Elementarily Equivalent Domains for Topological Languages over Regions in Euclidean Space

Ernest Davis*
Dept. of Computer Science
New York University
davise@cs.nyu.edu

January 25, 2011

Abstract

We prove that the class of rational polyhedra and the class of topologically regular regions definable in an o-minimal structure are each elementarily equivalent to the class of polyhedra for topological languages.

Keywords: Elementarily equivalent, first-order equivalent, topological language, RCC.

1 Introduction

The study of qualitative spatial reasoning using topological relations over spatial regions has flourished since the seminal papers of Egenhofer and Franzosa [4] and of Randell, Cui, and Cohn [11, 12]. (See [2] for a recent survey.) One frustrating aspect of this research programme, however, is that the important logical characteristics of the theories involved often depend very sensitively on rather fine details of the language or the model. For example Kontchakov et al. [7, 8] study a variety of existential languages, with different dimensionalities of space, domains of regions, and different collections of predicates, and they demonstrate many differences between these in terms of expressivity and of computational complexity. Since there is often no very principled way of deciding which particular language is most reasonable, one ends up with a large number of equally plausible theories, each with its own characteristics.

In this paper we present some results in the opposite direction, discussing some distinctions that do *not* make a difference. We show that a number of collections of spatial regions are *elementarily equivalent* to the space of polyhedra, for topological languages over regions. That is, suppose you have a collection of topological relations over spatial regions, such as "Regions **P** and **Q** are externally connected," or "Region **Z** is the union of **X** and **Y**", and you have a first-order language $\mathcal{L}$ whose predicates refer to these relations. Then any sentence in $\mathcal{L}$ that is true over the domain of polyhedra in Euclidean space is also true over any of the other domains of regions we will discuss here. Specifically, we show that the class of rational polyhedra and the class of topologically regular regions definable in an o-minimal structure over the reals are each elementarily equivalent to the class of polyhedra in $\mathbb{R}^k$.

*Many thanks to Ian Pratt-Hartmann for helpful discussions, and to the reviewers for valuable suggestions. This research was supported in part by NSF grant #IIS-0534809.

It should be emphasized that these results apply to a language of *any* collection of topological relations — i.e. relations that are invariant under homeomorphisms of the entire space to itself — including, for what it is worth, non-computable relations such as "the number of connected components of region $\mathbf{P}$ is the index of a non-halting Turing machine."

Pratt-Hartmann has proved a strong result of this kind in [10], corollary 2.174, p. 89:

Theorem 1 *All splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ with curve-selection have the same $\mathcal{L}_\Sigma$-theory for any topological signature Σ.*

In other words, if $\mathcal{C}$ and $\mathcal{D}$ are mereotopologies satisfying the specified conditions and Σ is a collection of topological relations, then the structures $\langle \mathcal{C}, \Sigma \rangle$ and $\langle \mathcal{D}, \Sigma \rangle$ are elementarily equivalent. For the definitions of the terms "mereotopology". "splittable", "finitely decomposable", and "curve selection" see the cited paper. The key points here are (a) that these are topological features of a collection of regions; (b) that the theorem is only proven in two-dimensional space. The results in the current paper apply to Euclidean space of arbitrary finite dimension, but they place conditions on the collection of regions that are much more restrictive, and they are formulated in algebraic and structural terms rather than in geometric terms.

Section 2 presents a general meta-logical theorem giving sufficient conditions that two structures are elementarily equivalent. Section 3 presents the proof that the collection of rational polyhedra is elementarily equivalent to the collection of polyhedra, relative to a topological language. Section 4 presents the proof that any o-minimal collection of regions over the reals that includes the polyhedra is elementarily equivalent to the collection of polyhedra.

2 A general meta-logical theorem

In this section, we prove a general meta-logical theorem, giving sufficient conditions that two structures are elementarily equivalent. First, let us briefly discuss structures and elementary equivalence.

Definition 1 *An* L-structure *is a triple $\langle \mathcal{D}, \sigma, \mathcal{I} \rangle$ where $\mathcal{D}$ is a domain; $\sigma = \langle \sigma_1 \ldots \sigma_m \rangle$ is a signature of m formal symbols; and $\mathcal{I}$ is an interpretation mapping each σ_i to a relation over $\mathcal{D}$.*

Definition 2 *A* structure *is a tuple $\langle \mathcal{D}, \mathcal{P}_1 \ldots \mathcal{P}_m \rangle$ where $\mathcal{P}_i$ is a relation over $\mathcal{D}$. The L-structure $\langle \mathcal{D}, \langle \sigma_1 \ldots \sigma_m \rangle, \mathcal{I} \rangle$ corresponds to the structure $\langle \mathcal{D}, \mathcal{I}(\sigma_1) \ldots \mathcal{I}(\sigma_m) \rangle$*

Definition 3 *Let $\langle \mathcal{D}, \sigma, \mathcal{I} \rangle$ and $\langle \mathcal{E}, \sigma, \mathcal{J} \rangle$ be two L-structures with the same signature σ. These are* elementarily equivalent *if, for every sentence ϕ in the first-order language over σ, $\mathcal{I} \models \phi$ if and only if $\mathcal{J} \models \phi$.*

Two structures are elementarily equivalent if the corresponding L-structures with the same signature σ are elementarily equivalent.

Throughout the remainder of this section, let Ω be a set. Let $\mathcal{A}$ be a set of bijections from Ω to itself that forms a group; that is, $\mathcal{A}$ is closed under composition and inverse.

We will use boldface capitals such as $\mathbf{R}$ to denote elements of Ω. We will use the composition operator $\Gamma \circ \Phi$ in analysts' style rather than algebraicists'; that is, $(\Gamma \circ \Phi)(\mathbf{R}) = \Gamma(\Phi(\mathbf{R}))$. If $\mathcal{P}$ is a relation over Ω and $\mathcal{C} \subset \Omega$ then $\mathcal{P}|_\mathcal{C}$ will denote the restriction of $\mathcal{P}$ to $\mathcal{C}$.

Definition 4 *A relation $\mathcal{P}(\mathbf{R}_1 \ldots \mathbf{R}_n)$ over elements of Ω is an* invariant *of $\mathcal{A}$ if, for all $\Phi \in \mathcal{A}$, $\mathcal{P}(\mathbf{R}_1 \ldots \mathbf{R}_n)$ if and only if $\mathcal{P}(\Phi(\mathbf{R}_1) \ldots \Phi(\mathbf{R}_n))$.*

Definition 5 *Let $\mathcal{C}$ and $\mathcal{D}$ be subsets of Ω. $\mathcal{C}$ is* finitely embeddable *in $\mathcal{D}$ with respect to $\mathcal{A}$ if it satisfies the following condition: For any $m \geq 0$, let $\{\mathbf{P}_1 \ldots \mathbf{P}_m\}$ be a set of m elements in $\mathcal{C}$. Then there exists a bijection $\Gamma \in \mathcal{A}$ such that $\Gamma(\mathbf{P}_i) \in \mathcal{D}$ for $i = 1 \ldots m$.*

For readability, we will often omit the reference to $\mathcal{A}$ in using "finitely embeddable" and similar terms.

Definition 6 *Let $\mathcal{C}$ and $\mathcal{D}$ be subsets of Ω. $\mathcal{C}$ is* extensible *in $\mathcal{D}$ with respect to $\mathcal{A}$ if it satisfies the following condition. For any $m \geq 0$, let $\{\mathbf{P}_1 \ldots \mathbf{P}_m\}$ be a set of m elements in $\mathcal{C}$, and let $\Gamma \in \mathcal{A}$ be a bijection over Ω such that $\Gamma(\mathbf{P}_i) \in \mathcal{D}$ for $i = 1 \ldots m$. Then for any element $\mathbf{P}_{m+1} \in \mathcal{C}$ there exists a bijection $\Gamma' \in \mathcal{A}$ such that $\Gamma'(\mathbf{P}_i) = \Gamma(\mathbf{P}_i)$ for $i = 1 \ldots m$ and $\Gamma(\mathbf{P}_{m+1}) \in \mathcal{D}$.*

For the case $m = 0$, this condition asserts that for any $\mathbf{P} \in \mathcal{C}$ there exists $\Gamma \in \mathcal{A}$ such that $\Gamma(\mathbf{P}) \in \mathcal{D}$.

Note that we cannot simply choose $\Gamma' = \Gamma$, because $\Gamma(\mathbf{P}_{m+1})$ is not necessarily in $\mathcal{C}$.

