Projective Geometry

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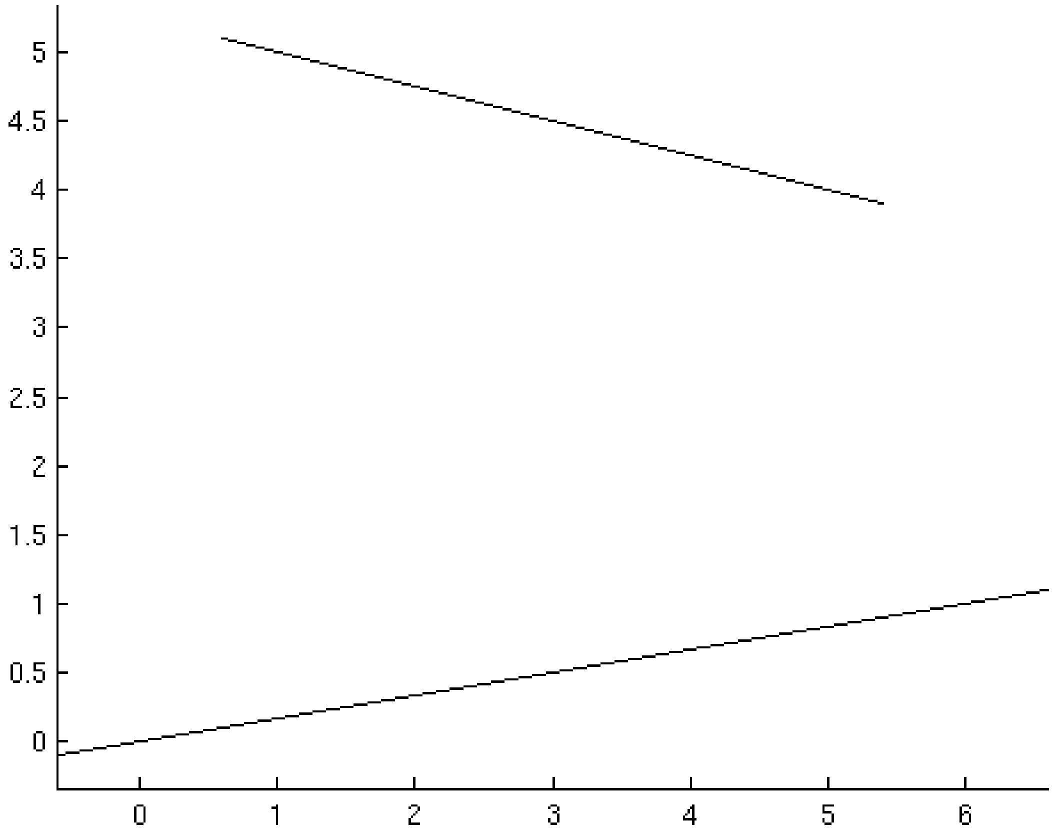
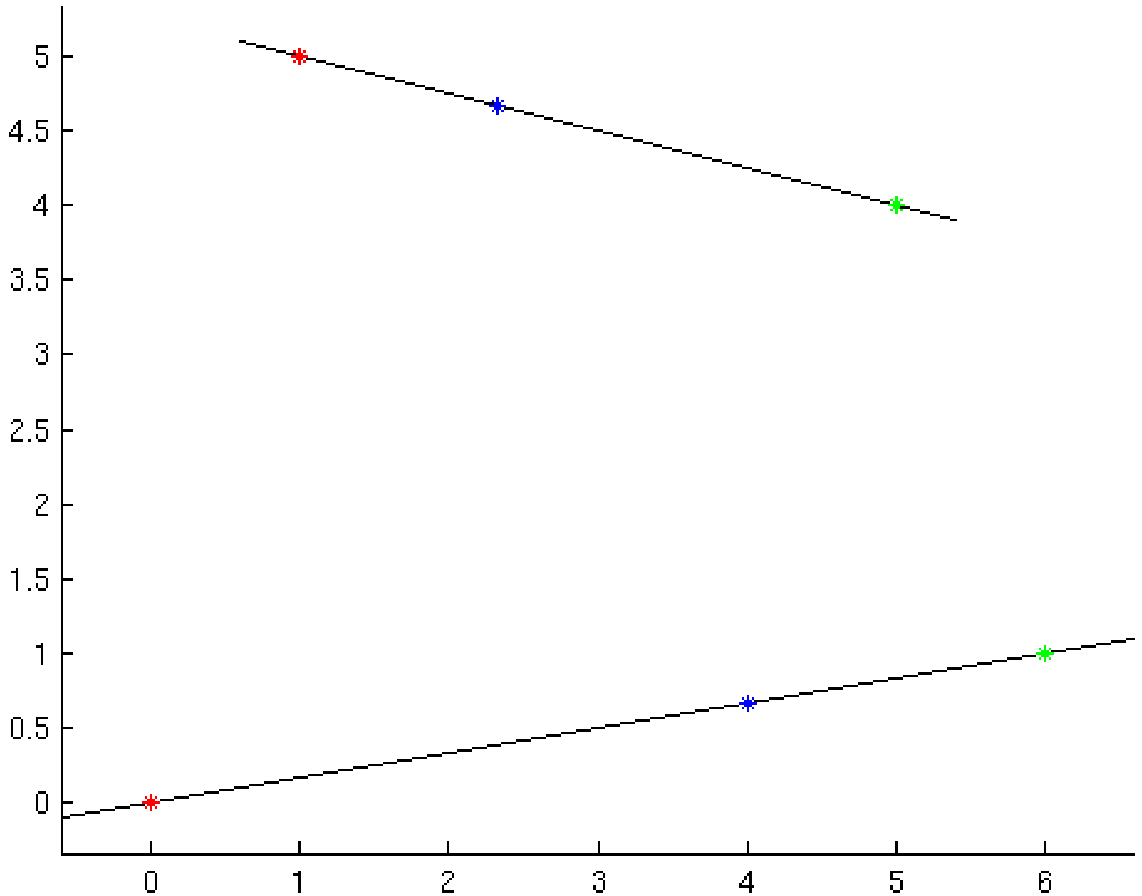
Csplash

April 26, 2014

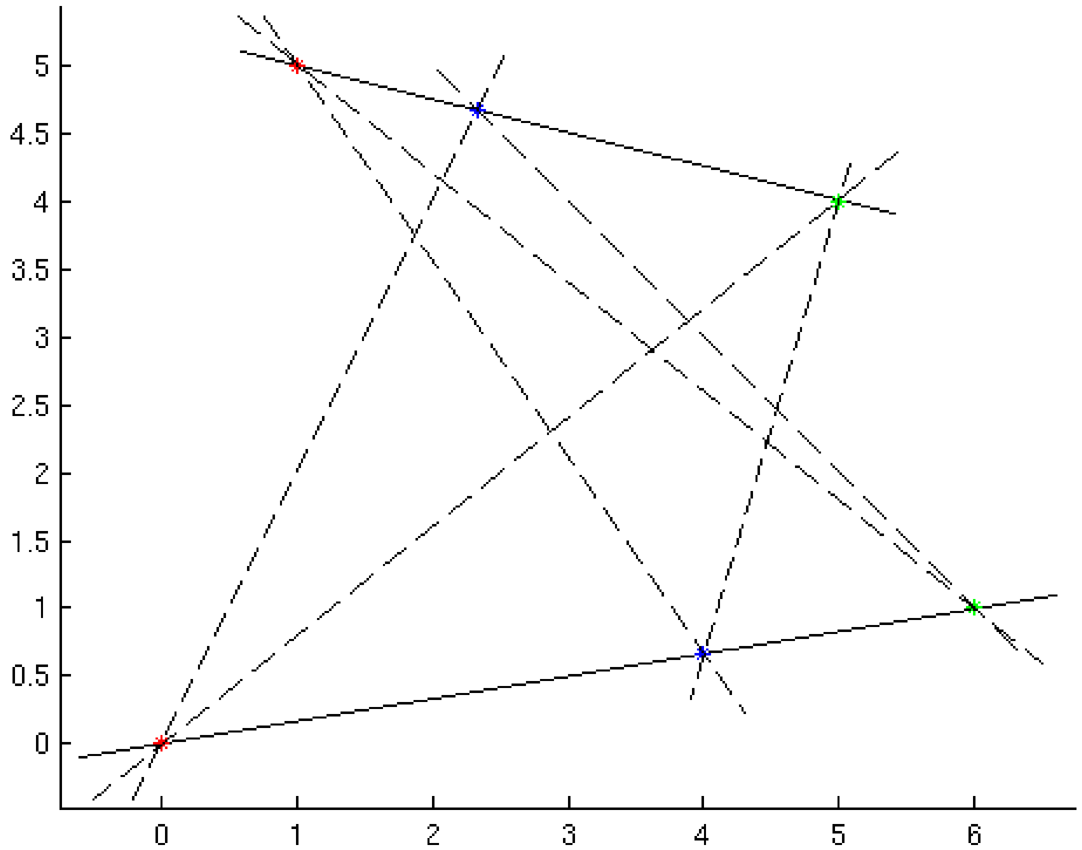
Pappus' theorem:
Draw two lines



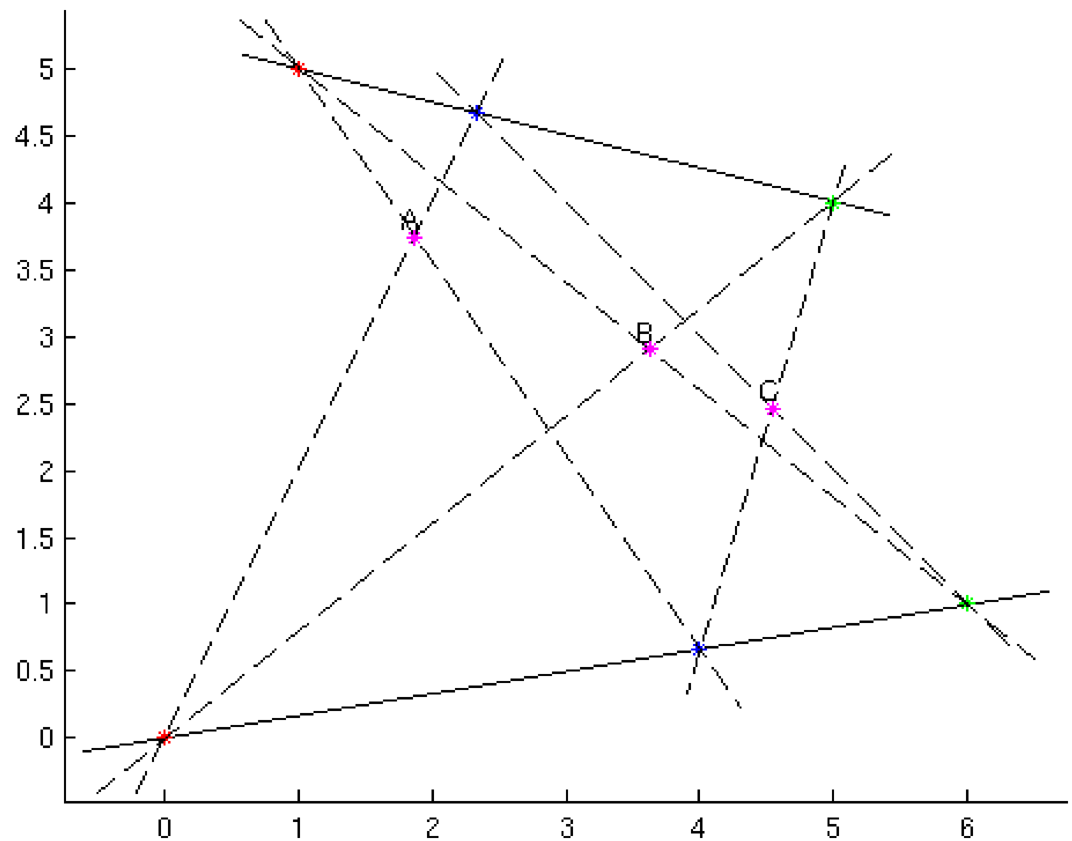
Draw red, green, and blue points on each line

