$ such that the line $\overline{uv}$ is a common tangent to the discs D_v and D_u, and the segment $[u, v]$ avoids other discs. The edge set E comprises such segments $[u, v]$ as well as arcs between pairs of nodes that are consecutive on the boundary of the same disc. We view p, q as special discs of radius 0 in this construction. If S has n discs, it is easy to see that there are $O(n^2)$ nodes and $O(n^2)$ edges in G. We can now use Dijkstra's algorithm to compute the shortest paths from p to all the nodes, including q, using the length of each edge as its weight.

Dijkstra's algorithm can be implemented to run in $O(|E| + |V| \log |V|) = O(n^2 \log n)$ time in the well-known **Real RAM computational model** where arithmetic on real numbers are error-free and unit time, and the exact comparison of numbers also take unit time. Note that in this case, the real RAM model needs the square-root and arcsin functions.

We assume the input data are algebraic: points p, q and the centers and radii of the discs in S are represented by algebraic numbers. We may take any standard representation of algebraic numbers (e.g., the isolating interval representation). Unfortunately, a Real RAM computation for our shortest path problem is not known to be realizable by Turing machines.

Let us clarify this remark. To carry out Dijkstra's algorithm on a Turing machine, we need to ensure that the comparison of two path lengths can be carried out without error – this is the principle of **exact geometric computation** [21], mentioned in the introductory paragraph. This principle ensures that our algorithm determines the exact combinatorial shortest path: for the path in (1), this combinatorial path is basically the

[1] We implicitly require our problems to be "combinatorially non-trivial". This means that the combinatorial input size parameter, n, is arbitrarily large.

[2] Cf. the quote from Titchmarsh in the introduction.

sequence $p_0, p_1, \dots, p_k$ of nodes. But since the path lengths involve transcendental quantities, it is not obvious how we can determine when two such quantities are equal. Certainly, no general algorithms are known.

1.2 What is EGC Computability?

The issue has to do with foundational questions about computation over real numbers. The model for computing over a countable domain ($\mathbb{N}, \mathbb{Q}$, algebraic numbers or finite strings Σ^*) is largely settled – it is widely accepted as the Turing model or any of its equivalent formulations. But over an uncountable domain such as the real numbers, the appropriate model is still a matter of debate. Currently, the two main approaches to real computation are the **algebraic approach** (Real RAM model or BSS model [2]) and the **analytic approach** (Turing machine with infinite input tapes [18]). In [20], it is noted that the zero problem is trivial in the algebraic approach, and undecidable in the analytic approach. Instead, an intermediate solution is proposed, whereby all real computation is to be replaced by real approximations in the standard Turing model. There is no hope to represent all real numbers, and so real approximations is a necessity. The fundamental principle here is that *all input and output numbers must be representable*. Thus, output numbers are necessarily an approximation. But what does "computing exactly" mean in the context of real approximations? We formulate this notion precisely here.

We first fix a countable set $\mathbb{F} \subseteq \mathbb{R}$ of real numbers with the following properties: (1) $\mathbb{F}$ is a ring that extends $\mathbb{Z}$, (2) if $x \in \mathbb{F}$ then $x/2 \in \mathbb{F}$ (so $\mathbb{F}$ is dense in $\mathbb{R}$), (3) there is an encoding τ of the members of $\mathbb{F}$ as binary strings, (4) all the ring operations, $x \mapsto x/2$, comparisons $x : y$ and deciding if a binary string encodes a number are all polynomial-time computable relative to this encoding. A real number x is said to be **representable** iff $x \in \mathbb{F}$. For instance, we can choose $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \{m2^n : m, n \in \mathbb{Z}\}$, using standard encodings. For our purposes, $\mathbb{F}$ and τ may be fixed.

Let Σ be a set of symbols. By a **real combinatorial object** (RCO) we mean a directed graph G with labels on its vertices and edges. The labels are either elements of Σ or real-tuples. Note that geometric objects can be modeled as real combinatorial objects [21]. We say G is **representable** if all its real numbers are representable. If G' is another RCO, we say G' and G are **combinatorially equivalent**, denoted $G \equiv G'$, if the two graphs are isomorphic when we remove all numerical labels. We say G' is a p**-bit absolute approximation** of G if $G \equiv G'$ and if $t = (t_1, \dots, t_k)$ and $t' = (t'_1, \dots, t'_\ell)$ are corresponding real tuples in G and G', then $k = \ell$ and $|t_i - t'_i| \le 2^{-p}$ for each i. There is a corresponding notion of p-bit relative approximation.

Clearly, the representable RCO's can be encoded as strings for Turing machine computation. A more natural approach is to model the combinatorial part by pointer structures as in Schönhage's pointer machines. This approach is taken in [20]. However, the present account is formulated entirely using standard Turing machines.

Let RC denote the set of all RCO's. A **real combinatorial problem** P is a multivalued function $P : RC \to 2^{RC}$. We say P is **EGC-solvable** (or EGC-computable) if there is a standard Turing machine which, given any representable RCO G, eventually halts with a representable output G' such that G' is combinatorially equivalent to some $G'' \in P(G)$. We say P is **absolutely approximable** if for all representable input G, and $p \in \mathbb{F}$, the output G' satisfies the additional property that G' is a p-bit absolute approximation of some $G'' \in P(G)$.

2. ARITHMETIC ON ARC LENGTHS

It is clear that the nodes p_i of feasible paths such as (1) are algebraic. Hence the length $d(\mu_i)$ is algebraic when μ_i is a straight-line segment. But when μ_i is an arc, it may be non-algebraic. For instance, if μ_i is exactly half the circumference of a unit disc then $d(\mu_i) = \pi$. The pairs of numbers to be compared in Dijkstra's algorithm are lengths of feasible paths. These lengths have the form

$$\alpha + \sum_{i=1}^{m} \theta_i r_i \tag{2}$$

where $\alpha \ge 0$ is an algebraic number, and $0 < r_1 < \cdots < r_m$ are the distinct radii among the discs. Here $\theta_i \ge 0$ is the total angle (in radians) that the path traverses around discs with radii r_i. The difference of two such numbers is also of this form, except that α and the θ_i's can now be negative. The exact comparison of lengths is reduced to determining the sign of such differences.