The following simple example illustrates the distinction between embeddability and extensibility. Let $\Omega = \mathbb{R}$ the set of real numbers. Let $\mathcal{A}$ be the collection of bijections that are monotonic under ordering; that is, for $\Gamma \in \mathcal{A}$ and for $x, y \in \mathbb{R}$, if $x < y$ then $\Gamma(x) < \Gamma(y)$. Let $\mathcal{C} = \mathbb{R}$, and let $\mathcal{D} = \mathbb{Z}$ the set of integers. Then $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$; given any set of real numbers $\{\mathbf{P}_1 \ldots \mathbf{P}_m\}$, sort them, and then map them to the corresponding integers in sequence. However $\mathcal{C}$ is not extensible in $\mathcal{D}$; if we have chosen $\mathbf{P}_1 = 0, \mathbf{P}_2 = 2$, and $\Gamma(x) = x/2$, then $\Gamma(\mathbf{P}_1)$ and $\Gamma(\mathbf{P}_2)$ are in $\mathcal{D}$, but there is no way to extend Γ so that $\Gamma(1) \in \mathcal{D}$. This same example illustrates that $\mathbb{Z}$ is not extensible in itself with respect to $\mathcal{A}$.

If we use the same Ω, $\mathcal{A}$ and $\mathcal{C}$ but let $\mathcal{D} = \mathbb{Q}$, the set of rational numbers, then $\mathcal{C}$ is extensible within $\mathbb{Q}$. If Γ is a function mapping the set of real numbers $S = \{\mathbf{P}_1 \ldots \mathbf{P}_m\}$ to a set of rationals, and we are given the next element $\mathbf{P}_{m+1}$, then we can extend Γ by finding the $\mathbf{P}_i$ and $\mathbf{P}_j$ to be the elements of S immediately above and below $\mathbf{P}_{m+1}$, and then choose $\Gamma'(\mathbf{P}_{m+1})$ to be a rational between $\Gamma(\mathbf{P}_i)$ and $\Gamma(\mathbf{P}_j)$. (Applying theorem 8 below, one can go on to show that the structures $\langle \mathbb{R}, < \rangle$ and $\langle \mathbb{Q}, < \rangle$ are elementarily equivalent.) Note that there is no injection as a whole from $\mathcal{C}$ as a whole into $\mathcal{D}$, since $\mathcal{C}$ has a larger cardinality than $\mathcal{D}$, so one cannot choose a fixed Γ at the start.

Lemma 2 *If $\mathcal{C}$ is extensible in $\mathcal{D}$ with respect to $\mathcal{A}$, then $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$.*

Proof: The fact that the set $\{\mathbf{P}_1 \ldots \mathbf{P}_m\}$ is embeddable is trivial by induction on m.

Lemma 3 *If $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$ with respect to $\mathcal{A}$, and $\mathcal{D}$ is extensible in $\mathcal{E}$, then $\mathcal{C}$ is extensible in $\mathcal{E}$.*

Proof: Let $\mathbf{P}_1 \ldots \mathbf{P}_m$ be m elements in $\mathcal{C}$; let $\Gamma \in \mathcal{A}$ be a bijection over Ω such that $\Gamma(\mathbf{P}_i) \in \mathcal{E}$ for $i = 1 \ldots m$; and let $\mathbf{P}_{m+1} \in \mathcal{C}$. Since $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$, choose $\Theta \in \mathcal{A}$ such that $\Theta(\mathbf{P}_i) \in \mathcal{D}$ for $i = 1 \ldots m+1$. Let $\mathbf{Q}_i = \Theta(\mathbf{P}_i)$ for $i = 1 \ldots m+1$. Let $\Phi = \Gamma \circ \Theta^{-1}$. Then, for $i = 1 \ldots m$, $\Phi(\mathbf{Q}_i) = \Gamma(\mathbf{P}_i) \in \mathcal{E}$. Since $\mathcal{D}$ is extensible in $\mathcal{E}$, there exists $\Phi' \in \mathcal{A}$ such that $\Phi'(\mathbf{Q}_i) = \Phi(\mathbf{Q}_i)$ for $i = 1 \ldots m$ and $\Phi'(\mathbf{Q}_{m+1}) \in \mathcal{E}$. Now let $\Gamma' = \Phi' \circ \Theta$. Then for $i = 1 \ldots m$, $\Gamma'(\mathbf{P}_i) = \Phi'(\mathbf{Q}_i) = \Phi(\mathbf{Q}_i) = \Gamma(\mathbf{P}_i)$; and $\Gamma(\mathbf{P}_{m+1}) = \Phi'(\mathbf{Q}_{m+1}) \in \mathcal{E}$. ∎

Corollary 4 *Extensibility is transitive: If $\mathcal{C}$ is extensible in $\mathcal{D}$ and $\mathcal{D}$ is extensible in $\mathcal{E}$, then $\mathcal{C}$ is extensible in $\mathcal{E}$.*

Proof: Immediate from lemmas 2 and 3.

Definition 7 *Two sets $\mathcal{C} \subset \Omega$ and $\mathcal{D} \subset \Omega$ are* mutually extensible *with respect to $\mathcal{A}$ if each is extensible in the other. A set $\mathcal{C}$ is* self-extensible *if it is extensible in itself.*

Corollary 5 *Mutual extensibility is an equivalence relation over the class of self-extensible sets.*

Proof: Immediate from lemma 3 and definition 7.

Corollary 6 *If $\mathcal{C}$ is finitely embeddable in $\mathcal{D}$ and $\mathcal{D}$ is self-extensible, then $\mathcal{C}$ is extensible in $\mathcal{D}$.*

Proof: Immediate from lemma 3 with $\mathcal{E} = \mathcal{D}$.

Lemma 7 *Let $\mathcal{C}$ and $\mathcal{D}$ be mutually extensible subsets of Ω with respect to $\mathcal{A}$. Let $\mathcal{P}_1 \ldots \mathcal{P}_m$ be relations over Ω that are invariants of $\mathcal{A}$. Let $\sigma = \langle \sigma_1 \ldots \sigma_m \rangle$ be a signature with m symbols. Let $\mathcal{I}$ and $\mathcal{J}$ be the interpretations of σ such that $\mathcal{I}(\alpha_i) = \mathcal{P}|_{\mathcal{C}}$ and $\mathcal{J}(\alpha_i) = \mathcal{P}|_{\mathcal{D}}$. Let ϕ be a prenex first-order formula over σ. Let $\mu_1 \ldots \mu_n$ be the free variables in ϕ. Let $\mathcal{U}$ be a valuation from $\mu_1 \ldots \mu_n$ to $\mathcal{C}$. Let $\Gamma \in \mathcal{A}$ be a bijection such that $\Gamma(\mathcal{U}(\mu_i)) \in \mathcal{D}$. Let $\mathcal{V}$ be the valuation from $\mu_1 \ldots \mu_m$ to $\mathcal{D}$, $\mathcal{V}(\mu_i) = \Gamma(\mathcal{U}(\mu_i))$. Then $\mathcal{C}, \mathcal{I}, \mathcal{U} \models \phi$ if and only if $\mathcal{D}, \mathcal{I}, \mathcal{V} \models \phi$.*

Proof by induction on the number of quantifiers in ϕ.

Base case : ϕ is a quantifier-free formula with variables $\mu_1 \ldots \mu_n$. Let $\mathcal{U}$ be a valuation of $\mu_1 \ldots \mu_n$ onto $\mathcal{C}$ such that $\mathcal{C}, \mathcal{I}, \mathcal{U} \models \phi$. Since $\mathcal{C}$ is embeddable in $\mathcal{D}$, there is a bijection $\Gamma \in \mathcal{A}$ such that $\Gamma(\mathcal{U}(\mu_i)) \in \mathcal{D}$ for $i = 1 \ldots n$. Let $\mathcal{V}(\mu_i) = \Gamma(\mathcal{U}(\mu_i))$. Since $\mathcal{P}_i$ is invariant under $\mathcal{A}$ for $i = 1, \ldots, m$, it follows that each atomic formula occurring in ϕ has the same value under $\mathcal{J}, \mathcal{V}$ as under $\mathcal{I}, \mathcal{U}$. Hence each subformula of ϕ, including ϕ itself has the same value under $\mathcal{J}, \mathcal{V}$ as under $\mathcal{I}, \mathcal{U}$.

The proof of the reverse implication is symmetric.

Recursive case: Suppose that the statement is true for all formulas with no more than q quantifiers. Let ϕ be a formula with $q + 1$ quantifiers; thus ϕ has either the form $\exists \mu \psi$ or $\forall \mu \psi$ where ψ is a formula with m quantifiers. Let $\mu_1 \ldots \mu_n$ be the free variables in ϕ; then the free variables in ψ are $\mu_1 \ldots \mu_n, \mu$.

Let $\mathcal{U}$, Γ and $\mathcal{V}$ be as in the statement of the lemma.