- 

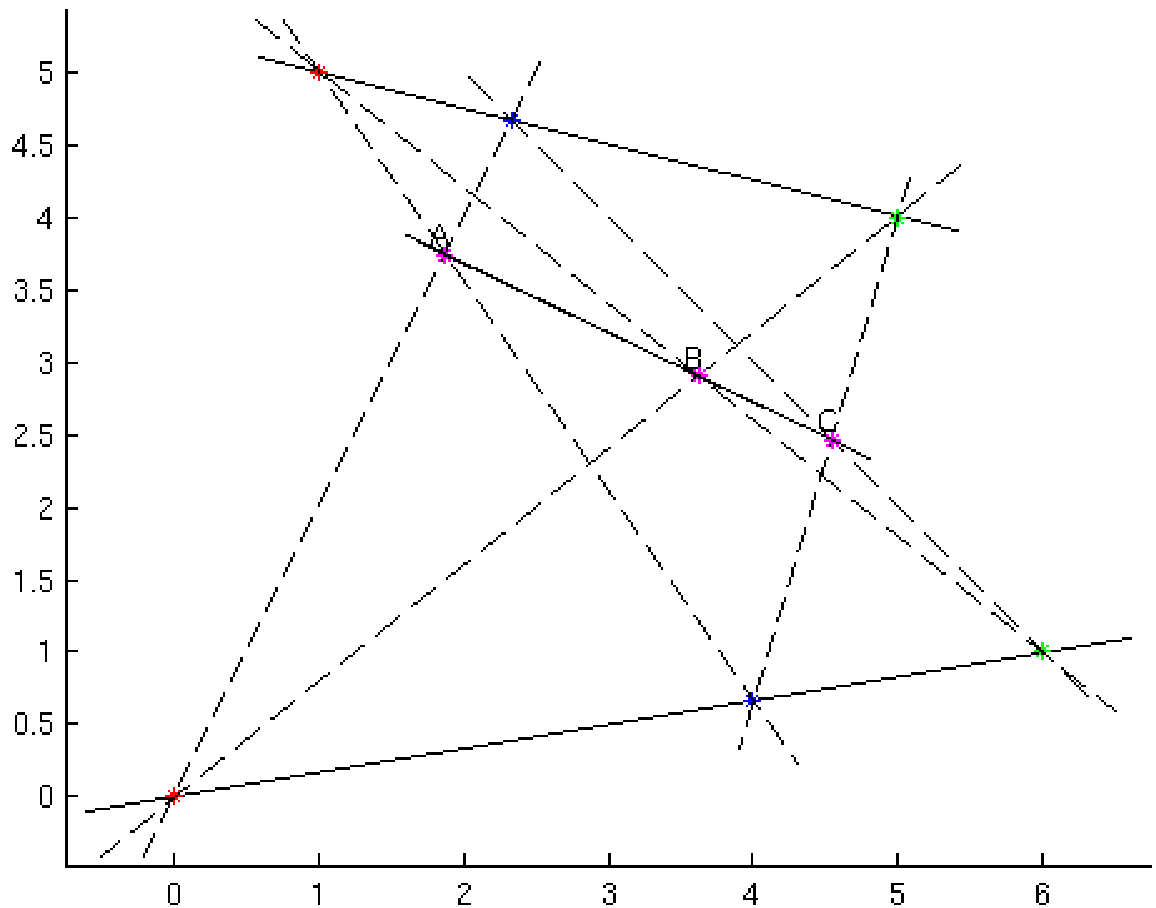
Connect all pairs of points with different colors.



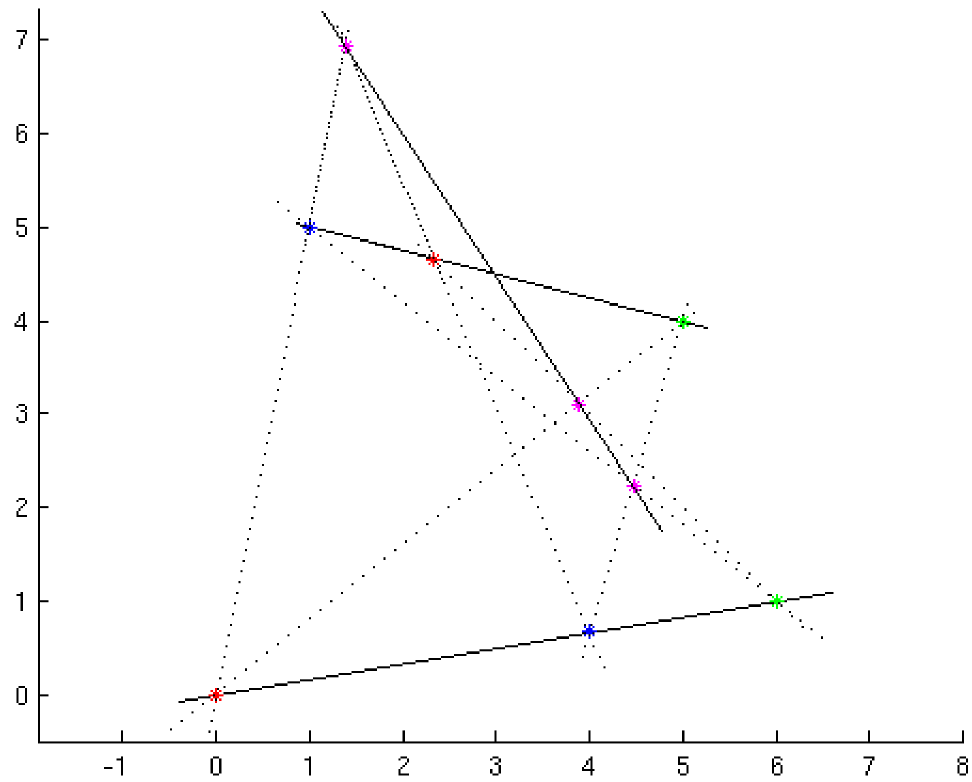
A = crossing of two red-green lines. B = crossing of red-blues. C=crossing of green-blues.



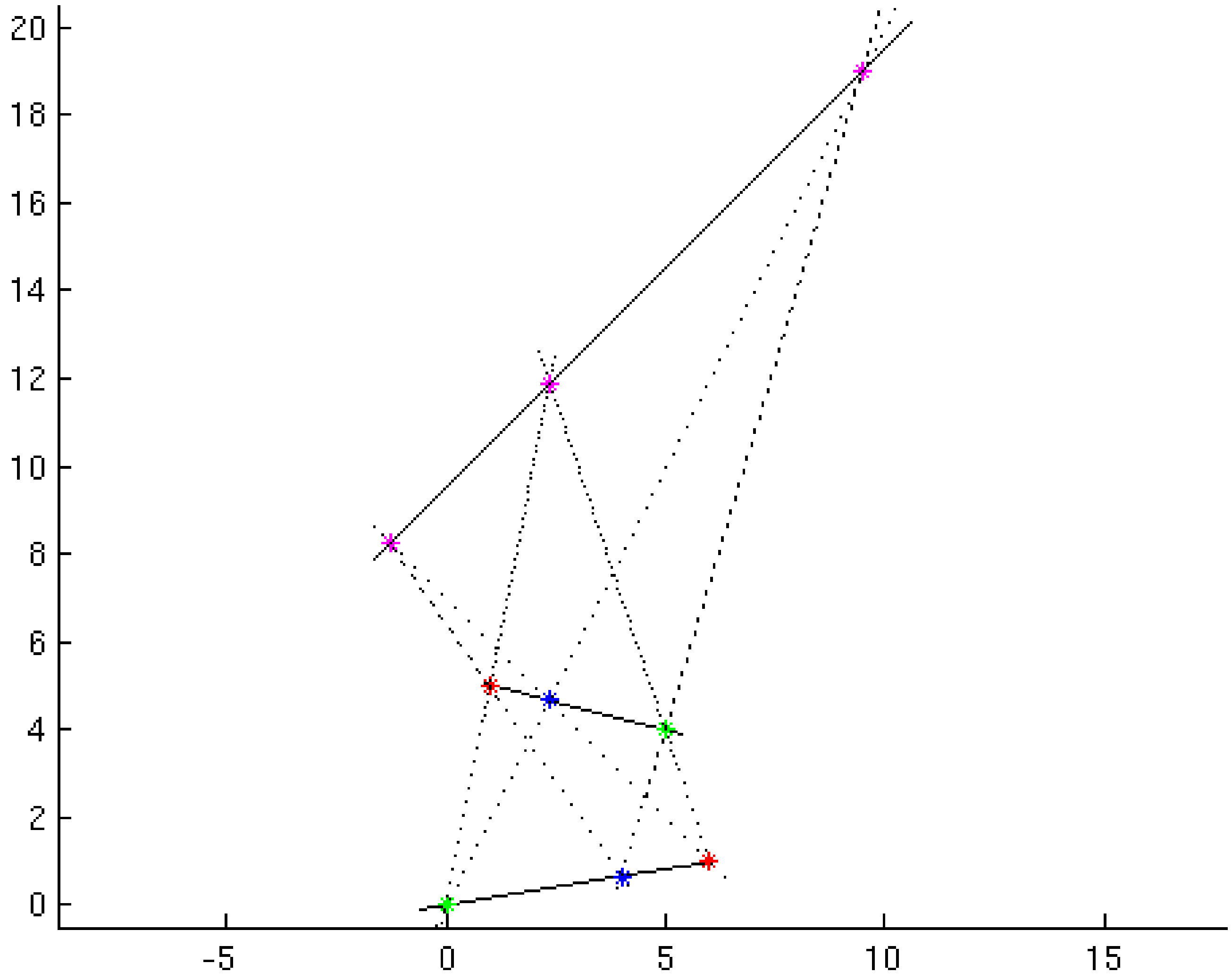
Theorem: A, B, and C are collinear.