LEMMA 1. *In the path length (2), each* $\cos\theta_i$ *is computable and algebraic.*

Proof. The cosines and sines of angles spanned by individual arcs in a feasible path is algebraic. The cosine and sine of a sum of such angles can be computed from the cosines and sines of the individual angles using the ring operations. Hence each $\cos\theta_i$ in (2) is computable and algebraic. **Q.E.D.**

Lindemann's theorem says that if x is algebraic then e^x is transcendental. Note that $\mathbf{i} = \sqrt{-1}$ is algebraic. Thus:

LEMMA 2. *In (2),* $\theta_i \neq 0$ *implies* θ_i *is transcendental.*

Proof. Since $\cos\theta$ is algebraic, so is $e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$. So $\mathbf{i}\theta$, and hence θ, must be transcendental. **Q.E.D.**

Ultimately, our computations are reduced to two sets of tools: computing with algebraic numbers and approximation of transcendental functions.

- Algebraic number computation. To be specific, we assume that a real algebraic number α is represented using the isolating interval representation $\alpha \sim [P(X), a, b]$ where $P(X) \in \mathbb{Z}[X]$, $a, b \in \mathbb{Q}$, $a < b$ and α is the unique real root of $P(X)$ in $[a, b]$. Algorithms to perform arithmetic operations and to compare such numbers are available (See e.g., [19]).

- Transcendental approximation: All the well-known elementary functions such as $\sin x$, $\arccos x$ are simple transformations of hypergeometric functions. Let $\mathbf{a} = (a_1, \ldots, a_p), \mathbf{b} = (b_1, \ldots, b_q)$ where $a_i, b_j \in \mathbb{C}$ and $p+1 \geq q \geq 0$. Each $\mathbf{a}, \mathbf{b}$ defines a hypergeometric function $f(x)$ usually denoted by

$$f(x) = {}_pF_q(\mathbf{a}; \mathbf{b}|x).$$

For any algebraic x, we can approximate $f(x)$ to any desired absolute error bound. In fact, [5, 6] shows that we can achieve this approximation *uniformly* in $\mathbf{a}, \mathbf{b}, x$. In other words, the *general hypergeometric function*,

$$\begin{aligned} & H(p, q, a_1, \ldots, a_p, b_1, \ldots, b_q, x) \\ = & \; {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q|x). \end{aligned}$$

can be approximated to any desired absolute error bound. But it is not known how to approximate H with *relative* error bounds [6]. Note that H is anadic, i.e., it does not have a fixed arity.

2.1 Representation of Signed Arc Lengths

A computable representation of arc lengths is the key in showing the computability of the shortest path problem. Suppose C is a circle represented by its center $o(C)$ and radius $r(C)$. Let A be a *directed* arc of C that spans an angle θ. Note that θ here is an arbitrary real number; it is a negative number iff the arc A is counter-clockwise. We represent A by the quadruple

$$A = [C, p, q, n]$$

where $n \in \mathbb{Z}$ and p, q are points on C such that

$$\theta = n\pi + \phi(p, q) \tag{3}$$

where $\phi(p, q) \in (-\pi, \pi)$ is the angle spanned by the arc from p *counterclockwise* to q. We call C the **carrier** of representation A. The signed length of the directed arc A is denoted

$$\mathrm{val}(A) = \mathrm{val}(C, p, q, n)$$

with $\mathrm{val}(C, p, q, n) = \pm r(C)\theta$. A directed arc $[C, p, q, n]$ is said to be **algebraic** if each number involved in its description is an algebraic number. It follows from the previous lemma that a non-zero $\mathrm{val}(A)$ is transcendental when A is algebraic.

We mainly use A as a representation of the real value $\mathrm{val}(A)$; when we talk of "computing $A \pm A'$", we mean to compute some algebraic directed arc A'' where $\mathrm{val}(A'') = \mathrm{val}(A) \pm \mathrm{val}(A')$. The notation "$\phi(p, q)$" is well-defined provided p, q are not diametrically opposite each other. That is, $\phi(p, q)$ determines a unique arc A that spans an angle $0 \leq \theta < \pi$. Then $\phi(p, q) = \theta$ if the path from p to q along A is clockwise, and otherwise $\phi(p, q) = -\theta$. Thus we have $\phi(p, p) = 0$ and $\phi(q, p) = -\phi(p, q)$. We write $\phi(p, q) \uparrow$ in case p, q are diametrically opposite each other.

We now define several operations involving such representations.

2.2 Negation

Given a directed arc $A = [C, p, q, n]$, we define

$$-[C, p, q, n] = [C, q, p, -n].$$

We may verify from (3) that $\mathrm{val}(C, q, p, -n) = -\mathrm{val}(C, p, q, n)$. Sometimes, we need a second form of negation, namely,

$$\ominus[C, p, q, n] = [C, p, q', -n]$$

where q' is the reflection of q about the line $\overline{o(C)p}$. In fact, let $\alpha = \langle q - o(C), p - o(C)\rangle/(r(C))^2$ where $\langle\cdot,\cdot\rangle$ denotes dot product. Then $q' = 2(\alpha p + (1-\alpha)o(C)) - q$. Thus q' can be computed using only rational operations.

2.3 Normalization

We say $A = [C, p, q, n]$ is **normalized** if $\mathtt{sign}(n \cdot \phi(p, q)) \geq 0$. We now define an operation $N(A)$ on A:

$$N[C, p, q, n] = [C', p', q', n']$$

where $C' = C$ and $p' = p$. If $\mathtt{sign}(n) \cdot \phi(p, q) \geq 0$, then $q' = q, n' = n$. Otherwise, let $q' = 2o(C) - q$. Also, set $n' = n - \mathtt{sign}(n)$.