I. Suppose that ϕ has the form $\exists \mu \psi$, and suppose that this is true under $\mathcal{I}, \mathcal{U}$. Then there exists $\mathbf{R} \in \mathcal{C}$ such that ψ is true under $\mathcal{I}, \mathcal{U}'$ where $\mathcal{U}' = \mathcal{U} \cup \{\mu \rightarrow \mathbf{R}\}$. Since $\mathcal{C}$ is extensible in $\mathcal{D}$, there exists a bijection $\Gamma' \in \mathcal{A}$ such that $\Gamma'(\mathcal{U}(\mu_i)) = \mathcal{V}(\mu_i)$ for $i = 1 \ldots n$. and such that $\Gamma'(\mathbf{R}) \in \mathcal{D}$. Let $\mathcal{V}' = \mathcal{V} \cup \{\mu \rightarrow \Gamma'(\mathbf{R})\}$. Then $\psi, \mathcal{U}', \Gamma'$, and $\mathcal{V}'$ satisfy the induction hypothesis, so $\mathcal{J}, \mathcal{V}' \models \psi$. Therefore $\mathcal{J}, \mathcal{V} \models \phi$.

II. Suppose that ϕ has the form $\exists \mu \psi$, and suppose that this is true under $\mathcal{J}, \mathcal{V}$. Then there exists $\mathbf{Q} \in \mathcal{D}$ such that ψ is true under $\mathcal{J}, \mathcal{V}'$ where $\mathcal{V}' = \mathcal{V} \cup \{\mu \rightarrow \mathbf{Q}\}$. Since $\mathcal{D}$ is extensible in $\mathcal{C}$, using the bijection $\Theta = \Gamma^{-1}$, there exists a bijection $\Theta' \in \mathcal{A}$ such that $\Theta'(\mathcal{V}(\mu_i)) = \mathcal{U}(\mu_i)$ for $i = 1 \ldots n$. and such that $\Theta'(\mathbf{Q}) \in \mathcal{C}$. Let $\mathcal{U}' = \mathcal{U} \cup \{\mu \rightarrow \Theta(\mathbf{Q})\}$. Let $\Gamma' = \Theta'^{-1}$. Then $\psi, \mathcal{U}', \Gamma'$, and $\mathcal{V}'$ satisfy the induction hypothesis, so $\mathcal{I}, \mathcal{U}' \models \psi$. Therefore $\mathcal{I}, \mathcal{U} \models \phi$.

III. Suppose that ϕ has the form $\forall \mu \psi$, and suppose that this is true under $\mathcal{I}, \mathcal{U}$. Then $\exists \mu \neg \psi$ is false under $\mathcal{I}, \mathcal{U}$, so by the contrapositive of (II) $\exists \mu \neg \psi$ is false under $\mathcal{J}, \mathcal{V}$, so $\forall \mu \psi$ is true under $\mathcal{J}, \mathcal{V}$.

IV. Suppose that ϕ has the form $\forall \mu \psi$, and suppose that this is true under $\mathcal{J}, \mathcal{V}$. Then by the contrapositive of (I), $\forall \mu \psi$ is true under $\mathcal{I}, \mathcal{U}$.

∎

Theorem 8 *Let $\mathcal{C}$ and $\mathcal{D}$ be mutually extensible subsets of Ω with respect to $\mathcal{A}$. Let $\mathcal{P}_1 \ldots \mathcal{P}_m$ be relations over Ω that are invariants of $\mathcal{A}$. Then the structures $\langle \mathcal{C}, \mathcal{P}_1|_\mathcal{C}, \ldots \mathcal{P}_m|_\mathcal{C} \rangle$ and $\langle \mathcal{D}, \mathcal{P}_1|_\mathcal{D}, \ldots \mathcal{P}_m|_\mathcal{D} \rangle$ are elementarily equivalent.*

Proof: Let σ be a signature of m symbols and let ϕ be a sentence over σ. By lemma 7, since ϕ has no free variables, it holds in the structure $\langle \mathcal{D}, \sigma, \mathcal{J} \rangle$ if and only if it holds in the structure $\langle \mathcal{C}, \sigma, \mathcal{I} \rangle$. ∎

It will be convenient to abbreviate the structure $\langle \mathcal{C}, \mathcal{P}_1|_\mathcal{C}, \ldots \mathcal{P}_m|_\mathcal{C} \rangle$ as $\langle \mathcal{C}, \mathcal{P}_1 \ldots \mathcal{P}_m \rangle$; this is unambiguous, as the relations in a structure are necessarily limited to its domain.

Lemma 9 below gives a set of sufficient conditions for mutual extensibility.

Definition 8 *Let $\mathcal{D} \subset \Omega$. Let $\mathcal{A}$ and $\mathcal{B}$ be groups of bijections from Ω to itself. We say that $\mathcal{A}$ is* rectifiable *to $\mathcal{B}$ over $\mathcal{D}$ if the following condition holds: for all $\Gamma \in \mathcal{A}$, and all $\mathbf{D}_1 \ldots \mathbf{D}_m \in \mathcal{D}$, if $\Gamma(\mathbf{D}_i) \in \mathcal{D}$ for $i = 1 \ldots m$, then there exists $\Phi \in \mathcal{B}$ such that $\Phi(\mathbf{D}_i) = \Gamma(\mathbf{D}_i)$ for $i = 1 \ldots m$.*

Lemma 9 *Let $\mathcal{D} \subset \mathcal{C} \subset \Omega$. Let $\mathcal{A}$ be a group of bijections from Ω to itself and let $\mathcal{B}$ be a subgroup of $\mathcal{A}$. If the following conditions hold:*

a. $\mathcal{C}$ is closed under $\mathcal{A}$.

b. $\mathcal{D}$ is closed under $\mathcal{B}$.

c. $\mathcal{C}$ is embeddable in $\mathcal{D}$ under $\mathcal{A}$.

d. $\mathcal{A}$ is rectifiable to $\mathcal{B}$ over $\mathcal{D}$.

Then $\mathcal{C}$ and $\mathcal{D}$ are mutually extensible with respect to $\mathcal{A}$.

Proof: We first show that $\mathcal{D}$ is self-extensible under $\mathcal{A}$. Let $\Gamma \in \mathcal{A}$, and $\mathbf{D}_1 \ldots \mathbf{D}_m \in \mathcal{D}$ such that $\Gamma(\mathbf{D}_i) \in \mathcal{D}$ for $i = 1 \ldots m$. Let $\mathbf{D}_{m+1} \in \mathcal{D}$. Then since $\mathcal{A}$ is rectifiable to $\mathcal{B}$ over $\mathcal{D}$, there exists $\Gamma' \in \mathcal{B}$ such that $\Gamma'(\mathbf{D}_i) = \Gamma(\mathbf{D}_i)$ for $i = 1 \ldots m$. Since $\mathcal{D}$ is closed under $\mathcal{B}$, $\Gamma'(\mathbf{D}_{m+1}) \in \mathcal{D}$.

Therefore, by corollary 6, $\mathcal{C}$ is extensible in $\mathcal{D}$.

The fact that $\mathcal{D}$ is extensible in $\mathcal{C}$ under $\mathcal{A}$ is immediate from the facts that $\mathcal{D} \subset \mathcal{C}$ and $\mathcal{C}$ is closed under $\mathcal{A}$. Thus, if $\Gamma \in \mathcal{A}$, and $\mathbf{D}_{m+1} \in \mathcal{D}$, then $\Gamma(\mathbf{D}_{m+1}) \in \mathcal{C}$. ∎

3 Rational polyhedra

In this section we prove that the domain of rational polyhedra in $\mathbb{R}^k$ (i.e. polyhedra with rational coordinates) is elementarily equivalent to the domain of general polyhedra for a topological language over regions.

The main lemma we will need for this proof is the following (this will be lemma 15 below):

Lemma: *The space of piecewise linear (PL) mappings is rectifiable to the space of rational piecewise linear mappings over the space of rational polyhedra.*

That is: Let $\mathbf{D}_1 \ldots \mathbf{D}_m$ be rational polyhedra and let Φ be a PL homeomorphism with *real* coefficients such that $\Phi(\mathbf{D}_i)$ is a rational polyhedron for $i = 1 \ldots m$. We need to show that there exists a PL *rational* homeomorphism Ψ such that $\Psi(\mathbf{D}_i) = \Phi(\mathbf{D}_i)$ for $i = 1 \ldots m$.