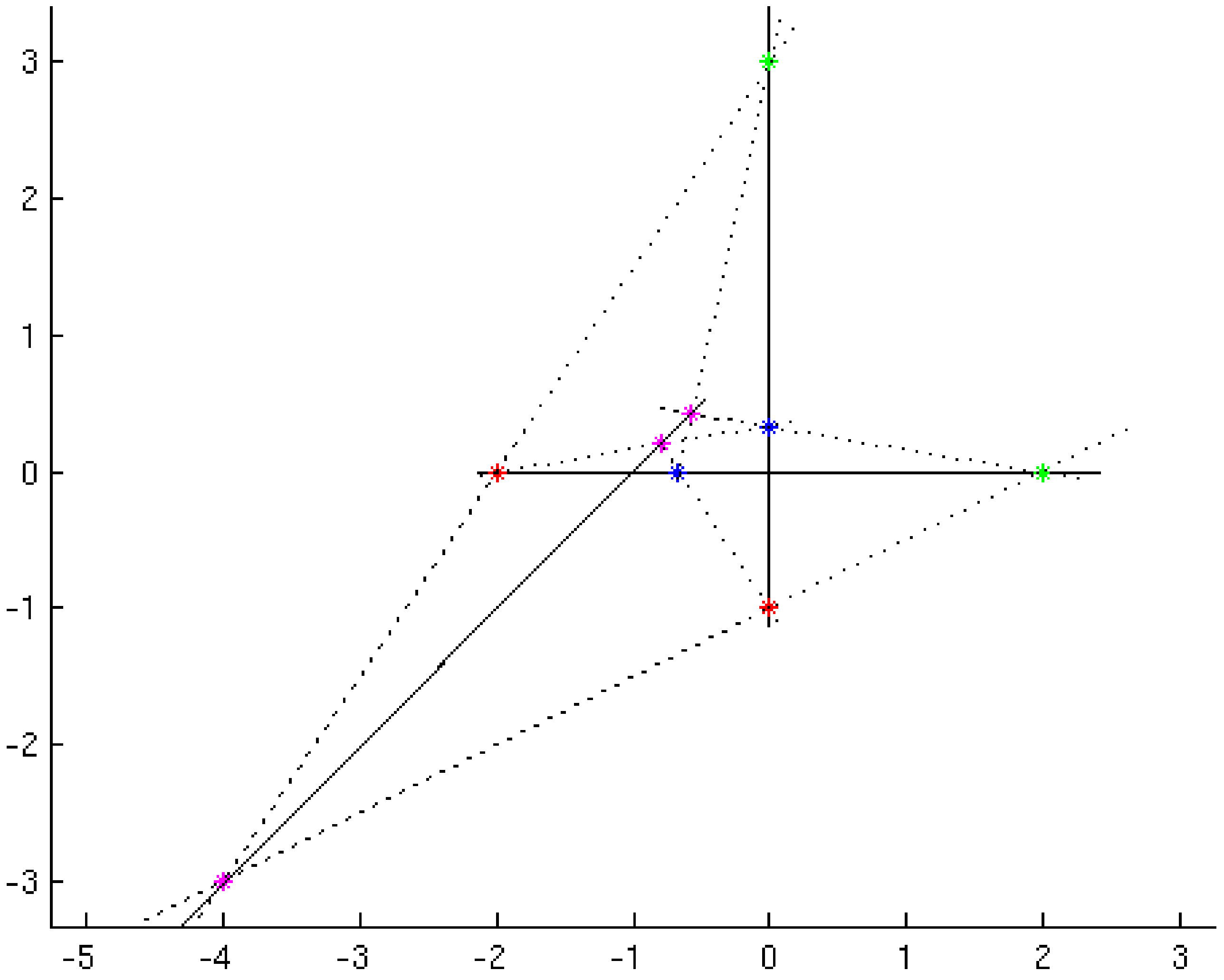
More Pappus diagrams



and more



and more



Pappus' theorem

The theorem has only to do with points lying on lines.

No distances, no angles, no right angles, no parallel lines.

You can draw it with a straight-edge with no compass.

The simplest non-trivial theorem of that kind.

Outline

- The projective plane =

  Euclidean plane + a new line of points
- Projection
  - Fundamental facts about projection
  - The projective plane fixes an bug in projection.
- Pappus’ theorem

Time permitting:

- Perspective in art
- Point/line duality

PART I: THE PROJECTIVE PLANE

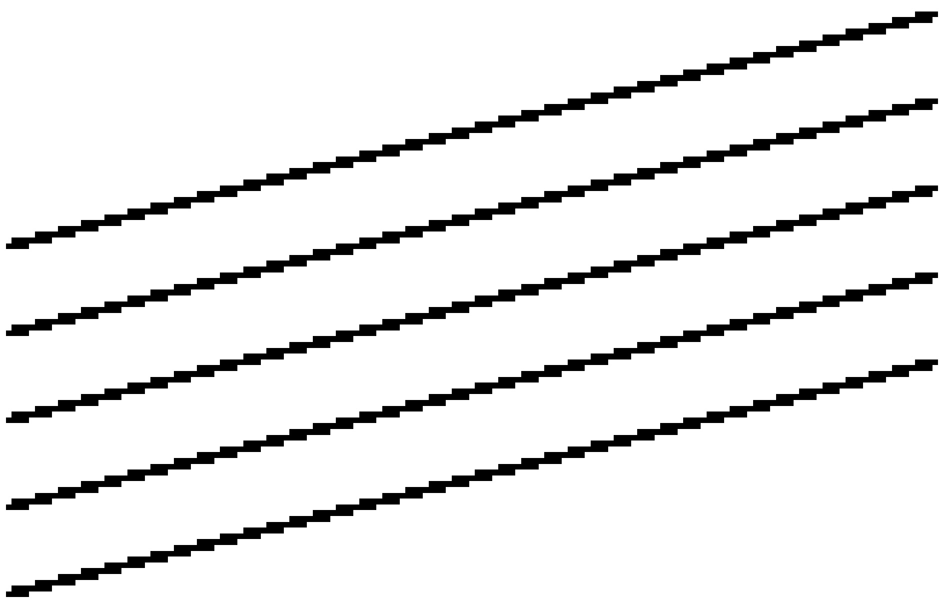
Euclidean geometry is unfair and lopsided!

- Any two points are connected by a line.
- Most pairs of lines meet in a point.
- But parallel lines don’t meet in a point!

To fix this unfairness

Definition: A *sheaf* of parallel lines is all the lines that are parallel to one another.

Obvious comment: Every line L belongs to exactly one sheaf (the set of lines parallel to L).



Projective plane

For each sheaf *S* of parallel lines, construct a new point p "at infinity". Assert that p lies on every line in *S.*

All the "points at infinity" together comprise the "line at infinity"

The projective plane is the regular plane plus the line at infinity.

Injustice overcome!

Every pair of points *U* and *V* is connected by a single line.

Case 1: If *U* and *V* are ordinary points, they are connected in the usual way.

Case 2. If U is an ordinary point and V is the point on sheaf S, then the line in S through U connects U and V.

Case 3. If U and V are points at infinity they lie on the line at infinity.

Injustice overcome (cntd)

If *L* and *M* are any two lines, then they meet at a single point.

Case 1: *L* and *M* are ordinary, non-parallel lines: as usual.

Case 2: *L* and *M* are ordinary, parallel lines: they meet at the corresponding point at infinity.

Case 3: *L* is an ordinary line and *M* is the line at infinity: they meet at the point at infinity for *L.*

Topology

As far as the projective plane is concerned, there is no particular difference between the points at infinity and ordinary points; they are all just points.

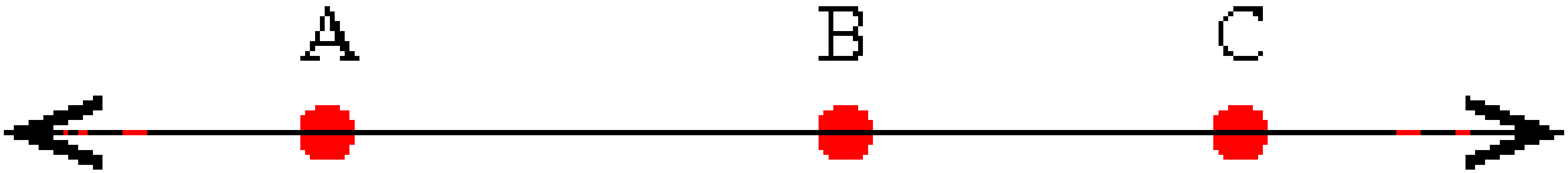
If you follow line L out to the point at infinity, and then continue, you come back on L from the other direction. (Note: there is a *single* point at infinity for each sheaf, which you get to in *both* directions.)

The price you pay

- No distances. There is no reasonable way to define the distance between two points at infinity.
- No angles

More price to pay: No idea of "between"

- B is between A and C; i.e. you can go from A to B to C.
- Or you can start B, pass C, go out to the point at infinity, and come back to A the other way. So C is between B and A.



Non-Euclidean Geometry

- The projective plane is a non-Euclidean geometry.
- (Not the famous one of Bolyai and Lobachevsky. That differs only in the parallel postulate --- less radical change in some ways, more in others.)