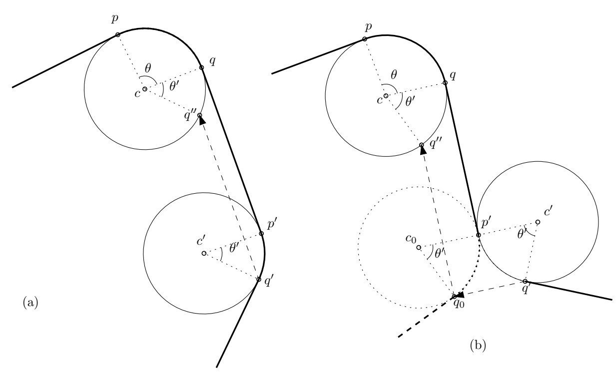

Figure 2: Compatible Sum: (a) Parallel (b) Antiparallel

2.4 Compatible Sum

Let $A' = [C', p', q', n']$ be another directed arc. We say A and A' are **compatible** provided the radii of C and C' are the same, and the vectors $q - o(C)$ and $p' - o(C')$ are compatible, *i.e.*, $q - o(C) = \pm(p' - o(C'))$. There are two cases: **parallel** if $q - o(C) = (p' - o(C'))$, and **antiparallel** if $q - o(C) = -(p' - o(C'))$. Define the **compatible sum**

$$[C, p, q, n] \oplus [C', p', q', n']$$

which results in a new directed arc representation, $[C, p'', q'', n'']$ with the property

$$\mathrm{val}(C, p'', q'', n'') = \begin{cases} \mathrm{val}(C, p, q, n) + \mathrm{val}(C', p', q', n') & \text{if parallel} \\ \mathrm{val}(C, p, q, n) - \mathrm{val}(C', p', q', n') & \text{if anti-parallel.} \end{cases}$$

In the shortest path problem, a special case of compability arises: A and A' are compatible if the line $\overline{qp'}$ is a common tangent of C and C'. This is illustrated by Figure 2.

- Parallel: See Figure 2(a). Suppose $\phi(p', q')\phi(p, q) \geq 0$. In this case, q'' is equal to $q' + (c - c')$. To determine n'' and p'', we first define

  $\delta = 0$ iff $\phi(p, q)\phi(q, q'') \leq 0$ and $\phi(p, q)\phi(p, q'') \geq 0$

  and otherwise, $\delta = 1$. If $\delta = 0$ then $n'' = n+n'$ and $p'' = p$. If $\delta = 1$ then $n'' = n + n' + \mathtt{sign}(\phi(p, q))$ and $p'' = 2o(C)-p$. Remark that in case $\delta = 1$, we could have $\phi(p, q'') \uparrow$, i.e., $p, q'', o(C)$ are collinear.

- Antiparallel: See Figure 2(b). In this case, we must first "reflect" the arc $A' = [C', p', q', n']$ across the tangent line $\overline{qp'}$ to get a new directed arc $[C_0, p_0, q_0, n_0]$ where the center of C_0 is denoted c_0 in Figure 2(b). Also, $p_0 = p'$ and $n_0 = -n'$. We now proceed as the parallel case.

We summarize the results of the preceding development:

LEMMA 3. *Let the oriented arcs A, B be compatible. The operations*

$$-A, \quad \ominus(A), \quad N(A), \quad A \oplus B$$

can be computed using only rational operations.

It should be remembered that these are rational operations on algebraic numbers, so the computational effort may still be substantial.

Finally, we address the question of determining the sign of arc lengths. For any vector $v = (x, y)$, let $v^{\perp} = (-y, x)$ denote its counter clockwise rotation by 90 degrees.

LEMMA 4. *If $A = [C, p, q, n]$ is normalized, then*

$$\mathtt{sign}(\mathrm{val}(A)) = \begin{cases} \mathtt{sign}(n) & \text{if } n \neq 0 \\ \mathtt{sign}(\langle p - o(C), (q - o(C))^{\perp}\rangle) & \text{if } n = 0. \end{cases}$$

Thus, $\mathtt{sign}(\mathrm{val}(A))$ is computable when A is algebraic.

REMARK. The results in this section already imply the computability of the shortest path problem when all the discs have the same radius r_0: for any feasible path μ, we can compute an algebraic α and directed A such that $d(\mu) = \alpha + \mathrm{val}(A)$. To compare $d(\mu)$ and $d(\mu') = \alpha' + \mathrm{val}(A')$, we use the fact that $d(\mu) = d(\mu')$ iff $\alpha = \alpha'$ and $\mathrm{val}(A) = \mathrm{val}(A')$. We know how to determine if $\alpha = \alpha'$. To determine the sign of $\mathrm{val}(A) - \mathrm{val}(A')$, we first normalize $A = [C, p, q, n], A' = [C, p', q', n']$. Then this sign is $n - n'$ if $n \neq n'$. Otherwise, it is the sign of the algebraic number $\|p-q\| - \|p'-q'\|$. Finally, suppose we know $d(\mu) \neq d(\mu')$. Then the sign of $d(\mu) - d(\mu')$ can be obtained by computing more and more precise approximations.

3. ALGEBRAIC OPERATIONS

We now want to add (the lengths of) arbitrary directed arcs A and A', not necessarily compatible. This can be achieved with combinations of the following more involved operations on directed arcs: unlike the previous operations, these are no longer rational.

3.1 Restricted Addition

Let $A = [C, p, q, n]$, $A' = [C', p', q', n']$ be two directed arcs. We want to form $A'' = [C'', p'', q'', n'']$ such that $\mathrm{val}(A'') = \mathrm{val}(A) + \mathrm{val}(A')$, i.e., we want to add directed arc lengths. We describe this under the *restriction* $r(C) = r(C')$. In turn, we may assume $C = C'$. In view of the compatible sum $\oplus$ operation, it suffices to show how to make A compatible with A'. This reduces to the next operation, rotation.

Observe that "restricted subtraction", $\mathrm{val}(A) - \mathrm{val}(A')$, is easily reduced to restricted addition, by using negation.

3.2 Rotation

Let $A = [C, p, q, n]$ and $A' = [C, p', q', n']$ (sharing the same carrier). To make A compatible with A', we "rotate" both p and q by a common angle θ. The angle θ may be specified by two points a, b on C such that $\phi(a, b) = \theta$. If p, a, b are points on C, let $Rot(C, p, a, b)$ denote the "rotated point" r on C such that $\phi(p, r) = \phi(a, b)$. Suppose $Rot(A, A') = [C, p'', q'', n'']$ denote the transformation of A so that it becomes compatible with A'. Then $q'' = p$, $p'' = Rot(p, q, p')$ and $n'' = n$.