A simple example will help clarify the issues here and point the way toward a solution (figure 1). Let $m = 1$, let $\mathbf{D}$ be the triangle with vertices $\langle 0, 0 \rangle$ $\langle 1, 0 \rangle$, $\langle 0, 1 \rangle$ and let $\mathbf{P}$ be the square $[2, 3] \times [0, 1]$. Now define the following 9 points in $\mathbf{D}$

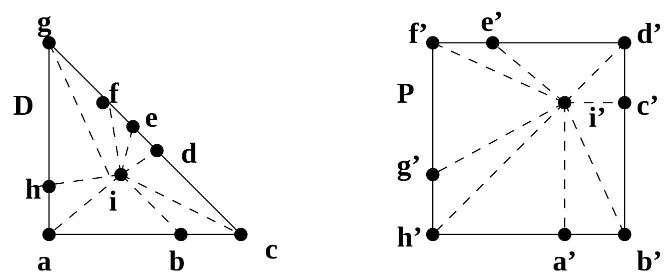

Figure 1: Irrational mapping between rational regions

$\mathbf{a} = \langle 0, 0\rangle$; $\mathbf{b} = \langle 1/\sqrt{2}, 0\rangle$; $\mathbf{c} = \langle 1, 0\rangle$; $\mathbf{d} = \langle 1/\sqrt{3}, 1 - 1/\sqrt{3}\rangle$; $\mathbf{e} = \langle 1/\sqrt{5}, 1 - 1/\sqrt{5}\rangle$; $\mathbf{f} = \langle 1/\sqrt{7}, 1 - 1/\sqrt{7}\rangle$; $\mathbf{g} = \langle 0, 1\rangle$; $\mathbf{h} = \langle 0, 1/\sqrt{11}\rangle$; $\mathbf{i} = \langle \sqrt{6/13}, \sqrt{8/17}\rangle$

And 9 points in $\mathbf{P}$:

$\mathbf{a}' = \langle 2 + \sqrt{10/19}, 0\rangle$; $\mathbf{b}' = \langle 3, 0\rangle$; $\mathbf{c}' = \langle 3, \sqrt{11/23}\rangle$; $\mathbf{d}' = \langle 3, 1\rangle$; $\mathbf{e}' = \langle 2 + \sqrt{14/29}, 1\rangle$; $\mathbf{f}' = \langle 2, 1\rangle$; $\mathbf{g}' = \langle 2, \sqrt{15/31}$; $\mathbf{h}' = \langle 2, 0\rangle$; $\mathbf{i}' = \langle 2 + \sqrt{18/37}, \sqrt{20/41}\rangle$.

One can now construct a PL mapping Φ that maps $\mathbf{D}$ into $\mathbf{P}$ and maps each vertex $\mathbf{v}$ into $\mathbf{v}'$ (see lemma 13). Of course, since Γ maps rational points into irrational points, and vice versa, it is not a rational PL mapping.

Our task now is to construct a rational PL mapping from $\mathbf{D}$ into $\mathbf{P}$. Of course, the temptation in this case is just to chuck Φ overboard and construct a sensible mapping by inspection. We are not going to do that; instead, we will use Φ as a starting point and modify it so that it becomes a rational mapping.

We need to fix three kinds of problems with Φ. First, Φ maps the vertices of $\mathbf{D}$ into irrational points. Second, Φ^{-1} maps the vertices of $\mathbf{P}$ into irrational points. Third, the vertices of the cells of Φ itself, such as $\mathbf{i}$, are at irrational points, and Φ maps them to irrational points.

Our solution is to move all these irrational points to nearby rational points and then build the new Φ with the same structure as Φ based on all these rational point associations. But we have to do this in a way that maintains the mapping of $\mathbf{D}$ to $\mathbf{P}$. Thus, for example, $\Phi(\mathbf{c})$ has to remain on the edge $\mathbf{b}'\mathbf{d}'$ and $\Phi^{-1}(\mathbf{b}')$ has to remain on the edge $\mathbf{ac}$.

With this example in mind, we proceed to the proof.

Not surprisingly the proof works in the same way for any subfield of the reals, not just the rationals.

We begin by defining conventions of notation and standard terminology. The index k will be the dimension of the Euclidean space, throughout. We will use boldface lower-case letters like $\mathbf{x}$ for geometric points in $\mathbb{R}^k$, and boldface upper-case letters like $\mathbf{R}$ for sets of points, also called *regions*. We will use vector notation $\vec{v}$ for vectors and $\hat{v}$ for unit vectors. We will use upper-case Greek letters like Γ for homeomorphisms and other functions. We will use calligraphic letters like $\mathcal{C}$ for collections of regions or functions. $\mathbb{F}$ will be a subfield of the reals. For other entities, we used italicized letters.

For any sets, including regions, the set difference U minus V is denoted $U \setminus V$. The distance between points $\mathbf{x}$ and $\mathbf{y}$ is denoted $d(\mathbf{x}, \mathbf{y})$. The closed disk of radius r around $\mathbf{x}$ is denoted $\bar{B}(\mathbf{x}, r)$; the open disk is denoted $B(\mathbf{x}, r)$.

If $\mathbf{R}$ is a region then the topological boundary of $\mathbf{R}$, denoted $\partial\mathbf{R}$, is defined as
$\partial\mathbf{R} = \text{closure}(\mathbf{R}) \setminus \text{interior}(\mathbf{R})$.

Definition 9 *A subset* $\mathbf{R}$ *of* $\mathbb{R}^k$ *is* topologically closed regular *(generally abbreviated to "regular") if* $\mathbf{R} = \text{closure}(\text{interior}(\mathbf{R}))$.

Definition 10 *A relation* $\mathcal{P}(\mathbf{R}_1 \ldots \mathbf{R}_n)$ *over regions in* $\mathbb{R}^k$ *is* topological *if it is invariant over the space of homeomorphisms of* $\mathbb{R}^k$ *to itself.*

We use the standard theory of simplices, complexes, abstract complexes, and piecewise-linear (PL) mappings [13, 9].

Definition 11 *For* $q \leq k$*, a set of points* $\{\mathbf{p}_0 \ldots \mathbf{p}_q\}$ *is* affine-independent *if the vectors* $\mathbf{p}_1 - \mathbf{p}_0 \ldots \mathbf{p}_q - \mathbf{p}_0$ *are linearly independent. Let* $\mathbf{P} = \langle \mathbf{p}_0 \ldots \mathbf{p}_q \rangle$ *be a* $q+1$*-tuple of affine-independent points. For any point* $\mathbf{x}$*, if there exist coefficients* $t_0 \ldots t_q$ *such that* $\sum_{i=0}^{q} t_q = 1$ *and* $\sum_{i=0}^{q} t_i \mathbf{p}_i = \mathbf{x}$*, then* $\langle t_0 \ldots t_q \rangle$ *are the* barycentric coordinates *of* $\mathbf{x}$ *with respect to* $\mathbf{P}$*. (If these exist, they are unique.) The set of points that have barycentric coordinates with respect to* $\mathbf{P}$ *is the* affine space spanned by $\mathbf{P}$*. The* open simplex *spanned by* $\mathbf{P}$*, denoted "*$S(\mathbf{P})$*" is the set of points* $\mathbf{p}$ *such that all the barycentric coordinates of* $\mathbf{p}$ *are in (0,1). The corresponding* closed simplex*, denoted* $\bar{S}(\mathbf{P}) = \text{closure}(S(\mathbf{P}))$*. The* dimension *of both the affine space and the open simplex is* q*. If* $\mathbf{W} \subset \mathbf{P}$*, then* $S(\mathbf{W})$ *is a* face *of* $S(\mathbf{P})$ *and* $\bar{S}(\mathbf{W})$ *is a face of* $\bar{S}(\mathbf{P})$*.*

The elements $\langle \mathbf{p}_0 \ldots \mathbf{p}_m \rangle$ *are the* vertices *of* $S(\mathbf{P})$ *and* $\bar{S}(\mathbf{P})$*. A single vertex is considered both an open and a closed simplex of dimension 0.*

A point with coordinates in $\mathbb{F}$ will be called an "$\mathbb{F}$-point"; likewise "$\mathbb{F}$-simplex" and so on.

Definition 12 *An open* $\mathbb{F}$*-half-space is defined by a linear inequality* $a_0 + \Sigma_{i=1}^{k} a_i x_i > 0$ *where* $a_i \in \mathbb{F}$ *for* $i = 0 \ldots k$*. A* basic $\mathbb{F}$-polytope *is the intersection of finitely many open* $\mathbb{F}$*-half-spaces. A* closed $\mathbb{F}$-polytope *is the closure of the union of finitely many basic* $\mathbb{F}$*-polytopes. An* $\mathbb{F}$*-polyhedron is a compact* $\mathbb{F}$*-polytope.*

We denote the collection of $\mathbb{F}$-polyhedra as $Poly[\mathbb{F}]$. The set of all polyhedra, $Poly[\mathbb{R}]$ will be abbreviated as $Poly$.