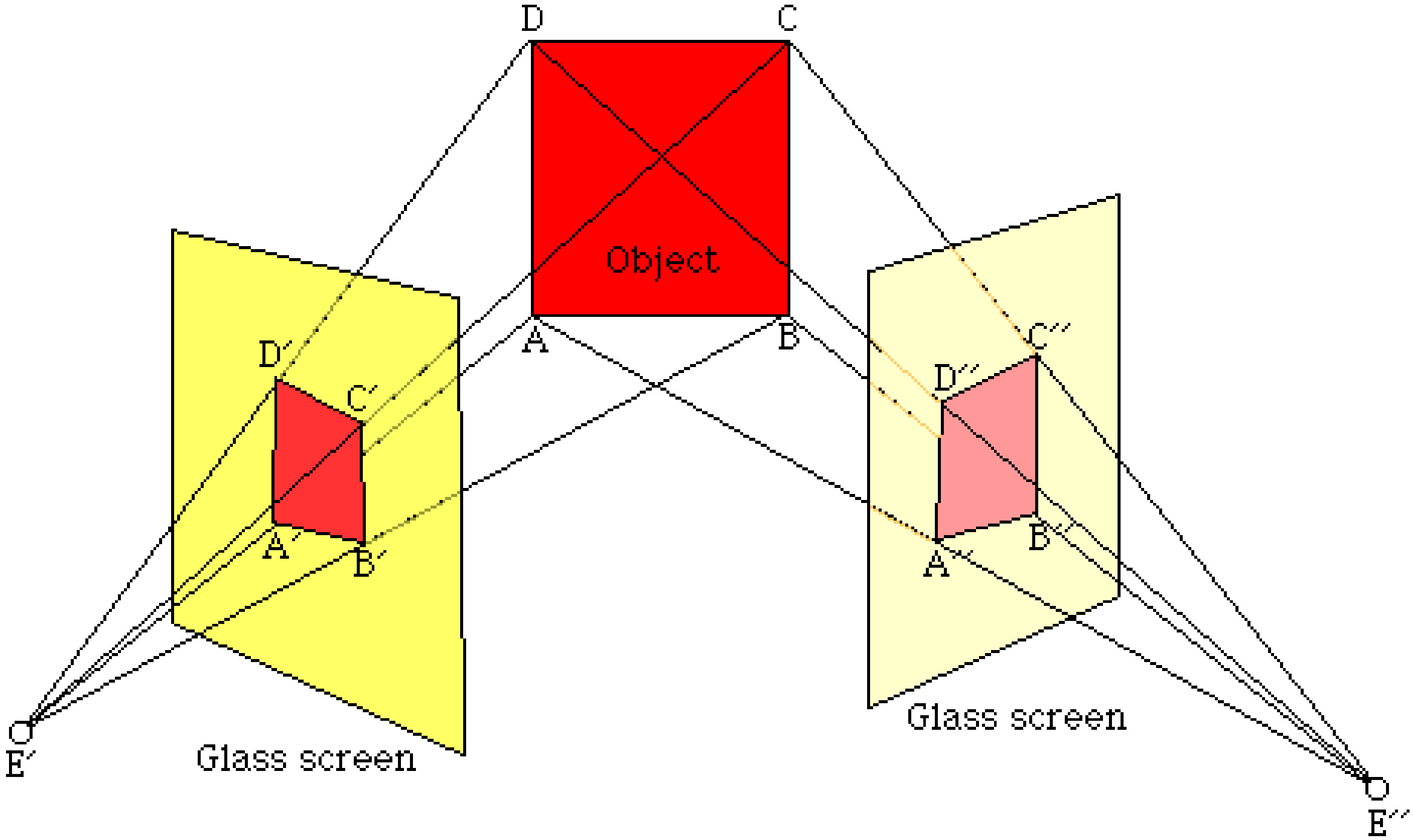
PART II: PROJECTION

Projection

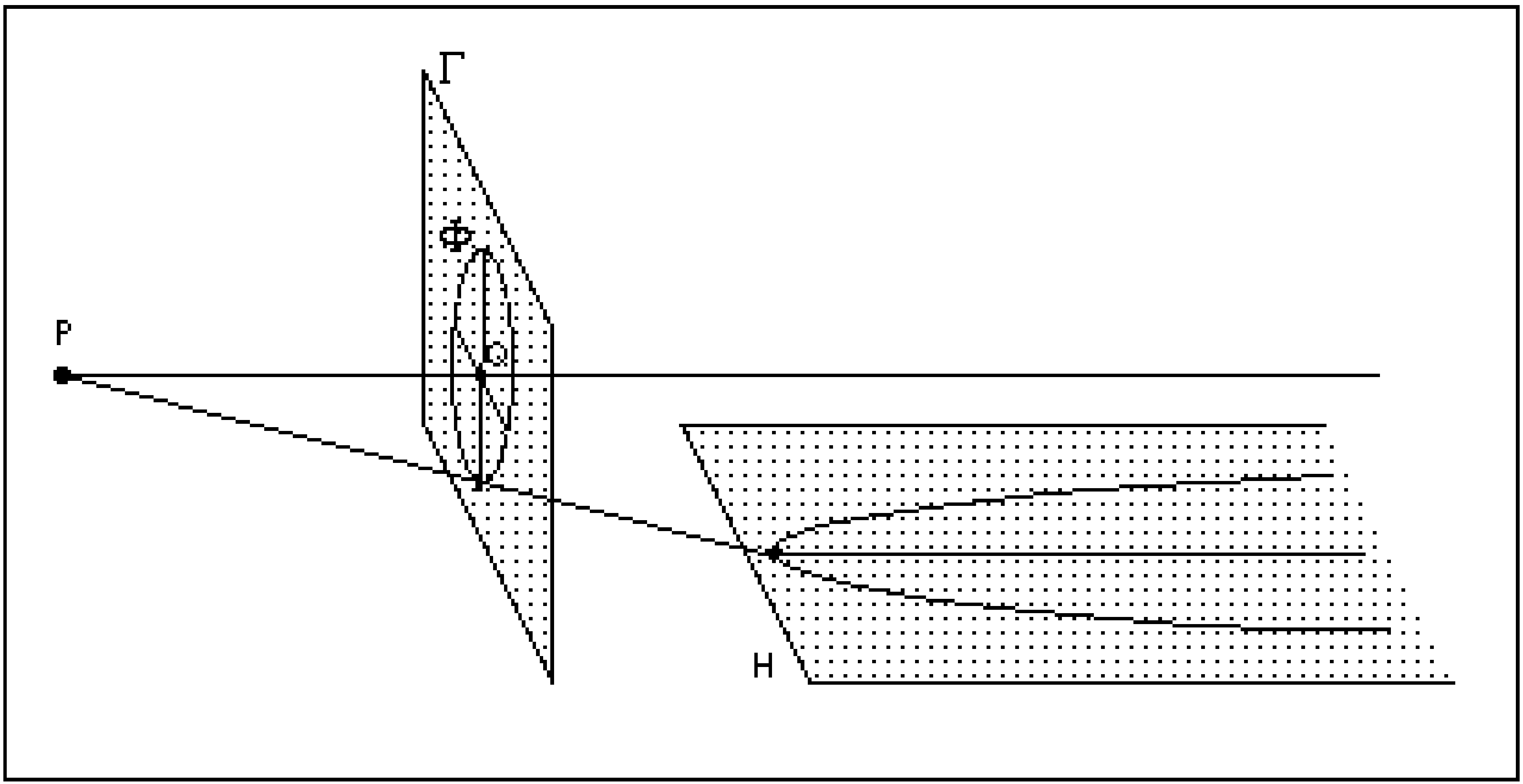
- Two planes: a *source plane* S and an *image plane* I. (Which is which doesn't matter.)
- A *focal point* f which is not on either S or I.
- For any point x in S, the *projection* of x onto I, $P_{f,I}(x)$ is the point where the line fx intersects I.

Examples

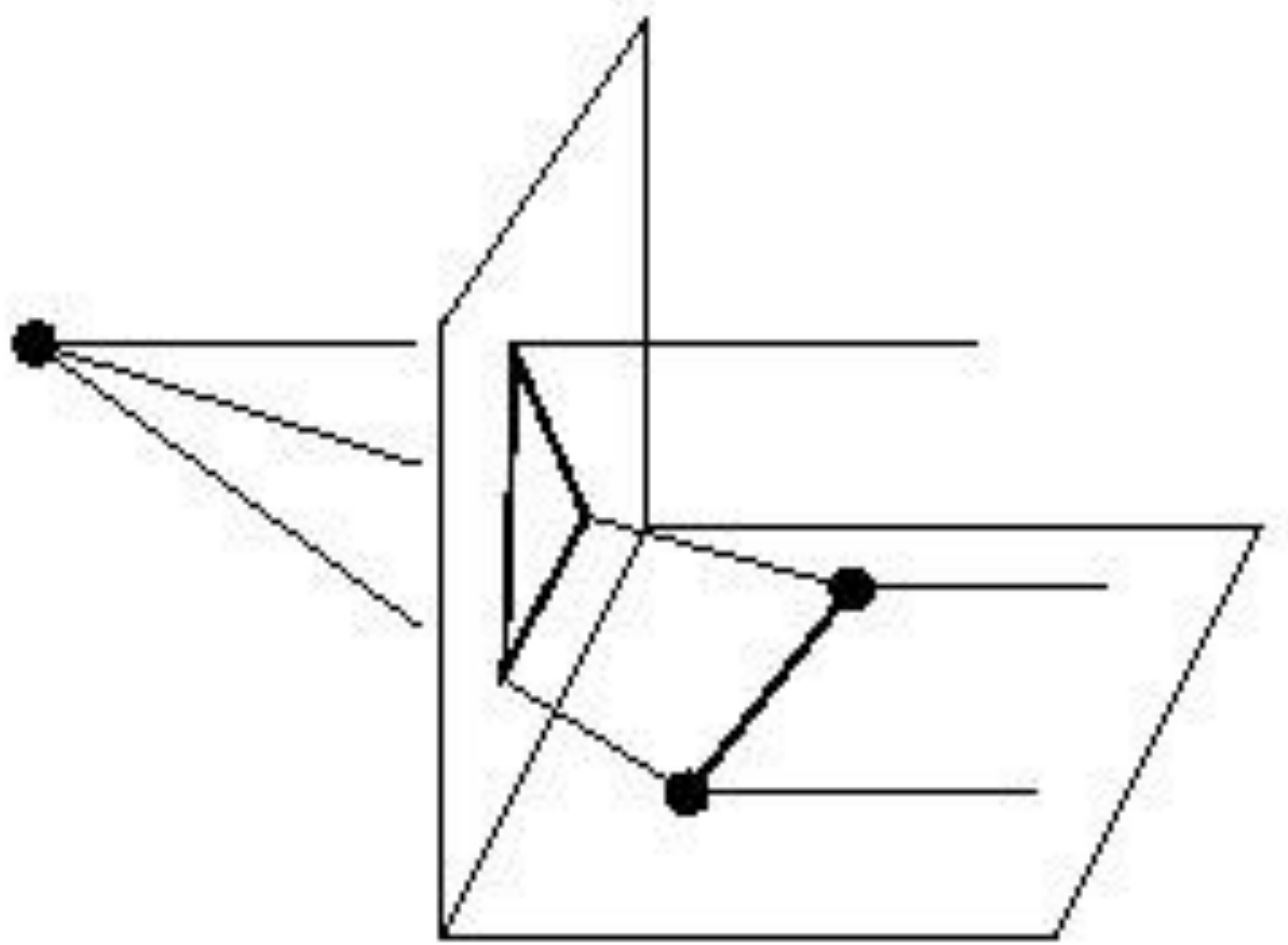
From http://www.math.utah.edu/~treiberg/Perspect/Perspect.htm