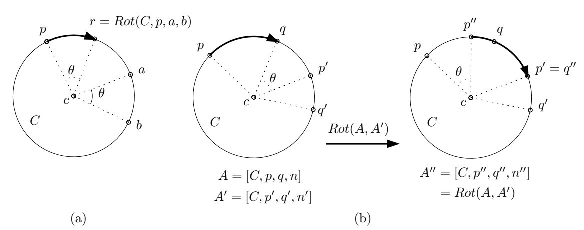

Figure 3: Rotation: (a) $Rot(C, p, a, b)$**, (b)** $Rot(A, A')$

If $Rot(C, p, a, b) = r$, then each coordinate of r can be computed using rational operations and a single square-root operation.

3.3 Scalar Multiplication

The most complicated operation we address is scalar multiplication: given a real number γ and a directed arc $A = [C, p, q, n]$, we want to compute another directed arc denoted

$$\begin{aligned} A' &= \gamma \cdot [C, p, q, n] \\ &= [C', p', q', n'] \end{aligned} \quad (4)$$

where $C' = C$, $p' = p$ and $\mathrm{val}(A') = \gamma \cdot \mathrm{val}(A)$. Hence we only need to determine p', q', n'. We only consider the case where $\gamma = m/k$ is rational. This multiplication will be carried out in two steps: first divide by a positive integer k and then multiply by a positive integer m. We may assume that A is normalized and angle represented by A is positive, given by

$$\theta = \theta^* + n\pi, \qquad \theta^* \in [0, \pi), \tag{5}$$

We can compute $\cos\theta = (2r^2 - \|p-q\|^2)/2r^2$. The crux of our problem is to determine $\cos(\theta/k)$.

REMARK: The requirement that the carriers of A and $A' = \gamma A$ have the same radius is what makes the scalar multiplication operation nontrivial. Without this restriction, we can easily compute an A' such that $\mathrm{val}(A') = \gamma \mathrm{val}(A)$. The idea is to choose a carrier for A' whose radius is γ times the radius of A.

3.4 Division by an Integer

First consider the problem of computing $A' = (1/k) \cdot A$ where $k \geq 2$ is an integer. Let $T_k(x)$ be the k-th Chebyshev polynomial of the first kind [11, p. 315–317]. These polynomials are defined recursively with $T_0(x) = 1, T_1(x) = x$, and for $k \geq 1$,

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x). \tag{6}$$

Each $T_k(x)$ is an integer polynomial of degree k satisfying the relation $T_k(\cos(\theta/k)) = \cos\theta$ for any θ. In our problem, we know $\cos\theta$ and we need to compute $\cos(\theta/k)$ as the zero of the polynomial

$$T_k(x) - \cos\theta. \tag{7}$$

It is easy to see that the set of k zeros of (7) is

$$Z = \{\cos\frac{\theta + 2\pi\ell}{k}, \qquad \ell = 0, \ldots, k-1\}. \tag{8}$$

As we will see, the polynomial (7) has multiple roots iff $\cos\theta = \pm 1$, i.e., when $\theta^* = 0$ in (5).

LEMMA 5. *Define $w = 1, 2, \ldots, k$, by*

$$w = \begin{cases} 1 + (n \,\mathbf{mod}\, k), & \text{if } \left\lfloor \frac{n}{k} \right\rfloor \text{ is even,} \\ k - (n \,\mathbf{mod}\, k), & \text{if } \left\lfloor \frac{n}{k} \right\rfloor \text{ is odd.} \end{cases}$$

Then we have:

(1) If $\theta^* > 0$, then $T_k(x) - \cos\theta$ has k distinct zeros, and $\cos(\theta/k)$ is the w-th largest among them.

(2) If $\theta^* = 0$, then $T_k(x) - \cos\theta$ has $\left\lceil \frac{k+1}{2} \right\rceil$ distinct zeros, and $\cos(\theta/k)$ is the $\left\lfloor \frac{w+1}{2} \right\rfloor$-th largest among them.

Moreover, if we write $\theta/k = \theta' + n'\pi$ for $\theta' \in [0, \pi)$ and $n' = 0, 1, 2, \ldots$, then

$$n' = \left\lfloor \frac{n}{k} \right\rfloor, \quad \text{and} \quad \sin(\theta/k) \begin{cases} \geq 0, & \text{if } n' \text{ is even} \\ \leq 0, & \text{if } n' \text{ is odd.} \end{cases}$$

PROOF. From (5) and (8), we know that

$$Z = \left\{\cos\frac{\theta^* + n\pi + 2\pi l}{k} \; : \; l = 0, 1, \ldots, k-1\right\}.$$

(A) Suppose $\left\lfloor \frac{n}{k} \right\rfloor$ is even. Then we can write $n = 2Nk + w - 1$ for some $N = 0, 1, 2, \ldots$ Actually, $\left\lfloor \frac{n}{k} \right\rfloor = 2N$. Now

$$\begin{aligned} \frac{\theta}{k} &= \frac{\theta^* + n\pi}{k} \\ &= \frac{\theta^* + \left(\left\lfloor \frac{n}{k} \right\rfloor k + w - 1\right)\pi}{k} \\ &= \left(\frac{\theta^*}{k} + \frac{w-1}{k}\pi\right) + \left\lfloor \frac{n}{k} \right\rfloor \pi. \end{aligned}$$

Since $0 \leq \frac{\theta^*}{k} < \frac{\pi}{k}$ and $0 \leq \frac{w-1}{k}\pi \leq \frac{k-1}{k}\pi$, we have $0 \leq \frac{\theta^*}{k} + \frac{w-1}{k}\pi < \pi$. Thus $n' = \left\lfloor \frac{n}{k} \right\rfloor$.

There are four possibilities: $w-1$ can be even or odd, and θ^* can be positive or 0.