Definition 13 *A homeomorphism* Γ *from* $\mathbb{R}^k$ *to itself is a* piecewise-linear (PL) mapping *if for some integer* m *there exist a sequence of* m *polytopes* $\mathbf{P}_1 \ldots \mathbf{P}_m$*; a sequence of* m *matrices* $M_1 \ldots M_m$*; and a sequence of* m *vectors* $\vec{c}_1 \ldots \vec{c}_m$ *such that:*

- $\cup_{i=1}^{m} \mathbf{P}_i = \mathbb{R}^k$.
- *for* $i = 1 \ldots m$*, if* $\mathbf{p} \in \mathbf{P}_i$ *then* $\Gamma(\mathbf{p}) = M_i \cdot \mathbf{p} + \vec{c}_i$.

Γ *is a* bounded PL mapping *if* $\Gamma(\mathbf{p})$ *is the identity over each unbounded* $\mathbf{P}_i$*.*

The polytopes $\mathbf{P}_1 \ldots \mathbf{P}_m$ *are the* cells *of* Γ*.*

An $\mathbb{F}$*-PL mapping is a PL mapping such that all the cells are* $\mathbb{F}$*-polytopes, and all the elements of* M_i *and* $\vec{c}_i$ *are in* $\mathbb{F}$*.*

Definition 13 departs slightly from the definition in [13] in that here the collection of cells is required to be finite, whereas in [13] it is only required to be locally finite.

We will denote the class of all bounded $\mathbb{F}$-PL mappings as $\mathcal{M}[\mathbb{F}]$.

Definition 14 *A* closed complex *is a set C of closed simplices such that*

a. If $\mathbf{P}$ *is in* C *then any face of* $\mathbf{P}$ *is in* C*.*

b. If $\mathbf{P}, \mathbf{Q} \in C$*, then* $\mathbf{P} \cap \mathbf{Q}$ *is either empty or a face of both* $\mathbf{P}$ *and* $\mathbf{Q}$*.*

An open complex *is the set of open simplices corresponding the simplices of a closed complex.*

Definition 15 *Let Z be a finite set. An* abstract simplex *over Z is a subset of Z. An* abstract complex *C over Z is a collection of abstract simplices over Z such that, if $A \in C$ and $B \subset A$ then $B \in C$. A* realization *of C is a function Γ from Z to $\mathbb{R}^k$. A realization is* proper *if $\Gamma(C)$ is a complex satisfying definition 14.*

Lemma 10 *Let Γ be a realization of abstract complex C over Z. Γ is proper if the following condition is met: For every pair of abstract simplices $F, G \in C$, if $F \cap G = \emptyset$ then $\bar{S}(\Gamma(F)) \cap \bar{S}(\Gamma(G)) = \emptyset$.*

Proof: Immediate from lemma 2.1, p. 8 of [9].

Since a realization Γ of Z is an assignment of a point in $\mathbb{R}^k$ to each element of Z, it can be considered a point in $\mathbb{R}^{k|Z|}$.

Definition 16 *Let Z be a set of abstract points. We define the following metric over the realizations of Z (the $\mathcal{L}^\infty$ metric): For any two realizations Φ and Γ, $d^\infty(\Phi, \Gamma) = \max_{z \in Z} d(\Phi(z), \Gamma(z))$.*

The associated metric topology is, of course, the standard topology on $\mathbb{R}^{k|Z|}$.

Lemma 11 *Let C be an abstract complex over Z. The set of proper realizations of C is an open set within $\mathbb{R}^{k|Z|}$.*

Proof: Define the function $f_C : \mathbb{R}^{k|Z|} \mapsto \mathbb{R}$ as $f(\Gamma)$ is the minimum distance from $\bar{S}(\Gamma(U))$ to $\bar{S}(\Gamma(V))$ over all pairs of simplices $U, V \in C$. Clearly this is a continuous function. By lemma 10, the set of proper realizations is equal to the set of Γ where $f_C(\Gamma) > 0$, and hence an open set. ∎

Definition 17 *Let $\mathcal{P}$ be a set of closed simplices. A* triangulation *of $\mathcal{P}$ is a complex C such that, for any simplex $\mathbf{P} \in \mathcal{P}$ and for any face $\mathbf{Q}$ of $\mathbf{P}$, $\mathbf{Q}$ is the union of simplices in C.*

Lemma 12 *Let $\{\mathbf{P}_1 \ldots \mathbf{P}_m\}$ be a finite set of $\mathbb{F}$-polyhedra. Let $\mathbf{B}$ be an $\mathbb{F}$-polyhedron such that $\mathbf{P}_i$ is disjoint from $\partial\mathbf{B}$ for $i = 1 \ldots m$. Let $\mathcal{P} = \{\mathbf{B}, \mathbf{P}_1 \ldots \mathbf{P}_m\}$. Then there exists an $\mathbb{F}$-triangulation $\mathcal{T}$ of $\mathcal{P}$ such that, if $\mathbf{t}$ is a vertex of $\mathcal{T}$ and $\mathbf{t} \in \partial\mathbf{B}$ then $\mathbf{t}$ is a vertex of $\mathbf{B}$.*

Proof: By definition 11, each polyhedron $\mathbf{P}_i$ is the finite union of simplices. Let $\mathcal{V}$ be the set of all intersections between component simplices of polyhedra in $\mathcal{P}$. Then $\mathcal{V}$ is a collection of cells, as defined in [13] p. 13. Since $\mathbb{F}$ is a field, all the vertices of cells in $\mathcal{V}$ are $\mathbb{F}$-vertices. By [13], p. 16, proposition 2.9, $\mathcal{V}$ has a triangulation whose vertices are exactly the vertices of $\mathcal{V}$. Since the only vertices of $\mathcal{V}$ in $\partial\mathbf{B}$ are vertices of $\mathbf{B}$ itself, the conclusion of the lemma follows. ∎.

Lemma 13 *Let $\mathcal{P} = \{\mathbf{B}, \mathbf{P}_1 \ldots \mathbf{P}_m\}$ be a set of $\mathbb{F}$-polyhedra such that $\mathbf{P}_i \subset \text{interior}(\mathbf{B}), i = 1 \ldots m$. Let $\mathcal{T}$ be an $\mathbb{F}$-triangulation of $\mathcal{P}$. Let C be an abstract complex and Γ be a realization such that $\Gamma(C) = \mathcal{T}$. Let Δ be a proper realization of C such that, for every abstract vertex $z \in Z$, if $\Gamma(z) \in \partial\mathbf{B}$, then $\Delta(z) = \Gamma(z)$. Then there exists a PL $\mathbb{F}$-homeomorphism Ψ such that for every z, $\Psi(\Gamma(z)) = \Delta(z)$ and such that Ψ is the identity outside $\mathbf{B}$.*

Proof: We use barycentric coordinates to construct Ψ. For any point $\mathbf{x} \in \mathbf{B}$, let $\mathbf{X}$ be the open simplex in $\mathcal{T}$ containing $\mathbf{x}$. Let $\mathbf{x}_1 \ldots \mathbf{x}_q$ be the vertices of $\mathbf{X}$, and let $\langle t_1 \ldots t_q \rangle$ be the barycentric coordinates of $\mathbf{x}$. Define $\Psi(\mathbf{x})$ such that, for $\mathbf{x} \in \mathbf{B}$, $\Psi(\mathbf{x}) = \sum_{i=1}^{q} t_i \Delta(\Gamma^{-1}(\mathbf{x}_i))$; for $\mathbf{x} \notin \mathbf{B}$, $\Psi(\mathbf{x}) = \mathbf{x}$.

Since Δ and Γ map the vertices of each simplex to an affine independent set, Ψ is a bijection within the simplex. Since Δ and Γ are proper realizations, the open simplices of $\mathcal{T}$ are disjoint, and the open simplices of $\Psi(\mathcal{T})$ are disjoint, so Ψ is overall a bijection. As a point $\mathbf{x}$ in $\mathbf{X}$ approaches a face $\mathbf{F}$ of $\mathbf{X}$, the barycentric coordinates corresponding to vertices of $\mathbf{X}$ outside $\mathbf{F}$ go to 0, so Ψ is continuous in moving from simplices to their faces and back. The same argument shows that Ψ^{-1} is continuous. Thus Ψ is a homeomorphism from $\mathbf{B}$ to $\Psi(\mathbf{B})$.

It is obvious that Ψ is piecewise linear. Since every vertex of $\mathcal{T}$ is an $\mathbb{F}$-vertex and is mapped onto an $\mathbb{F}$-vertex, Ψ is an $\mathbb{F}$-mapping. Clearly Ψ is the identity on $\partial\mathbf{B}$, so its continuation as the identity outside $\mathbf{B}$ is continuous. ∎

Lemma 14 *For any subfield $\mathbb{F}$ of $\mathbb{R}$, Poly is embeddable in Poly($\mathbb{F}$) under $\mathcal{M}[\mathbb{R}]$.*

Proof: Let $\mathbf{D}_1 \ldots \mathbf{D}_m$ be polyhedra in *Poly*. Let $\mathbf{B}$ be an $\mathbb{F}$-polyhedron such that $\mathbf{D}_i \subset \text{interior}(\mathbf{B})$ for $i = 1 \ldots m$. Let $\mathcal{D} = \{\mathbf{B}, \mathbf{D}_1 \ldots \mathbf{D}_m\}$. Since none of the $\mathbf{D}_i$ intersect $\partial\mathbf{B}$, we can use lemma 12 with $\mathbb{F} = \mathbb{R}$ to construct a triangulation $\mathcal{T}$ of $\mathcal{D}$ such that the only vertices of $\mathcal{T}$ in $\partial\mathbf{B}$ are the vertices of $\mathbf{B}$ itself.