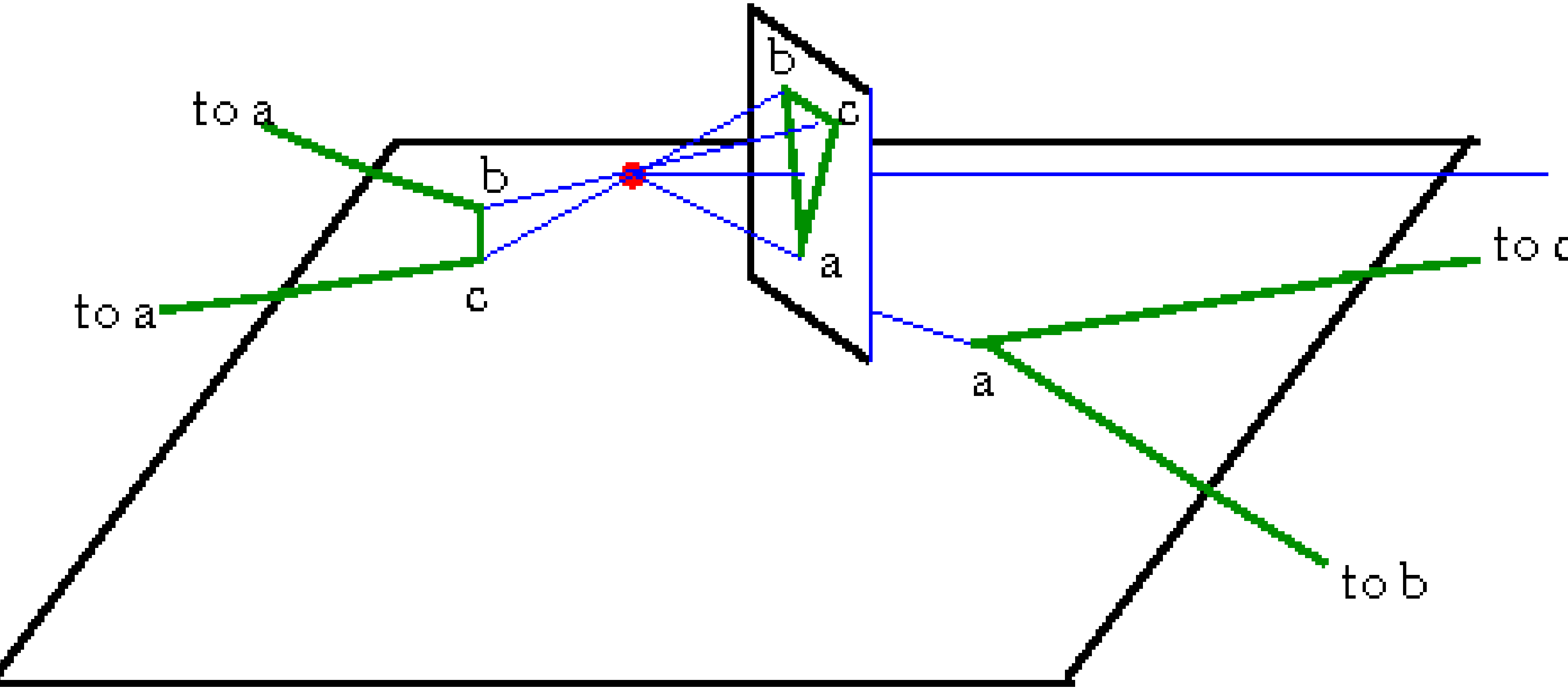


From Stanford Encyclopedia of Philosophy, “Nineteenth Century Geometry”, http://plato.stanford.edu/entries/geometry-19th/



From
http://www.math.poly.edu/~alvarez/teaching/projective-geometry/Inaugural-Lecture/page_2.html





b
c
to a
b
to a
c
a
to c
a
to b

Properties of projection

1. For any point x in S, there is at most projection $P_{f,l}(x)$.

   Proof: The line fx intersects l in at most 1 point.

2. For any point y in l, there is at most one point x in S such that $y = P_{f,l}(x)$.

   Proof: x is the point where fy intersects S.

3. If L is a line in S, then $P_{f,I}(L)$ is a line in I.

Proof: $P_{f,I}(L)$ is the intersection of I with the plane containing f and L.

4. If x is a point on line L in S, then $P_{f,I}(x)$ is a point on line $P_{f,I}(L)$.

Proof: Obviously.

Therefore, if you have a diagram of lines intersecting at points and you project it, you get a diagram of the same structure.

E.g. the projection of a Pappus diagram is another Pappus diagram.

More properties of projection

5. If S and I are not parallel, then there is one line in S which has no projection in I.

   Proof: Namely, the intersection of S with the plane through f parallel to I.

6. If S and I are not parallel, then there is one line in I which has no projection in S.

   Proof: Namely, the intersection of I with the plane through f parallel to S.

Call these the “lonely lines” in S and I.

Using the projective planes takes care of the lonely lines!

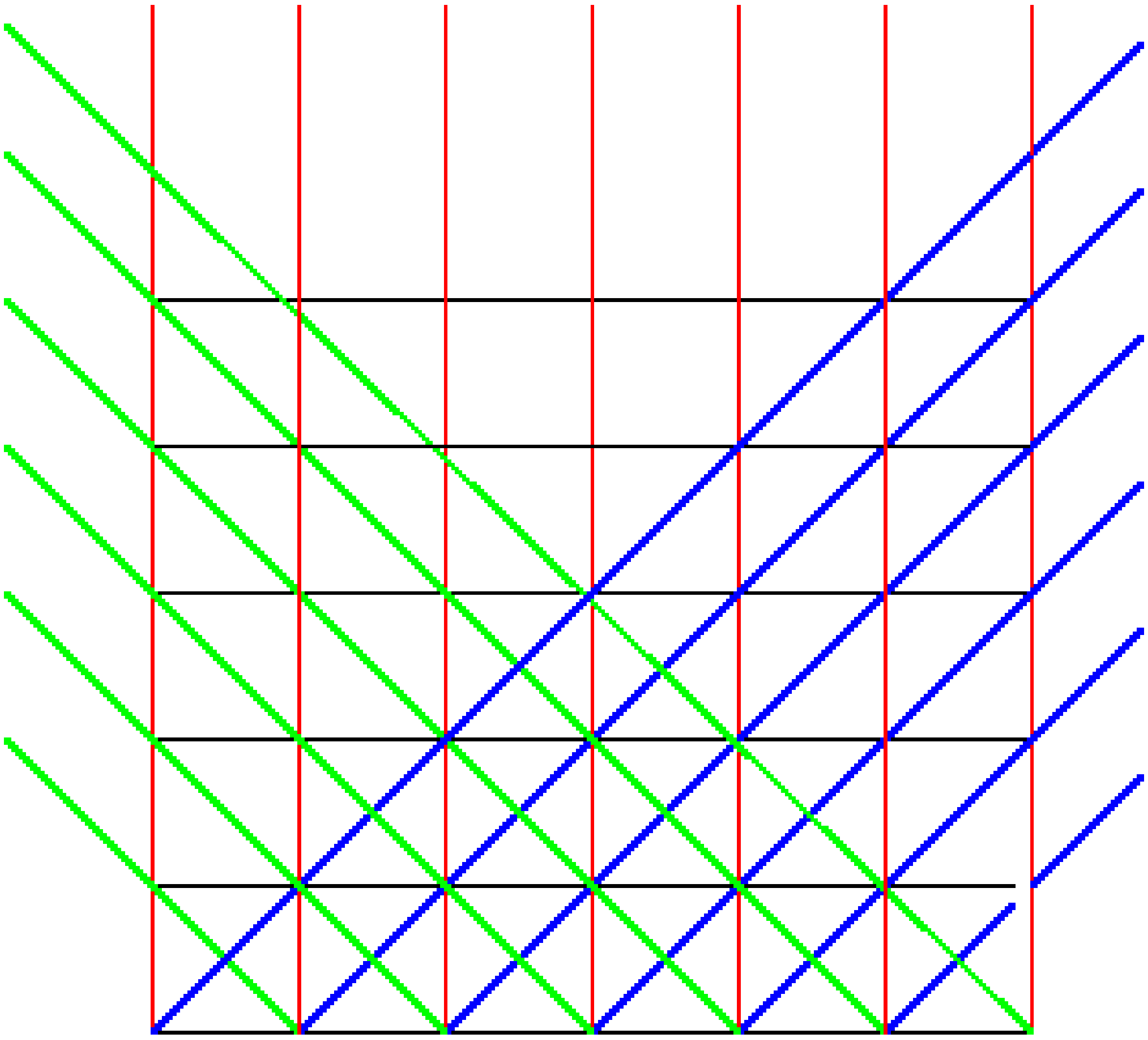
Suppose H is a sheaf in S.

The images of H in I all meet at one point h on the lonely line of I.

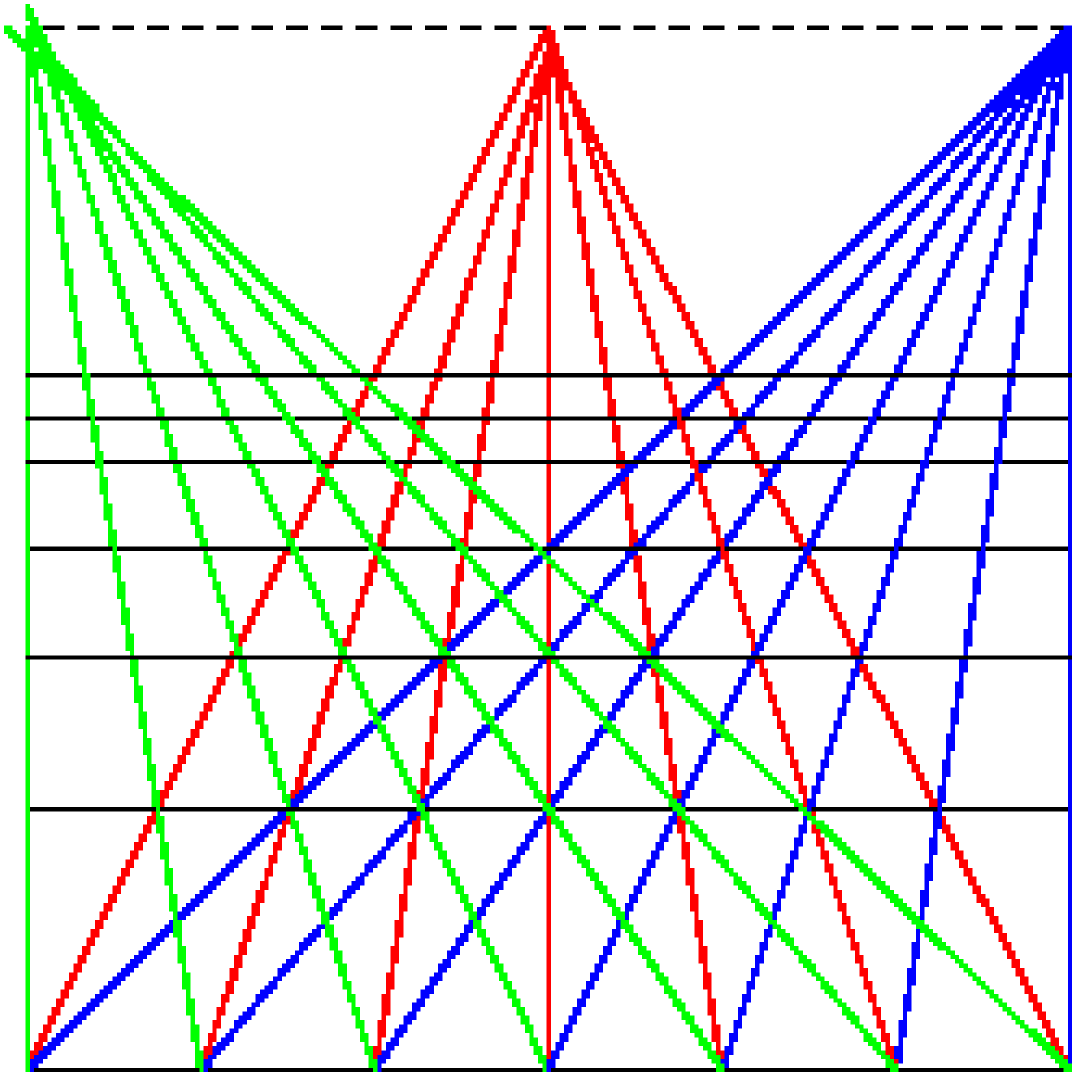
Any two different sheaves meet at different points on the lonely line of I.