(i) Even $w - 1$:

$$\begin{aligned} Z &= \left\{\cos\frac{\theta^* + 2Nk\pi + (w-1+2l)\pi}{k} \; : \; l = 0, 1, \ldots, k-1\right\} \\ &= \left\{\cos\left(\frac{\theta^*}{k} + \frac{2l}{k}\pi\right) \; : \; l = 0, 1, \ldots, k-1\right\}. \end{aligned}$$

$$\cos\frac{2l+1}{k}\pi < \cos\left(\frac{\theta^*}{k} + \frac{2l}{k}\pi\right) \leq \cos\frac{2l}{k}\pi,$$
$$l = 0, 1, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor,$$
$$\cos\frac{2l}{k}\pi \leq \cos\left(\frac{\theta^*}{k} + \frac{2l}{k}\pi\right) < \cos\frac{2l+1}{k}\pi,$$
$$l = \left\lfloor \frac{k-1}{2} \right\rfloor + 1, \ldots, k-1.$$

(ii) Odd $w - 1$:

$$Z = \left\{\cos\left(\frac{\theta^*}{k} + \frac{2l+1}{k}\pi\right) \; : \; l = 0, 1, \ldots, k-1\right\}.$$

$$\cos\frac{2l+2}{k}\pi < \cos\left(\frac{\theta^*}{k} + \frac{2l+1}{k}\pi\right) \leq \cos\frac{2l+1}{k}\pi,$$
$$l = 0, 1, \ldots, \left\lceil \frac{k-3}{2} \right\rceil;$$
$$\cos\frac{2l+1}{k}\pi \leq \cos\left(\frac{\theta^*}{k} + \frac{2l+1}{k}\pi\right) < \cos\frac{2l+2}{k}\pi,$$
$$l = \left\lceil \frac{k-3}{2} \right\rceil, \ldots, k-1.$$

(1) Suppose $\theta^* > 0$. Then for both (i) and (ii), we can see easily that Z has k distinct elements and $\cos(\theta/k) = \cos\left(\frac{\theta^*}{k} + \frac{w-1}{k}\pi\right)$ is the w-th largest in Z.
(2) Suppose $\theta^* = 0$. Then Z has $\left\lceil \frac{k+1}{2} \right\rceil$ distinct elements and $\cos(\theta/k)$ is the $\left\lfloor \frac{w+1}{2} \right\rfloor$-th largest in Z.

(B) Suppose $\left\lfloor \frac{n}{k} \right\rfloor$ is odd. Then we can write $n = (2N+1)k + (k-w)$ for some $N = 0, 1, 2, \ldots$ Here, $\left\lfloor \frac{n}{k} \right\rfloor = 2N + 1$. Now

$$\begin{aligned} \theta/k &= \frac{\theta^* + n\pi}{k} \\ &= \frac{\theta^* + \left(\left\lfloor \frac{n}{k} \right\rfloor k + (k-w)\right)\pi}{k} \\ &= \left(\frac{\theta^*}{k} + \frac{k-w}{k}\pi\right) + \left\lfloor \frac{n}{k} \right\rfloor \pi. \end{aligned}$$

Since $0 \leq \frac{\theta^*}{k} < \frac{\pi}{k}$ and $0 \leq \frac{k-w}{k}\pi \leq \frac{k-1}{k}\pi$, we have $0 \leq \frac{\theta^*}{k} + \frac{k-w}{k}\pi < \pi$. Thus $n' = \left\lfloor \frac{n}{k} \right\rfloor$.

Again by considering the two cases (i′) even w and (ii′) odd w, we can also show the same result as when $\left\lfloor \frac{n}{k} \right\rfloor$ is even.

Now, clearly, the point $(\cos(\theta/k), \sin(\theta/k))$ is in the upper (resp., lower) half plane, if n' is even (resp., odd), which implies the last statement. $\square$

Using Sturm sequences, we can now easily identify the appropriate zero of (7) as $\cos(\theta/k)$. We can also compute $\sin(\theta/k) = \pm\sqrt{1-\cos^2(\theta/k)}$, since we know the sign of $\sin(\theta/k)$. Finally, taking $p' = p$, we can compute

$$q' = o(C) + \begin{bmatrix} \cos(\theta/k) & -\sin(\theta/k) \\ \sin(\theta/k) & \cos(\theta/k) \end{bmatrix} \cdot (p - o(C)).$$

3.5 Multiplication by an Integer

Again consider the problem of computing $A' = \gamma \cdot A$ in (4) where γ is an integer $m \geq 2$. This involves Chebyshev polynomials $U_k(x)$ of the second kind. These polynomials are slightly harder to deal with than Chebyshev polynomials of the first kind (e.g., there is no direct recursive definition of $U_k(x)$ analogous to (6)). Details are omitted in this abstract.

REMARK: For the purposes of achieving our main result, it is enough to have division by an integer (see below). Multiplication by an integer is not logically necessary.

3.6 Scaling

Let $k \geq 2$ be an integer and $A = [C, p, q, n]$. The operation of **scaling** A **by** k produces a directed arc $A'' = [C'', p'', q'', n'']$ such that $\mathrm{val}(A) = \mathrm{val}(A'')$ and $r(C'') = k \cdot r(C)$.

Thus, scaling (like rotation) does not change the value of a directed arc. To scale A by k, we first divide A by k, to obtain $A' = (1/k)A = [C', p', q', n']$. Now, C'' is obtained from $C' = C$ by increasing its radius to $k \cdot r(C)$ but keeping the center $o(C'') = o(C')$. We compute p'', q'' to be the intersection of the rays $\overrightarrow{o(C')p'}$ and $\overrightarrow{o(C'),q'}$ with C''. Finally, let $n'' = kn'$.

3.7 Unrestricted Addition

Let $A = [C, p, q, n]$ and $A' = [C', p', q', n']$ where their radii are commensurable, $r(C) = (k/m)r(C')$ for some rational k/m. To perform unrestricted addition $A + A'$, we first scale A by m, and scale A' by k. Now we can perform restricted addition on the result.

Summarizing the above constructions:

THEOREM 6. *Let $A = [C, p, q, n]$, $A' = [C', p', q', n']$ be algebraic, where $r(C)/r(C')$ is rational. The following operations are computable, by a reduction to algebraic number operations:*
(1) Scalar multiplication $(m/k)A$ where m/k is rational.
(2) Addition and subtraction $A \pm A'$.