Let C be an abstract complex and Γ be a realization such that $\Gamma(C) = \mathcal{T}$. Let N be the number of vertices in C. By lemma 11 there exists a neighborhood U of Γ in realization-space such that all realizations in U are proper. Since $\mathbb{F}^{Nk}$ is dense within $\mathbb{R}^{Nk}$, there exists $\Delta \in U$ such that $\Delta(z)$ is an $\mathbb{F}$-point for all z, and further $\Delta(z) = \Gamma(z)$ for all vertices z of $\mathbf{B}$,

By lemma 13 there exists a bounded homeomorphism Ψ from $\mathbb{R}^k$ to itself such that $\Psi(\Gamma(v)) = \Delta(v)$. Since each of the polyhedra $\mathbf{D}_i$ is the union of the simplices in T, each of the polyhedra $\Psi(\mathbf{D}_i)$ is the union of simplices in $\Phi(C)$, and hence is in *Poly*[$\mathbb{F}$]. ∎

Lemma 15 *For any subfield $\mathbb{F}$ of $\mathbb{R}$, $\mathcal{M}[\mathbb{R}]$ is rectifiable to $\mathcal{M}[\mathbb{F}]$ over Poly[$\mathbb{F}$].*

Proof: Let $\mathcal{D} = \mathbf{D}_1 \ldots \mathbf{D}_m$ be in *Poly*[$\mathbb{F}$]. Let Φ be a bounded PL mapping such that $\Phi(\mathbf{D}_i) \in$ *Poly*[$\mathbb{F}$] for $i = 1 \ldots m$. Let $\mathbf{P}_i = \Phi(\mathbf{D}_i)$ for $i = 1 \ldots m$. We need to show that there exists an $\mathbb{F}$-PL mapping Ψ such that $\Psi(\mathbf{D}_i) = \Phi(\mathbf{D}_i)$ for $i = 1 \ldots m$.

Let $\mathcal{C}$ be the bounded cells of Φ. Let $\mathbf{B}$ be an $\mathbb{F}$-polyhedron such that, for each $\mathbf{R}$ in $\mathcal{C} \cup \mathcal{D}$, $\mathbf{R} \subset \text{interior}(\mathbf{B})$. Note that since $\partial\mathbf{B}$ is outside all the cells of Φ, Φ is the identity on $\partial\mathbf{B}$. Using lemma 12, let $\mathcal{W}$ be a triangulation of $\mathcal{C} \cup \mathcal{D} \cup \{\mathbf{B}\}$ such that all the vertices of $\mathcal{W}$ in $\partial\mathbf{B}$ are vertices of $\mathbf{B}$. Then $\Phi(\mathcal{W})$ is a triangulation of $\{\mathbf{B}, \mathbf{P}_1 \ldots \mathbf{P}_m\}$, though not in general an $\mathbb{F}$-triangulation.

Let C be an abstract complex and let Γ be a realization such that $\Phi(\mathcal{W}) = \Gamma(C)$; obviously Γ is a proper realization of C. Let Z be the set of abstract vertices in C. Using lemma 11, let U be the open set of proper realizations of C. Let $e > 0$ be such that the box in realization space $[\Gamma_1 - e, \Gamma_1 + e] \times \ldots \times [\Gamma_{k|Z|} - e, \Gamma_{k|Z|} + e] \subset U$.

Let $\mathcal{P}$ be an $\mathbb{F}$-triangulation of $\{\mathbf{P}_i | i = 1 \ldots m\}$. Construct a new realization Δ of Z as follows: Let $\mathbf{q}$ be a point in $\Gamma(Z)$ (a vertex of $\Phi(\mathcal{W})$). If $\mathbf{q}$ is a vertex of $\mathcal{P}$, then $\Delta(\mathbf{q}) = \mathbf{q}$. Otherwise let $\mathbf{Q}$ be the open simplex in $\mathcal{P}$ containing $\mathbf{q}$. Let the coordinates of $\mathbf{q}$ be $\langle q_1 \ldots q_k \rangle$. Since Q is an $\mathbb{F}$-simplex, it is possible to find an $\mathbb{F}$-point $\mathbf{r} = \langle r_1 \ldots r_k \rangle \in Q$ such that $|r_i - q_i| < e$ for $i = 1 \ldots k$. Let $\Delta(z) = \mathbf{r}$. Once this is done, Δ is an $\mathbb{F}$-realization of Z which is a proper realization of C. Note that, since any vertices in $\partial\mathbf{B}$ are all $\mathbb{F}$-points, they are the same under Δ as under Γ.

Since the vertices of the cells of Φ are not necessarily $\mathbb{F}$-points, like the vertex $\mathbf{i}$ in figure 1, we need to carry out the same construction on the domain side of Φ as well. Let Λ be the realization of C corresponding to $\mathcal{W}$. As above, construct a box in realization space centered at Λ and contained in U; and let f be the distance from Λ to the faces of the box. Let $\mathcal{T}$ be an $\mathbb{F}$-triangulation of $\{\mathbf{P}_1 \ldots \mathbf{P}_m, \mathbf{B}\}$. Construct a realization Σ of Z as follows: for each $z \in Z$, let $\mathbf{R}$ be the open simplex in $\mathcal{T}$ such that $\Lambda(\mathbf{z}) \in \mathbf{R}$. Let $\Sigma(z)$ be an $\mathbb{F}$-point in $\mathbf{R}$ within f of $\Lambda(z)$ in each coordinate.

Thus $\Sigma(C)$ is an $\mathbb{F}$-triangulation of $\{\mathbf{P}_1 \ldots \mathbf{P}_m, \mathbf{B}\}$, and $\Delta(C)$ is an $\mathbb{F}$-triangulation of $\{\Phi(\mathbf{P}_1) \ldots \Phi(\mathbf{P}_m), \mathbf{B}\}$. Using lemma 13 construct PL $\mathbb{F}$-homeomorphism Ψ such that $\Psi(\Gamma(z)) = \Delta(z)$ for all $z \in Z$ and such that Ψ is the identity outside $\mathbf{B}$.

What remains to be shown is that for $\mathbf{D} \in \mathcal{D}$, $\Psi(\mathbf{D}) = \Phi(\mathbf{D})$, that all of our original polyhedra have the same image under Ψ as under Φ. Proof by induction over the dimensionality of the faces of $\mathbf{D}$ (*not* the triangulation). If $\mathbf{x}$ is a vertex of $\mathbf{D}$ then $\mathbf{x}$ is an $\mathbb{F}$-vertex so $\Psi(\mathbf{x}) = \Phi(\mathbf{x})$ by construction. If $\mathbf{F}$ is a q-dimensional face, then by induction, each of its boundary faces is the same under Ψ as under Ψ; hence, so is $\mathbf{F}$. ▮

It may be noted that there is a proof of the central part of lemma 15 in [1], Theorem 1. However, the formulation here is sufficiently different — in particular, Beynon does not demonstrate that the mapping between the complexes can be extended to all of $\mathbb{R}^k$, and he asserts lemma 11 without proof — that it seemed worthwhile giving the entire proof here.

Theorem 16 *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Let $Poly[\mathbb{F}]$ be the collection of $\mathbb{F}$-polyhedra in $\mathbb{R}^k$. Let $\mathcal{P}_1 \ldots \mathcal{P}_n$ be topological relations over $\mathbb{R}^k$. Then the structures $\langle Poly, \mathcal{P}_1 \ldots \mathcal{P}_n \rangle$ and $\langle Poly[\mathbb{F}], \mathcal{P}_1 \ldots \mathcal{P}_n \rangle$ are elementarily equivalent.*

Proof: Let $\mathcal{A}$ be the group of bounded PL homeomorphisms, and let $\mathcal{B}$ be the group of bounded PL $\mathbb{F}$-homeomorphisms. It is immediate that $Poly$ is closed under $\mathcal{A}$ and that $Poly[\mathbb{F}]$ is closed under $\mathcal{B}$. By lemma 14 $Poly$ is embeddable in $Poly[\mathbb{F}]$ under $\mathcal{A}$ and by lemma 15 $\mathcal{A}$ is rectifiable to $\mathcal{B}$ over $Poly[\mathbb{F}]$. Hence by lemma 9, $Poly$ and $Poly[\mathbb{F}]$ are mutually extensible. The result then follows from theorem 8. (Note that in applying theorem 8, the universe Ω is the set of regions in $\mathbb{R}^k$, and thus, from the standpoint of theorem 8, $\mathcal{A}$ and $\mathcal{B}$ are viewed as sets of bijections over the space of regions, rather than over the space of points. The same applies in the proof of theorem 27 below.) ▮

We now extend theorem 16 to polytopes. Let $UPoly[\mathbb{F}]$ (U for "unbounded") be the collection of polytopes, bounded and unbounded, and let $\mathcal{L}[\mathbb{F}]$ be the collection of $\mathbb{F}$-PL mappings, bounded and unbounded.