So we define the projection of the point at infinity for H in S to be the point on the lonely line where the images meet.

Sheaves in the source plane, viewed head on



Projection of sheaves in the image plane



And vice versa

Suppose H is a sheaf in I.

The images of H in S all meet at one point h on the lonely line of S.

Any two different sheaves meet at different points on the lonely line of S.

So we define the projection of the point at infinity for H in I to be the point on the lonely line of S where the images meet.

So projection works perfectly for projective planes.

- For every point x in the projective plane of S there exists exactly one point y in the projective plane of I such that $y = P_{f,I}(x)$. And vice versa.

Redoing property 3

- If L is a line in the *projective plane* of S, then $P_{f,I}(x)$ is a line in the projective plane of I.

Proof by cases:

1. L is an ordinary line in S, not the lonely line of S. x is a point in L. We proved above that $P_{f,I}(L)$ is a line M in I.
   A. If x is an ordinary point in L, not on the lonely line, then $P_{f,I}(x)$ is on M.

Proof, cntd.

B. If x is the intersection of L with the lonely line, then $P_{f,l}(x)$ is the point at infinity for M

C. If x is the point at infinity for L, then $P_{f,l}(x)$ is the intersection of M with the lonely line in I.

2. If L is the lonely line in S, then $P_{f,l}(L)$ is the line at infinity in I.
3. If L is the line at infinity in S, then $P_{f,l}(L)$ is the lonely line in I.

One more fact

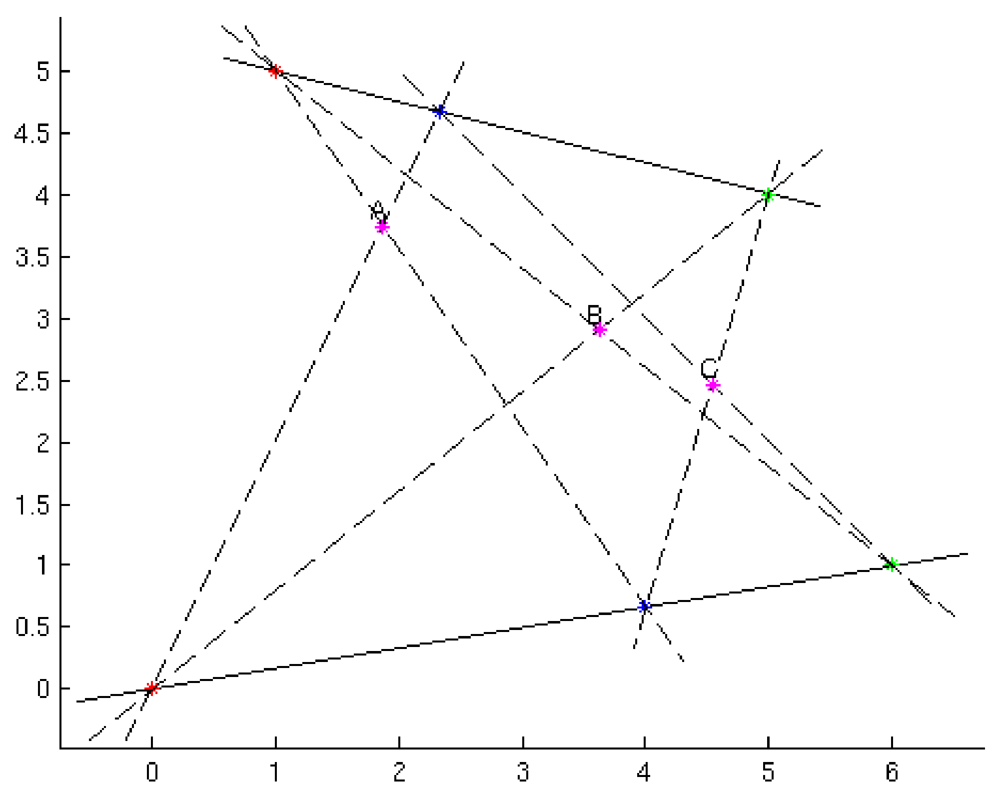
If L is any line in S, you can choose a plane I and a focus f such that $P_{f,I}(L)$ is the line at infinity in I.

Proof: Choose f to be any point not in S. Let Q be the plane containing f and L. Choose I to be a plane parallel to Q.

PART 3: NOW WE CAN PROVE PAPPUS' THEOREM!

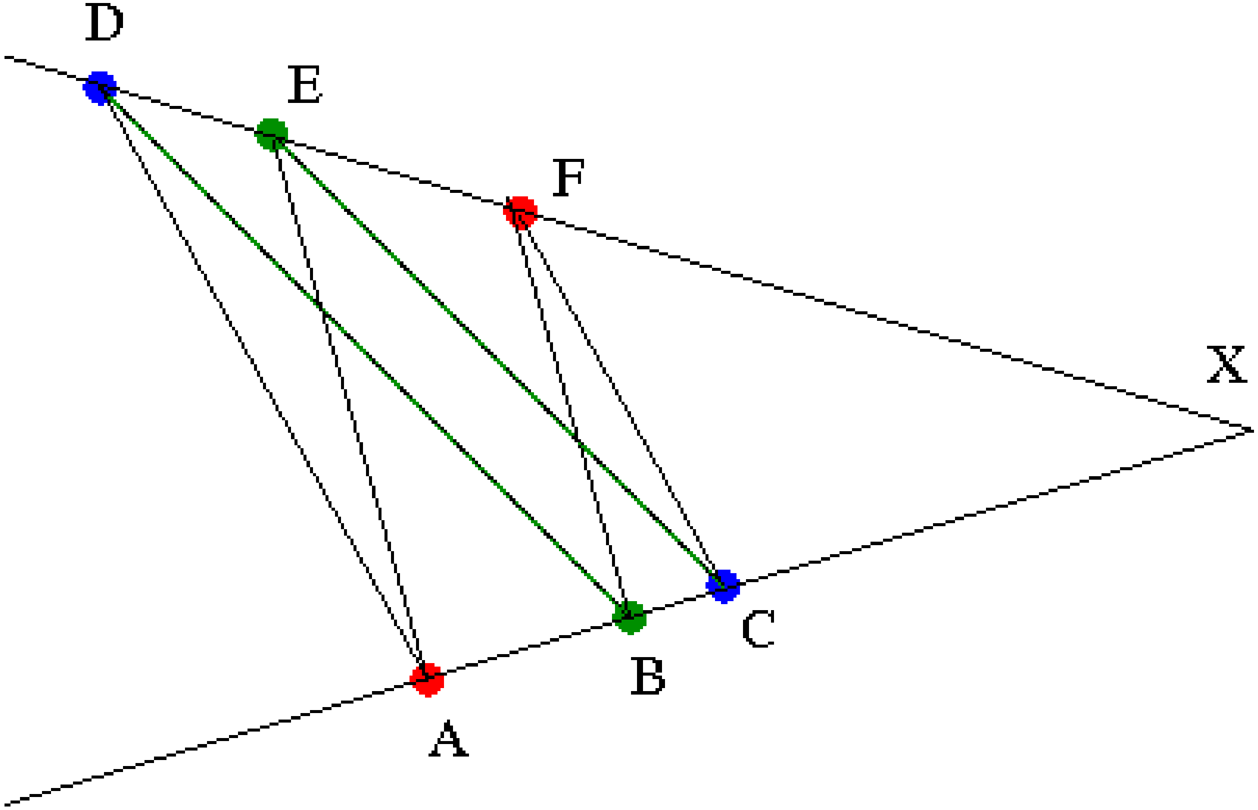
Now we can prove Pappus' theorem!

Proof: Start with a Pappus diagram



We're going to project the line AB to the line at infinity. That means that the two red-blue lines are parallel and the two red-green lines are parallel. We want to prove that C lies on the new line AB, which means that C lies on the line at infinity, which means that the two blue-green lines are parallel.

But this is a simple proof in Euclidean geometry.