4. COMPUTABILITY OF SHORTEST PATH AND ITS COMPLEXITY

It is now possible to implement Djikstra's algorithm on a Turing machine, and obtain our key theorem:

THEOREM 7. *The disc shortest path problem is EGC-computable for inputs with commensurable radii.*

Proof. For feasible paths μ, μ', the comparison $d(\mu) : d(\mu')$ is reduced determining the sign of a number $x = \alpha + \mathrm{val}(A)$ where α, A are algebraic. As before, $x = 0$ iff $\alpha = 0$ and $\mathrm{val}(A) = 0$. The latter is decidable by Lemma 4. Suppose x is non-zero. Now we appeal to the ability to approximate α and $\mathrm{val}(A)$ to within any desired absolute error $\varepsilon > 0$. For $i = 1, 2, \ldots$, we compute approximations $\widetilde{\alpha}, \widetilde{\mathrm{val}(A)}$ such that $|\alpha - \widetilde{\alpha}| \leq 2^{-i}$ and $|\mathrm{val}(A) - \widetilde{\mathrm{val}(A)}| \leq 2^{-i}$. We halt with the right decision when $|\widetilde{\alpha} - \widetilde{\mathrm{val}(A)}| > 2^{1-i}$. Halting of this procedure is guaranteed. **Q.E.D.**

We can consider a related decision problem: "Is the shortest path less than α?" where α is any given algebraic number. The above proof is easily modified to show the decidability of this question.

This computability result has no complexity bound. To this end, we need effective lower bounds on quantities of the form $|x| = |\alpha + \theta|$ where α and $\cos\theta$ are real algebraic numbers. Many results in transcendental number theory are non-effective. Fortunately for our application, we could adapt known bounds from transcendental number theory.

We need some definitions. For an algebraic number $\alpha \in \mathbb{C}$, let $m = \deg(\alpha)$ be the degree of α over $\mathbb{Q}$. If $p(x) = \sum_{i=0}^m a_i x^i \in \mathbb{Z}[x]$ is the minimal polynomial of α, then we denote by $M(\alpha) = |a_m| \prod_{i=1}^m \max\{1, |\alpha_i|\}$ the **Mahler measure** of α, where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_m$ are all the conjugates of α. The **height** of α is $\max_{i=0}^m |a_i|$. The **absolute logarithmic height** of α is given by

$$h(\alpha) = \frac{1}{\deg(\alpha)} \log M(\alpha).$$

The bound we need comes from:

THEOREM 8 (WALDSCHMIDT [16]). *Let $\alpha, \beta \in \mathbb{C}$ be nonzero algebraic numbers, and let $\log\beta$ be any determination of the logarithm of β. Assume*

$$D \geq [\mathbb{Q}(\alpha,\beta) : \mathbb{Q}], \qquad V \geq \max\{h(\beta), |\log\beta|/D, 1/D\},$$
$$V^+ = \max\{V, 1\}, \qquad 1 < E \leq \min\{e^{DV}, 4DV/|\log\beta|\}.$$

Then we have

$$|\alpha + \log\beta| > \exp\{-2^{35} D^3 V (h(\alpha) + \log(EDV^+))(\log(ED))(\log E)^{-2}\}.$$

COROLLARY 9. *Let α, $\theta \in \mathbb{C}$ be such that $\alpha, \cos\theta$ are nonzero algebraic numbers. Then*

$$|\alpha + \theta| > \exp\{-2^{35} D^3 V (h(\alpha) + \log(EDV^+))(\log(ED))(\log E)^{-2}\}.$$

where

$$D \geq [\mathbb{Q}(\mathbf{i}\alpha, e^{\mathbf{i}\theta}) : \mathbb{Q}], \qquad V \geq \max\{h(e^{\mathbf{i}\theta}), |\theta|/D, 1/D\},$$
$$V^+ = \max\{V, 1\}, \qquad 1 < E \leq \min\{e^{DV}, 4DV/|\theta|\}.$$

Proof. Note that $\cos\theta$ is algebraic if and only if $e^{\mathbf{i}\theta}$ is algebraic. Now choose α and β in the last theorem to be $\mathbf{i}\alpha$ and $e^{\mathbf{i}\theta}$. **Q.E.D.**

We will need several elementary bounds:

LEMMA 10. *Let α, β be nonzero and algebraic over $\mathbb{Q}$. Then $\deg(\zeta) \leq \deg(\alpha)\deg(\beta)$ for any $\zeta \in \mathbb{Q}[\alpha, \beta]$.*

Proof. Suppose $\deg(\alpha) = m$, $\deg(\beta) = n$ and v is the vector

$$(1, \alpha, \ldots, \alpha^{m-1}, \beta, \beta\alpha, \ldots, \beta\alpha^{m-1}, \ldots, \beta^{n-1}, \beta^{n-1}\alpha, \ldots, \beta^{n-1}\alpha^{m-1}).$$

Then given $\zeta \in \mathbb{Q}[\alpha, \beta]$, there exists $mn \times mn$ matrix $M \in \mathbb{Q}^{mn\times mn}$ such that $\zeta v = Mv$. Thus ζ is a zero of a polynomial $\det(M - xI) = 0$. **Q.E.D.**

LEMMA 11. *If* $\deg(\alpha) = m$ *and* $\deg(\cos\theta) = n$, *then* $D = [\mathbb{Q}(\mathbf{i}\alpha, e^{\mathbf{i}\theta}) : \mathbb{Q}] \leq 4mn^2$.

Proof. Note that $\deg(\sin\theta) \leq 2n$ since $\sin\theta = \sqrt{1 - \cos^2\theta}$. Hence

$$D \leq [\mathbb{Q}(\mathbf{i}, \alpha, \cos\theta, \sin\theta) : \mathbb{Q}].$$

Using the previous lemma, this is at most $\deg(\mathbf{i})\deg(\alpha)\deg(\cos\theta))\deg(\sin\theta)) = 2mn(2n) = 4mn^2$.

Q.E.D.