To transfer the above results on bounded polyhedra to the space of unbounded polytopes, we will use piecewise projective transformations.

Definition 18 *An* $\mathbb{F}$-projective mapping *is a function $\Gamma(\mathbf{x}) = (M \cdot \mathbf{x} + \vec{c})/(\vec{a} \cdot \mathbf{x} + b)$ where M is an $\mathbb{F}$-matrix, $\vec{c}$ and $\vec{a}$ are $\mathbb{F}$-vectors and $b \in \mathbb{F}$.*

Let $\mathbf{U}$ and $\mathbf{V}$ be $\mathbb{F}$-polytopes. Let $\mathbf{C}_1 \ldots \mathbf{C}_m$ be a set of $\mathbb{F}$-polytopes such that $\cup_{i=1}^{m} \mathbf{C}_i = \mathbf{U}$. Let Φ be a homeomorphism from $\mathbf{U}$ to $\mathbf{V}$, and let $\Phi_i, i = 1 \ldots m$ be $\mathbb{F}$ projective mappings, such that, for $\mathbf{x} \in \mathbf{C}_i$, $\Phi(\mathbf{x}) = \Phi_i(\mathbf{x})$. Then Φ is an
mathbbF piecewise projective mapping.

Lemma 17 *Let $\mathbf{P}$ be an unbounded closed polytope, and let Γ be a projective transformation such that $\Gamma(\mathbf{P})$ is also an unbounded closed polytope. Then Γ is a linear transformation.*

Proof of the contrapositive. Suppose that Γ is not a linear transformation; then Γ^{-1} is likewise not a linear transformation. Since Γ^{-1} is a projective transformation which is not a linear transformation, it has the form $\Gamma^{-1}(\mathbf{x}) = (M\mathbf{x} + \vec{c})/(\vec{a} \cdot \mathbf{x} + b)$, where $\vec{a} \neq \vec{0}$. Thus Γ^{-1} maps the hyperplane $\mathbf{H} = \{\mathbf{x} \mid \vec{a} \cdot \mathbf{x} + b = 0\}$ to the hyperplane at infinity, so Γ maps the hyperplane at infinity to $\mathbf{H}$. Considered as a function over the projective space, Γ is a continuous function. Since $\mathbf{P}$ borders the hyperplane at infinity, $\Gamma(\mathbf{P})$ borders but does not include $\mathbf{H}$, so $\Gamma(\mathbf{P})$ is not an unbounded closed polytope. ▮

In the proofs below, we will use extensively one particular piecewise projective mapping, which we will denote $\Theta(\vec{v})$. It is defined as follows. Let $\mathbf{B}$ be the open box of side 2 centered at the origin $\mathbf{B} = (-1,1)^k$. Let $\mathbf{v} = \langle v_1 \ldots v_k \rangle$ be a point in $\mathbf{B}$ and let $u = \max_{i=1}^{k} |v_i|$. Define $\Theta(\mathbf{v}) = \mathbf{v}/(1-u)$. Note that:

- Θ is a homeomorphism from $\mathbf{B}$ to $\mathbb{R}^k$.
- Θ is a piecewise projective transformation. It has $2k$ cells, $\mathbf{C}_{i+}$ and $\mathbf{C}_{i-}$ for $i = 1 \ldots k$, defined as follows:
  $\mathbf{C}_{i+} = \{\mathbf{p} \mid 0 \leq p_i < 1, p_i \geq p_j, p_i \geq -p_j, \text{ for all } j \neq i\}$
  $\mathbf{C}_{i-} = \{\mathbf{p} \mid -1 < p_i \leq 0, p_i \leq p_j, p_i \leq -p_j, \text{ for all } j \neq i\}$.
- Likewise, the cells of Θ^{-1} are the regions
  $\mathbf{D}_{i+} = \{\vec{p} \mid p_i \geq 0, p_i \geq p_j, p_i \geq -p_j, \text{ for all } j \neq i\}$ and
  $\mathbf{D}_{i-} = \{\vec{p} \mid -p_i \leq 0, p_i \leq p_j, p_i \leq -p_j, \text{ for all } j \neq i\}$.

For example in $\mathbb{R}^2$ the four cells of Θ are the left, right, up, and down quadrants of $\mathbf{B}$, bounded by the diagonals $x = \pm y$, and the cells of the Θ^{-1} are the quadrants of the plane.

Since Θ is piecewise projective, it maps any open polytope in $\mathbf{B}$ to an open polytope in $\mathbb{R}^k$. If $\mathbf{R}$ is an open polyhedron such that closure($\mathbf{R}$) $\subset \mathbf{B}$, then $\Theta(\mathbf{R})$ is a bounded polyhedron, and conversely. If $\mathbf{R}$ is an open polyhedron such that closure($\mathbf{R}$) $\subset$ closure($\mathbf{B}$) and closure($\mathbf{R}$) $\cap\, \partial\mathbf{B} \neq \emptyset$, then $\Theta(\mathbf{R})$ is an unbounded open polytope, and conversely.

The important feature of Θ is that it maps $\mathbf{B}$ to $\mathbb{R}^k$, while mapping polyhedra to polytopes.

The following lemma is analogous to lemma 13.

Lemma 18 *Let $\bar{\mathbf{B}} = [-1,1]^k$ (note that this is the closed box). Let $\mathbf{P}_1 \ldots \mathbf{P}_m$ be polyhedra such that for $i = 1 \ldots m$, $\mathbf{P}_i \subset \bar{\mathbf{B}}$, and let $\mathcal{P} = \{\bar{\mathbf{B}}, \mathbf{P}_i \ldots \mathbf{P}_m\}$. Let $\mathcal{T}$ be a triangulation of $\mathcal{P}$. Let C be an abstract complex and Γ be a realization such that $\Gamma(C) = \mathcal{T}$. Let Δ be a proper realization of C such that, for every abstract vertex $z \in Z$, if $\Gamma(z) \in \partial\bar{\mathbf{B}}$, then $\Delta(z)$ is on the same face of $\partial\bar{\mathbf{B}}$ as $\Gamma(z)$. Then there exists an PL homeomorphism Ψ from $\mathbf{B}$ to itself such that for every z, $\Psi(\Gamma(z)) = \Delta(z)$ and such that, if $\mathbf{F}$ is a face of $\bar{\mathbf{B}}$, then $\Psi(\mathbf{F}) = \mathbf{F}$. ($\Psi$ need not be defined outside $\bar{\mathbf{B}}$).*

Proof: As in the proof of lemma 13, for any point $\mathbf{x} \in \mathbf{B}$, let $\mathbf{x}_1 \ldots \mathbf{x}_q$ be the vertices of the open simplex in $\mathcal{T}$ containing $\mathbf{x}$, and let $\langle t_1 \ldots t_q \rangle$ be the barycentric coordinates of $\mathbf{x}$. Define $\Psi(\mathbf{x})$ such that, for $\mathbf{x} \in \mathbf{B}$, $\Psi(\mathbf{x}) = \sum_{i=1}^{q} t_i \Delta(\Gamma^{-1}(\mathbf{x}_i))$; The proof that Ψ is a homeomorphism proceeds as in lemma 13. If $\mathbf{x}$ is in a face $\mathbf{F}$ of $\bar{\mathbf{B}}$, let C be the abstract simplex $C = \Gamma^{-1}(\{\mathbf{x}_1 \ldots \mathbf{x}_q\})$. By assumption $\Delta(C) \subset \mathbf{F}$, so $\Phi(\mathbf{x}) \in \mathbf{F}$. ▮

Lemma 19 *Let χ be a piecewise projective transformation that is a homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$. Let $\{\mathbf{C}_1 \ldots \mathbf{C}_m\}$ be the cells of χ. Then there exists a PL homeomorphism Φ from $\mathbb{R}$ to itself such that, for $i = 1 \ldots m$, $\Phi(\mathbf{C}_i) = \chi(\mathbf{C}_i)$.*

Proof: Let χ_i be the restriction of χ to $\mathbf{C}_i$. Since χ is a homeomorphism of $\mathbb{R}^k$ to itself, for any unbounded cell $\mathbf{C}_i$, $\Gamma(\mathbf{C}_i)$ must be unbounded so, by lemma 17, χ_i is a linear transformation.