Given: D,E,F collinear
A,B,C collinear

DA ‖ FC

EA ‖ FB

Prove: DB ‖ EC

Proof: FX/DX = CX/AX

FX/EX = BX/AX

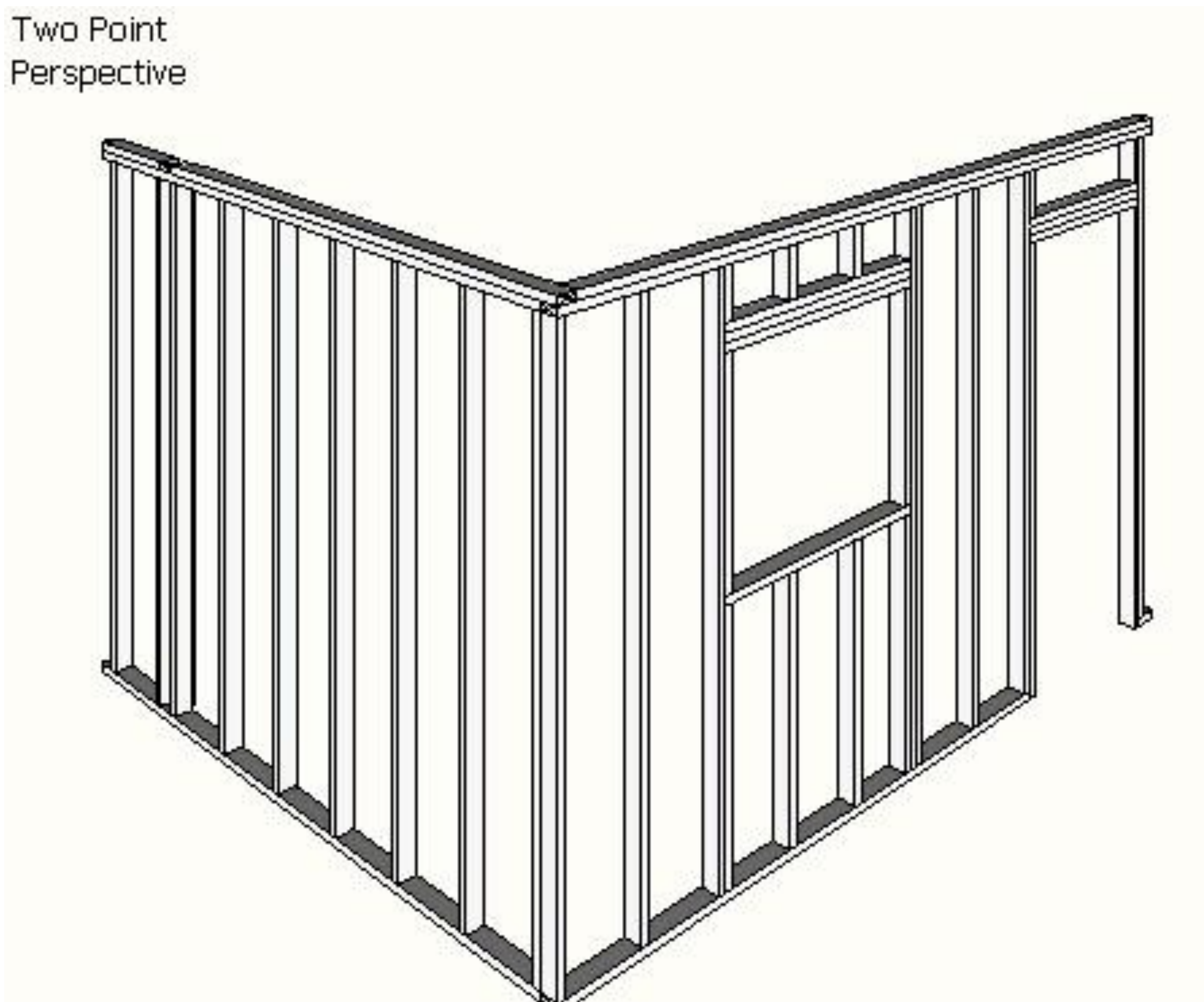
Hence EX/DX = CX/BX

DB ‖ EC

PART 3: PERSPECTIVE

One point perspective (Image plane is perpendicular to x axis)
Perugino, Delivery of the keys to St. Peter, 1481. From Wikipedia, Perspective

Two-point perspective:
Image plane is parallel to z axis.
(From Wikipedia, “Perspective”)



3-point perspective
Image plane is not parallel to any coordinate axis
From Wikipedia, “Perspective”

PART 4: POINT-LINE DUALITY

Numerical representation for ordinary points and lines

- A point is represented by a pair of Cartesian coordinates: <p,q>. e.g. <1,3>
- A line is an equation of the form Ax+By+C = 0 where A,B, and C are constants. E.g. 2x+y-5=0. A point <p,q> falls on the line if it satisfies the equation.

Multiple equation for lines

- The same line can be represented by multiple equations. Multiply by a constant factor.

  $2x + y - 5=0$

  $4x + 2y - 10 = 0$

  $6x + 3y - 15 = 0$

are all the same line.

Homogeneous coordinates for lines

Represent the line Ax+By+C =0 by the triple <A,B,C> with the understanding that any two triples that differ by a constant factor are the same line.

So, the triples <2,1,-5>, <4,2,-10>, <-6,-3,15>, <1, 1/2, -5/2> and so on all represent the line 2x+y-5=0.

Homogeneous coordinates for points

We want a representation for points that works the same way.

We will represent a point <p,q> by any triple <u,v,w> such that $w \neq 0$, u=p*w and v=q*w.

E.g. the point <1,3> can be represented by any of the triples <1,3,1>, <2,6,2>, <-3,9,-3>, <1/3,1,1/3> and so on.

So again any two triples that differ by a constant multiple represent the same point.

Point lies on a line

Point <u,v,w> lies on line <A,B,C> if Au+Bv+Cw=0.

Proof: <u,v,w> corresponds to the point <u/w, v/w>. If A*(u/w) + B*(v/w) + C = 0, then Au + Bv + Cw = 0.

Homogeneous coordinates for a point at infinity

- Parallel lines differ in their constant term.

$$2x + y - 5 = 0$$

$$2x + y - 7 = 0$$

$$2x + y + 21 = 0$$

The point at infinity for all these has homogeneous coordinates $\langle u,v,w\rangle$ that satisfy

$$2u + v - Cw = 0 \text{ for all } C$$

Clearly $v = -2u$ and $w = 0$.

Homogeneous coordinates for a point at infinity

Therefore, a point at infinity lying on the line

$$Ax + By + C = 0$$

has homogeneous coordinates $\langle -Bt, At, 0 \rangle$ where $t \neq 0$.

E.g. the triples $\langle -2,1,0 \rangle$, $\langle 4,-2,0 \rangle$ and so on all represent the point at infinity for the line

$$x + 2y - 5 = 0.$$

Homogeneous coordinates for a point at infinity

- Note that the points

Homogeneous	Natural
< -2, 1, 1>	<-2, 1>
< -2, 1, 0.1>	<-20, 10>
< -2, 1, 0.0001>	<-20000, 10000>

lie further and further out on the line x+2y=0, so it “makes sense” that <-2, 1, 0> lies infinitely far out on that line.

Homogeneous coordinates for the line at infinity

The line at infinity contains all points of the form

<u,v,0>. So if the homogeneous coordinates of the line at infinity are <A,B,C> we have

Au + Bv + 0C = 0, for all u and v. So A=B=0 and C can have any non-zero value.

Points in homogeneous coordinates

Any triple $\langle x,y,z \rangle$, not all equal to 0, with the rule that $\langle xr,yr,zr \rangle$ represents the same point for any $r \neq 0$.

Point $\langle x,y,z \rangle$ lies on line $\langle a,b,c \rangle$ if $ax+by+cz=0$.

Lines in homogeneous coordinates

Any triple <x,y,z>, not all equal to 0, with the rule that <xr,yr,zr> represents the same line for any $r \neq 0$.

Line <x,y,z> contains point <a,b,c> if $ax+by+cz=0$.

Point/Line duality

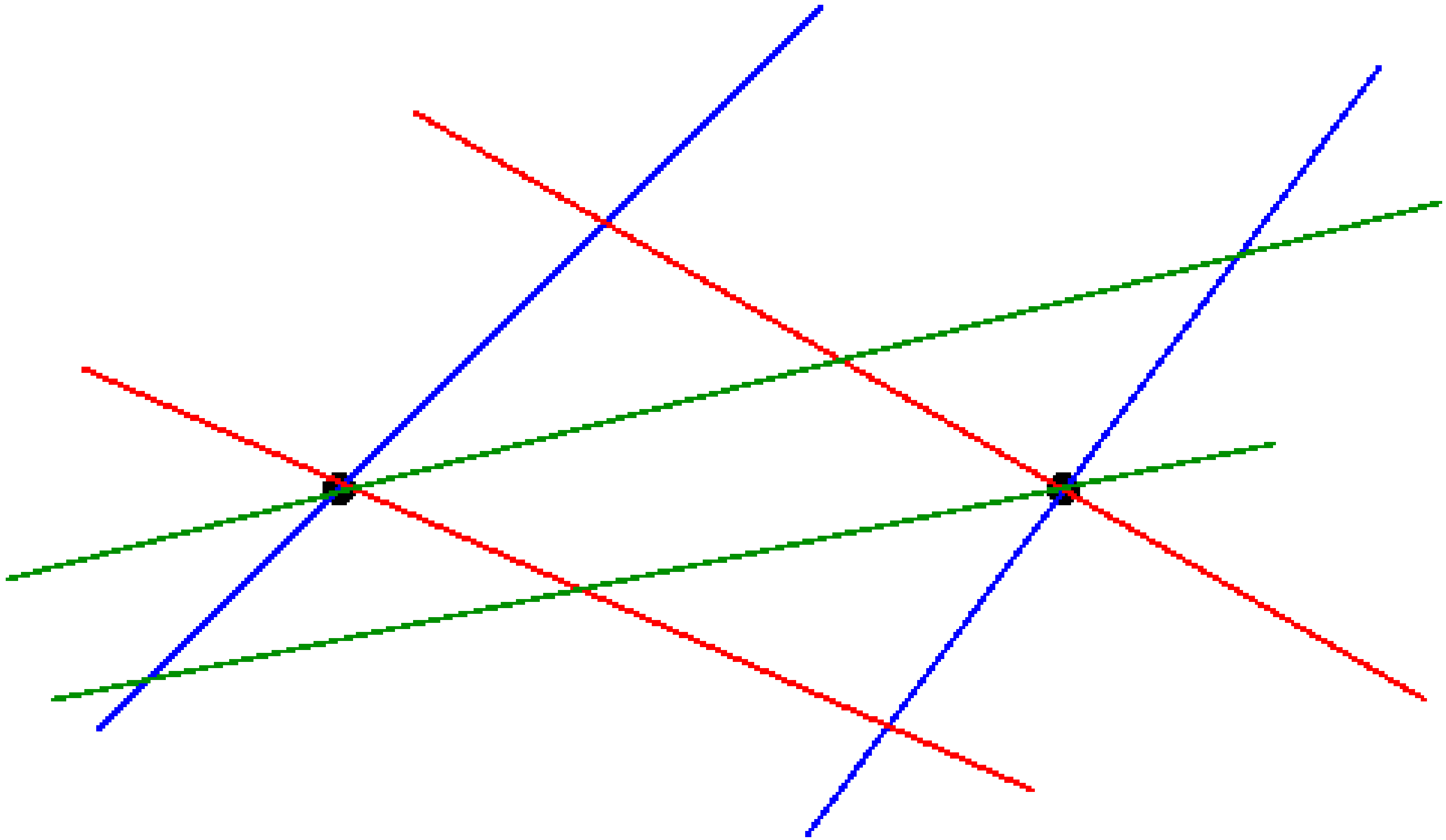
Therefore:

If you have any diagram of points and lines, you can replace every point with coordinates <a,b,c> with the line of coordinates <a,b,c> and vice versa, and you still have a valid diagram.