LEMMA 12 ([17]). *Let* $\alpha_1, \ldots, \alpha_n$ *be algebraic and* $k \in \mathbb{Z}$ *be nonzero. Then*

(1) $h(\alpha_1\alpha_2) \leq h(\alpha_1) + h(\alpha_2)$,

(2) $h(\alpha_1^k) = kh(\alpha_1)$,

(3) $h(\alpha_1 + \cdots + \alpha_n) \leq h(\alpha_1) + \cdots + h(\alpha_n) + \log n$.

COROLLARY 13. $h(e^{\mathbf{i}\theta}) \leq 2h(\cos\theta) + \frac{3}{2}\log 2$.

Proof.

$$\begin{aligned} h(e^{\mathbf{i}\theta}) &\leq h(\cos\theta) + h(\sin\theta) + \log 2 \\ &= h(\cos\theta) + \frac{1}{2}h(1 - \cos^2\theta) + \log 2 \\ &\leq h(\cos\theta) + \frac{1}{2}(h(\cos^2\theta) + \log 2) + \log 2 \\ &= 2h(\cos\theta) + \frac{3}{2}\log 2 \end{aligned}$$

Q.E.D.

To compute an explicit bound using the above results, we assume that all inputs are L-bit rational numbers, i.e., numbers of the form P/Q for integers P, Q with $|P|, |Q| < 2^L$. We have to bound (from below)

$$|\alpha + \theta| = \left|\sum_{k=1}^{m} \alpha_k + \sum_{k=1}^{m} r_k\theta_k\right|.$$

Here $m \leq 2N$, where N is the number of the discs. (Remember we have to compare two paths.) Here α_k, $\cos\theta_k$, and $\cos\theta$ are algebraic numbers. Let A be an upper bound of the degree of every α_k and $\cos\theta_k$. Similarly, let $B \cdot L$ be an upper bound of the absolute logarithmic height of every α_k and $\cos\theta_k$. Both A and B are constants not depending on L and N.

We need to bound $\deg(\alpha)$, $\deg(e^{\mathbf{i}\theta})$ and $h(e^{\mathbf{i}\theta})$. Note that

$$e^{\mathbf{i}\theta} = \prod_{k=1}^{m} \left(e^{\mathbf{i}\theta_k}\right)^{r_k}.$$

For each k, we have $\deg(e^{\mathbf{i}\theta_k}) \leq \deg(\cos\theta_k)\deg(\mathbf{i})\deg(\sin\theta_k)$ which is $\leq 4A^2$. We can write $r_k = P_k/Q_k$ for some (positive) integers $P_k, Q_k \leq 2^L$. Now

$$\begin{aligned} \deg(e^{\mathbf{i}r_k\theta_k}) &= \deg(e^{\mathbf{i}\frac{P_k}{Q_k}\theta_k}) \leq \deg(e^{\mathbf{i}\frac{\theta_k}{Q_k}}) \\ &\leq \quad Q_k \deg(e^{\mathbf{i}\theta_k}) \leq 2^L \deg(e^{\mathbf{i}\theta_k}) \leq 2^{L+2}A^2. \end{aligned}$$

Also, $h(e^{\mathbf{i}\theta_k}) \leq 2h(\cos\theta_k) + \frac{3}{2}\log 2 \leq 2BL + 1$, and thus $h(e^{\mathbf{i}r_k\theta_k}) = r_k h(e^{\mathbf{i}\theta_k}) \leq r_k(2BL + 1) \leq 2^L(2BL + 1)$. Therefore,

$$\begin{aligned} &\deg(e^{\mathbf{i}\theta}) \leq 2^{m(L+2)}A^{2m} \leq 2^{2N(L+2)}A^{4N}, \\ &h(e^{\mathbf{i}\theta}) \leq m2^L(2BL + 1)) \leq N2^{L+1}(2BL + 1). \end{aligned}$$

Similarly, $\deg(\alpha) \leq A^m \leq A^{2N}$ and $h(\alpha) \leq m \cdot BL + \log m \leq 2B \cdot LN + \log(2N)$.

Now we apply Corollary 9. Asymptotically, $D \leq 4 \cdot \deg(\alpha)\deg(\cos\theta)^2 \leq 4\deg(\alpha)\deg(e^{\mathbf{i}\theta})^2 \leq 2^{4N(L+2)+2}A^{10N}$, $V \leq N2^{L+1}(2BL+1)$, $E \geq 4DV/(2\pi \cdot 2N) \geq DV/(4N)$ and $V^+ = V$. Also

$$\begin{aligned} &(h(\alpha) + \log(EDV^+))(\log(ED))(\log E)^{-2} \\ &= \left(1 + \frac{\log(DV)}{\log E} + \frac{h(\alpha)}{\log E}\right)\left(1 + \frac{\log D}{\log E}\right) \leq C' \end{aligned}$$

and $D^3V \leq 2^{12N(L+2)+6}A^{30N} \cdot 2^{L+1}N(2BL+1) \leq LN \cdot (C'')^{LN}$ for some constants C', C''. These eventually give us:

$$|\alpha + \theta| > \exp(-C' \cdot D^3V) \geq \exp(-LN \cdot C^{LN}).$$

for some constant C and for large L and N. Thus:

THEOREM 14. *The number of digits we need to expand to compare the lengths of two paths is* $LN \cdot 2^{O(LN)}$, *where input is comprised of* L*-bit rational numbers and* N *is the number of the discs.*

We can deduce from this theorem that the complexity of the shortest path problem for N discs for inputs described by L-bit rational numbers is in single exponential time in L and N.

We can generalize the computability result in Theorem 7:

THEOREM 15. *The disc shortest path problem is computable for arbitrary algebraic inputs.*

In other words, we do not need commensurability of the input radii. To achieve this result, we invoke the general version of Baker's theorem on linear form in logarithms. We quote a simple version of this theorem [1, Theorem 3.1]: Let

$$\Lambda = \beta_0 + \beta_1 \log\alpha_1 + \cdots + \beta_n \log\alpha_n$$

where $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers of degree at most d and heights at most A, and where $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers with degrees at most d and heights at most $B \geq 2$. Then $\Lambda \neq 0$ implies

$$|\Lambda| > B^{-C}$$

where C is effectively computable from n, d, A and the determinations of the logarithms.

To apply this, we view the θ_i's in (2) to be suitable determinations of logarithms. To determine the sign of an expression of the form (2), where α and the θ_i's can be negative, we just need to approximate the expression with an absolute error less than $B^{-C}/2$. Indeed, this result can be achieved even without exploiting our directed arc representation, possibly at a huge complexity cost (the value of n in Baker's theorem can be up to twice the number of discs in the input). The determination of this complexity bound will be taken up in the final pape.

REMARK: While Theorem 7 is proved by elementary means (the only result from transcendental number theory is Lindemann's theorem), Theorem 15 requires a heavy weight theorem on linear forms in logarithms, whose complexity is less understood.

5. WHAT ANGLES HAVE ALGEBRAIC COSINES?

What can we say about θ if $\cos\theta$ is algebraic? Observe that for "standard angles", θ/π is rational. For instance, $\arccos(1) = 0$, $\arccos(\sqrt{3}/2) = \pi/12$, $\arccos(1/2) = \pi/6$, $\arccos(1/\sqrt{2}) = \pi/8$, $\arccos(0) = \pi/4$, etc. Jahnel [9] investigates the nature of $\cos\theta$ when θ/π is rational. These examples suggests the following:

$$\text{If } \cos\theta \text{ is algebraic, then } \theta/\pi \text{ is algebraic.} \tag{9}$$

If (9) is true, then the shortest path problem for arbitrary radii would be easy, because every feasible path length would have the form $\alpha + \pi\beta$ where α, β are algebraic. In support of (9), it is not hard to extend the "standard angles" to a countably infinite family of angles θ which satisfy the hypothesis and conclusion of (9). Unfortunately, (9) is falsified by the following argument.

1. First recall the Gelfond-Schneider theorem: let $a, b \in \mathbb{C}$ such that (1) b is algebraic and b is not 0 or 1, and (2) a is algebraic and irrational. Then b^a must be transcendental.

LEMMA 16. *If $\cos\theta$ and θ/π are both algebraic, then θ/π is rational.*

Proof. If $\cos\theta$ is algebraic, then so is $e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$. By assumption, $a = \theta/\pi$ is algebraic. Hence

$$e^{\mathbf{i}\theta} = e^{\mathbf{i}\pi a} = (e^{\mathbf{i}\pi})^a = (-1)^a$$

is algebraic. By the Gelfond-Schneider theorem, we conclude that a must be rational. **Q.E.D.**

COROLLARY 17. *(9) implies $A \subseteq B$ where*

$$A = \{\alpha \in [-1,1] : \alpha \text{ is algebraic}\}$$
$$B = \{\alpha \in [-1,1] : \alpha = \cos(m\pi/k) \text{ for some } m, k \in \mathbb{N}\}$$

2. We next characterize the set B using the Chebyshev polynomials of the first kind, $T_k(x)$.

LEMMA 18. *Let $B' = \{\alpha \in [-1,1] : T_k(\alpha) = \pm 1$ for some $k \in \mathbb{N}\}$. Then $B = B'$.*

Proof. If $\alpha \in B$ then for some $k, m \in \mathbb{N}$, $\alpha = \cos(m\pi/k)$ and so $T_k(\alpha) = \cos m\pi = \pm 1$. Thus $\alpha \in B'$. Conversely, if $\alpha \in B'$ then $T_k(\alpha) = \pm 1$ for some $k \in \mathbb{N}$. Let $\theta = \arccos(\alpha)$. Thus $T_k(\cos\theta) = \cos k\theta = \pm 1$. Thus $k\theta = m\pi$ for some $m \in \mathbb{N}$. This shows $\theta = m\pi/k$, and $\alpha = \cos\theta \in B$. **Q.E.D.**

3. Finally, we show

LEMMA 19. *The set B is a properly contained in A.*

Proof. If $\alpha \in B$ then $\alpha = \cos m\pi/k$. But $e^{\mathbf{i}\pi m/k}$ and $e^{-\mathbf{i}\pi m/k}$ are both zeros of $x^k - (-1)^m$. Hence these are algebraic, and so is $\alpha = (e^{\mathbf{i}\pi m/k} + e^{-\mathbf{i}\pi m/k})/2$. This proves $B \subseteq A$. To show proper containment, we consider the number $\alpha = \frac{-1+\sqrt{3}}{2} \in A$. Assuming the lemma is false, we have $\alpha \in B$. The minimal polynomial of α in $\mathbb{Z}[x]$ is $f(x) = 2x^2 + 2x - 1$. It follows from Lemma 18 that α satisfies the equation

$$T_k(\alpha) = \pm 1$$

for some $k \geq 1$. Hence, $f | T_k \pm 1$ for some k. Now it is well-known that all the zeros of $T_k \pm 1$ are real numbers in $[-1,1]$. But $\frac{-1-\sqrt{3}}{2} \notin [-1,1]$ is a zero of f. This is impossible since f divides $T_n \pm 1$. **Q.E.D.**

Thus, Corollary 17 and Lemma 19 show the falsity of (9).

REMARK: It is known (e.g., [4]) that the dihedral angle of a regular tetrahedron, $\theta = 70°31'44'' = \arccos(1/3)$, is not commensurable with π. This fact also shows that the containment $B \subseteq A$ is proper.

6. FINAL REMARKS

1. We have shown the first transcendental combinatorial problem that is EGC-solvable. Which other transcendental problems can we solve? Problems arising from differential equations, optimal control, nonholonomic motion, etc, are generally transcendental.

2. As shown in this paper, transcendental number theory can yield effective tools to attack such problems. There is some hope that such tools may yield general zero separation bounds for general classes of non-algebraic expressions [20]. This would then imply the Turing solvability of general classes of transcendental problems.

3. The exponential-time complexity of shortest path for discs can probably be improved to polynomial space. It is plausible that we could get better bounds for the case of unit discs. But even here, obtaining a subexponential time complexity seems to require a major breakthrough. Another intriguing direction is to understand angles whose trigonometric functions are algebraic. A special case of this has been developed by Conway et al [4] in their theory of geodetic angles.

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