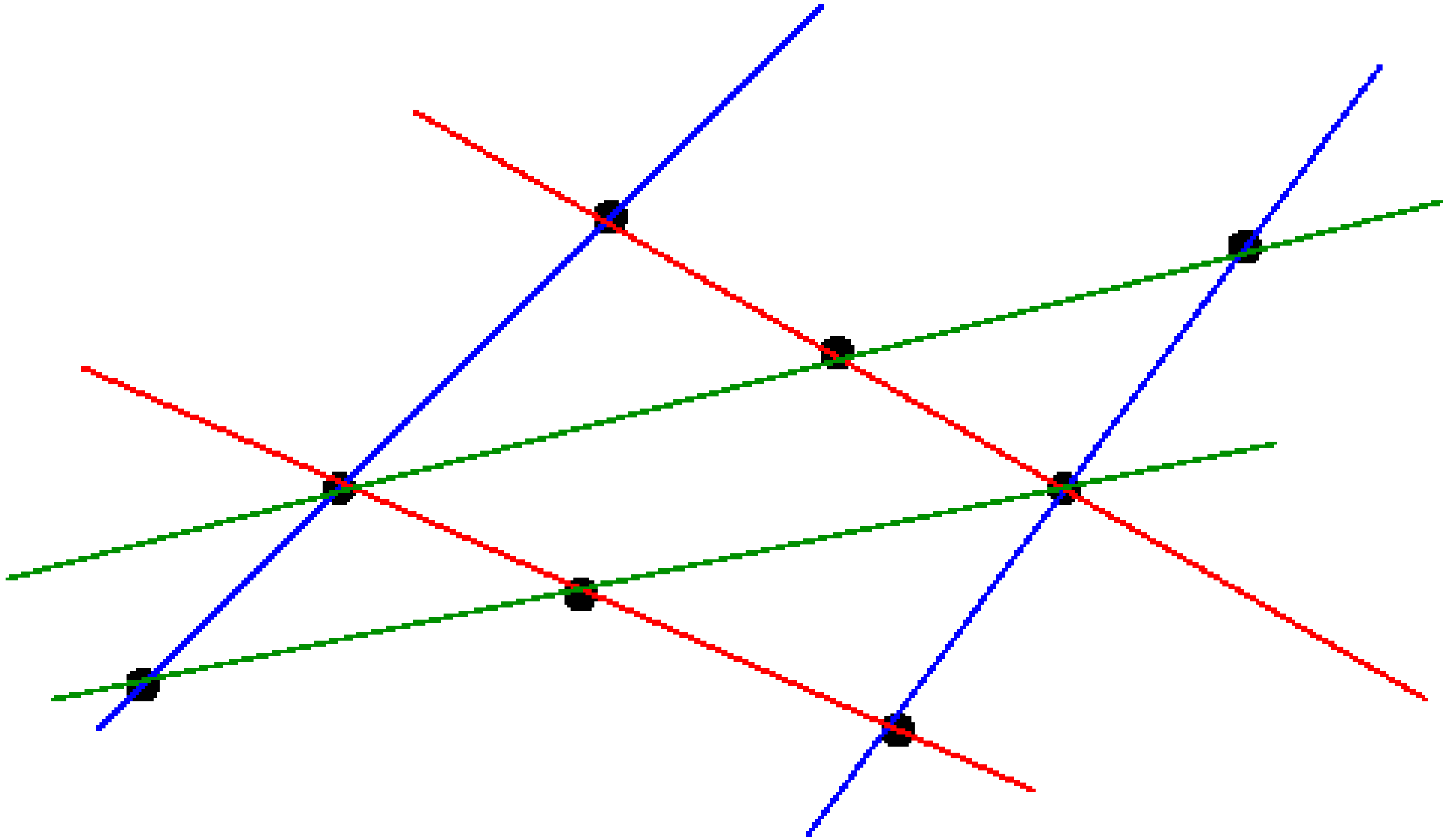
If you do this to Pappus' theorem, you get another version (called the "dual" version) of Pappus' theorem.

Pappus' theorem: Dual formulation

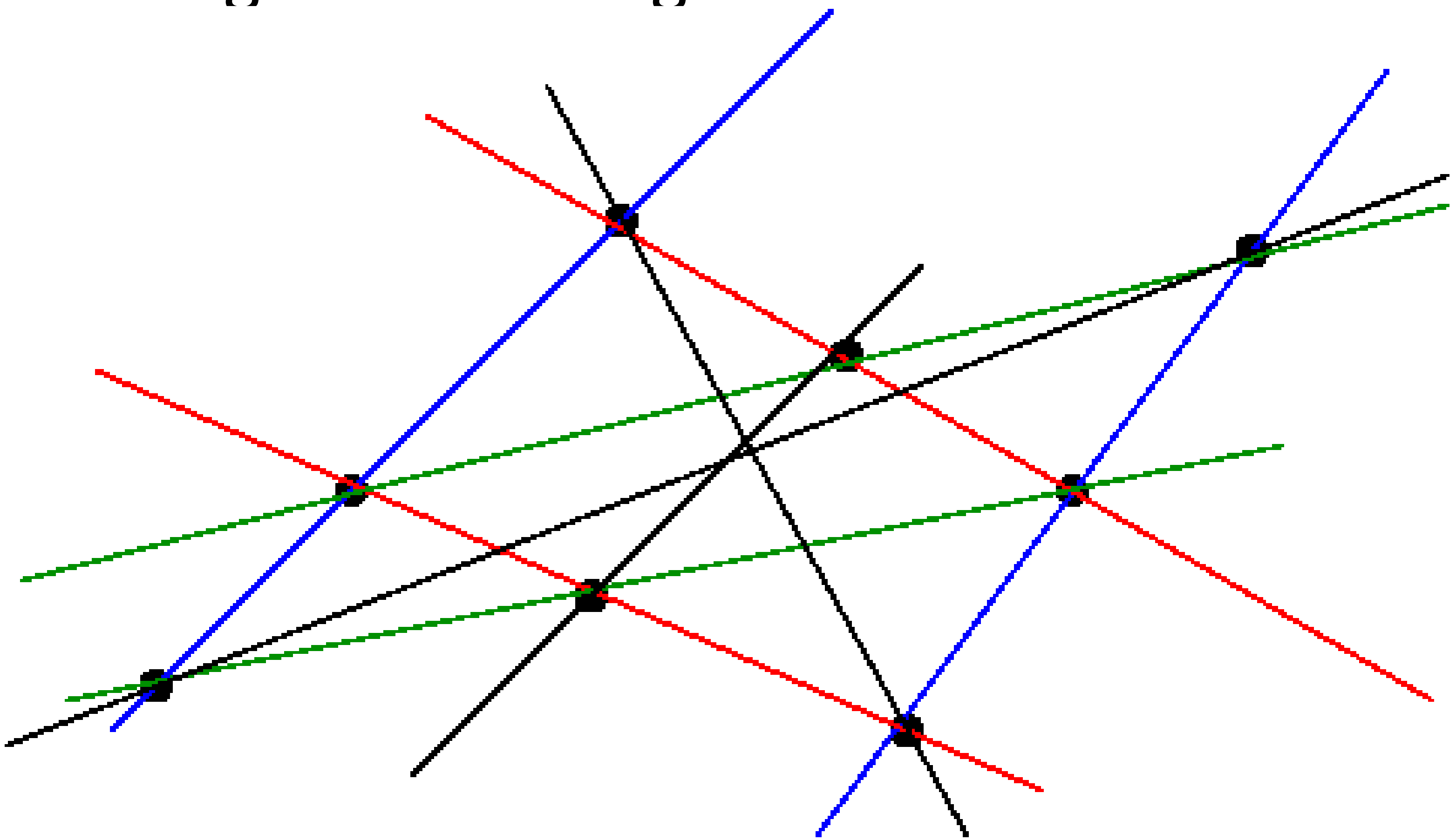
Pick any two points. Through each, draw a red line, a blue line, and a green line.



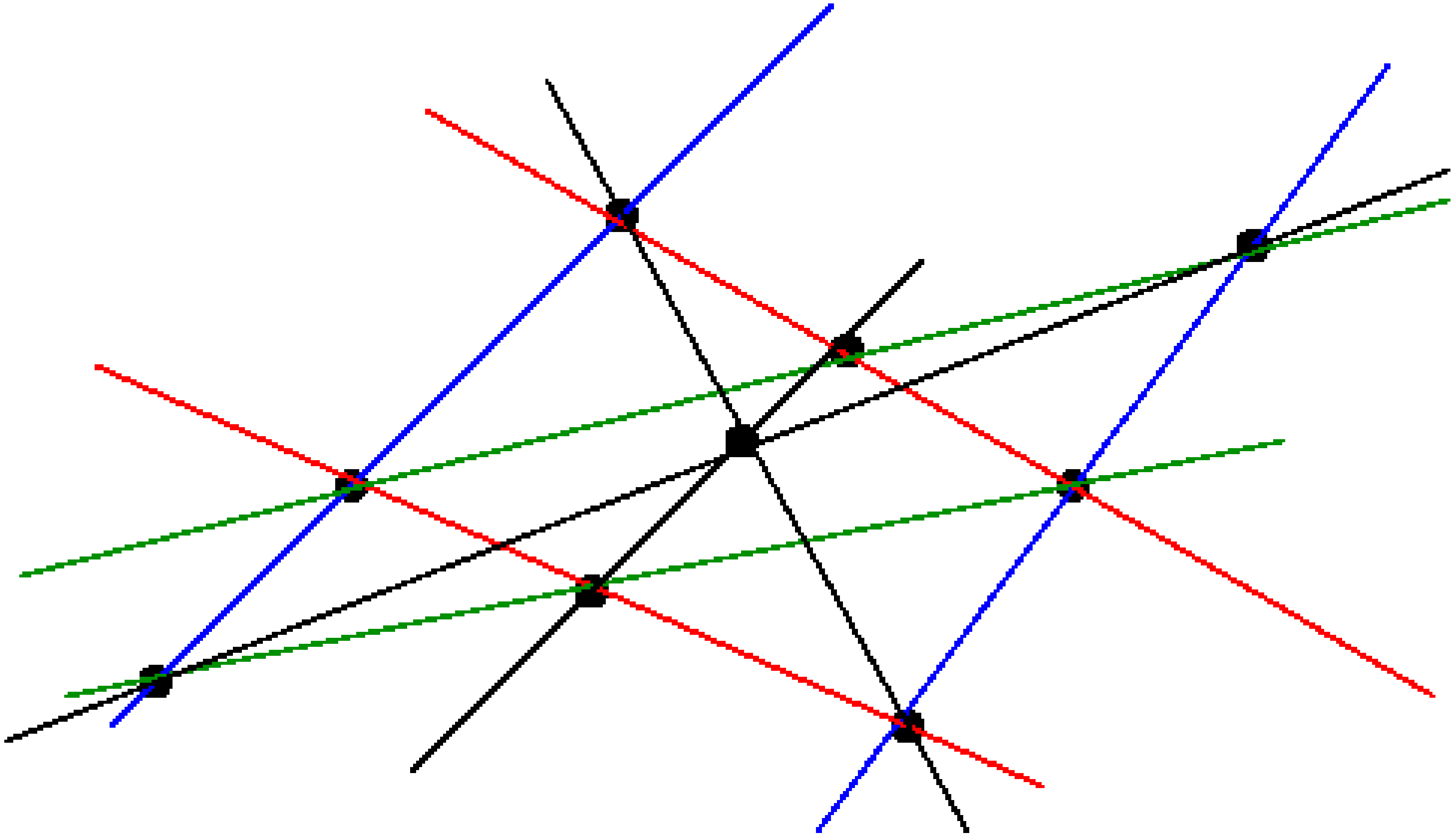
Find the intersection of the lines of different color.



Draw the lines that connects the two red-blue crossings, the two red-green crossings, and the two blue-green crossings.



These lines are coincident



Pappus' theorem: Original and dual

Draw two lines with red, blue and green points.

Draw the lines connecting points of different colors.

Find the intersections of the two red-blue, the two red-green, and the two blue-green lines.

These points are collinear.

Draw two points with red, blue, and green lines.

Find the intersection of lines of different colors.

Draw the lines connecting the two red-blue, the two red-green, and the two blue-green points.

These lines are coincident.