We define Φ as follows. Let $\mathcal{T}$ be a triangulation of the bounded polytopes in $\{\mathbf{C}_1 \ldots \mathbf{C}_m\}$.

- If $\mathbf{x}$ is in one of the simplexes in $\mathcal{T}$, let $\langle \mathbf{s}_1 \ldots \mathbf{s}_q \rangle$ be the open simplex of $\mathcal{T}$ containing $\mathbf{x}$; let $\langle t_1 \ldots t_q \rangle$ be the barycentric coordinates of $\mathbf{x}$; and let $\Phi(\mathbf{x}) = \sum_{i=1}^{q} t_i \cdot \chi(\mathbf{s}_i)$.
- Otherwise, let $\Phi(\mathbf{x}) = \chi(\mathbf{x})$.

The proof that Φ is continuous across bounded simplexes is same argument as in lemma 13. The fact it is consistently defined going from the bounded to the unbounded cells follows from the fact that Ψ_2 is linear on the unbounded cells, and hence must agree with the barycentric mapping on the bounded faces of the unbounded cells. ∎

Lemma 20 *$UPoly[\mathbb{R}]$ is embeddable in $UPoly[\mathbb{F}]$ under $\mathcal{L}[\mathbb{R}]$.*

Proof: Let $\langle \mathbf{P}_1 \ldots \mathbf{P}_m \rangle$ be a tuple of polytopes in $UPoly[\mathbb{R}]$. For $i = 1 \ldots m$ let $\mathbf{Q}_i = \text{closure}(\Theta^{-1}(\mathbf{P}_i))$. Thus $\langle \mathbf{Q}_1 \ldots \mathbf{Q}_m \rangle$ is a tuple of polyhedra in $\bar{\mathbf{B}}$. Let $\mathbb{Q} = \{\bar{\mathbf{B}}, \mathbf{Q}_1 \ldots \mathbf{Q}_m\}$. Let $\mathcal{T}$ be a triangulation of $\mathbb{Q}$. Let C be an abstract simplex and Γ a realization of C such that $\Gamma(C) = \mathcal{T}$. By lemma 11 there exists an proper realization Δ such that $\Delta(z)$ is an $\mathbb{F}$-point for all $z \in C$ and such that, if $\Gamma(z)$ is in a face $\mathbf{F}$ of $\bar{\mathbf{B}}$ then $\Delta(z) \in \mathbf{F}$. By lemma 18 there exists an PL homeomorphism Ψ from $\mathbf{B}$ to itself such that for every z, $\Psi(\Gamma(z)) = \Delta(z)$ and such that, if $\mathbf{F}$ is a face of $\bar{\mathbf{B}}$, then $\Phi(\mathbf{F}) = \mathbf{F}$.

Now let $\chi = \Theta \circ \Psi \circ \Theta^{-1}$. Clearly this is a homeomorphism from $\mathbb{R}^k$ to itself. Since it is the composition of piecewise projective mappings, χ is itself a piecewise projective mapping. Since χ satisfies the conditions of lemma 19, there exists an LP homeomorphism Φ such that $\Phi(\mathbf{P}_i) = \chi(\mathbf{P}_i)$. ∎

Lemma 21 *$\mathcal{L}[\mathbb{R}]$ is rectifiable to $\mathcal{L}[\mathbb{F}]$ over $UPoly[\mathbb{F}]$.*

Proof: Let $\mathbf{P}_1 \ldots \mathbf{P}_m$ be $\mathbb{F}$-polytopes and let Γ be a PL mapping such that $\Gamma(\mathbf{P}_i)$ is an $\mathbb{F}$-polytope for $i = 1 \ldots m$. Note that $\Gamma(\mathbb{R}^k) = \mathbb{R}^k$. For $i = 1 \ldots m$, let $\mathbf{Q}_i = \Theta^{-1}(\mathbf{P}_i)$ and let $\mathbf{W}_i = \Theta^{-1}(\Gamma(\mathbf{P}_i)$; these are all $\mathbb{F}$-polyhedra. Let $\Psi = \Theta^{-1} \circ \Gamma \circ \Theta$. Thus Ψ is a piecewise projective mapping from $\bar{\mathbf{B}}$ to itself such that $\Psi(\mathbf{Q}_i) = \mathbf{W}_i$. Moreover, since Ψ is a homeomorphism it preserves betweenness relations on the points in $\bar{\mathbf{B}}$. That is, if $\mathbf{c}_1 \ldots \mathbf{c}_q$ are in $\bar{\mathbf{B}}$ and point $\mathbf{x}$ is in the open simplex with vertices $\mathbf{c}_1 \ldots \mathbf{c}_q$, then $\Psi(\mathbf{x})$ is in the open simplex with vertices $\Psi(\mathbf{c}_1) \ldots \Psi(\mathbf{c}_q)$.

Therefore, let $\mathbb{Q} = \{\bar{\mathbf{B}}, \mathbf{Q}_1 \ldots \mathbf{Q}_m\}$. Let $\mathcal{W}$ be the set of all intersections of a region in $\mathbb{Q}$ with a cell of Ψ. Let $\mathcal{T}$ be a triangulation of $\mathcal{W}$. Define the PL-mapping Ψ_2 from $\bar{\mathbf{B}}$ to itself as follows: For any point $\mathbf{x}$ let $\mathbf{c}_1 \ldots \mathbf{c}_q$ be the open simplex in $\mathcal{T}$ containing $\mathbf{x}$, let $t_1 \ldots t_q$ be the barycentric coordinates of $\mathbf{x}$, and let $\Psi_2(\mathbf{x}) = \Sigma_{i=1}^{q} t_i \Psi(\mathbf{c}_i)$. Then Ψ_2 is a PL homeomorphism from $\bar{\mathbf{B}}$ to itself. Moreover, for any simplex $\mathbf{S}$ in $\mathcal{T}$, $\Psi_2(\mathbf{S}) = \Psi(\mathbf{S})$; hence $\Psi_2(\mathbf{Q}_i) = \Psi(\mathbf{Q}_i)$.

By lemma 15, Ψ_2 is rectifiable to an $\mathbb{F}$-PL mapping; that is, there exists an $\mathbb{F}$-PL mapping Φ_2 such that $\Phi_2(\mathbf{Q}) = \Phi(\mathbf{Q})$ for $\mathbf{Q} \in \mathbb{Q}$.

Now, let $\chi = \Theta \circ \Phi_2 \cup \Theta^{-1}$. This is an piecewise $\mathbb{F}$-projective mapping from $\mathbb{R}^k$ to itself. For any $\mathbf{P}_i$ we have

$$\chi(\mathbf{P}_i) = \Theta(\Phi_2(\Theta^{-1}(\mathbf{P}_i))) = \Theta(\Phi_2(\mathbf{Q}_i)) = \Theta(\Psi(\mathbf{Q}_i)) = \Theta(\Theta^{-1}(\Gamma(\Theta(\Theta^{-1}\mathbf{P}_i)))) = \Gamma(\mathbf{P}_i)$$

Since χ satisfies the conditions of lemma 19, there exists an $\mathbb{F}$-PL mapping Φ such that $\Phi(\mathbf{P}_i) = \chi(\mathbf{P}_i) = \Gamma(\mathbf{P}_i)$. ∎

Theorem 22 *Let $\mathcal{P}_1 \ldots \mathcal{P}_n$ be topological relations. Then the structures $\langle UPoly[\mathbb{R}], \mathcal{P}_1 \ldots \mathcal{P}_n \rangle$ and $\langle UPoly[\mathbb{F}], \mathcal{P}_1 \ldots \mathcal{P}_n \rangle$ are elementarily equivalent.*

Proof: Identical to the proof of theorem 16, replacing the classes of bounded homeomorphisms by classes of unbounded homeomorphisms, and replacing lemmas 14 and 15 by lemmas 20 and 21. ∎

4 O-minimal domains

Using the powerful theory of o-minimal domains [16] we can show that all o-minimal domains are elementarily equivalent with respect to topological languages over regions in $\mathbb{R}^k$.

We begin with the set-theoretic definition of an o-minimal domain (there is also an equivalent model-theoretic definition):

Definition 19 *For $m = 1, 2, \ldots$ let $\mathbb{O}_m$ be a collection of subsets of $\mathbb{R}^m$. The sequence $\mathbb{O} = \langle \mathbb{O}_1, \mathbb{O}_2 \ldots \rangle$ is an* o-minimal domain over $\mathbb{R}$ *if the following conditions are satisfied:*

- *$\mathbb{O}_1$ is the set of all finite unions of points and intervals in $\mathbb{R}$.*
- *The graph of the addition function $\{\langle x, y, z \rangle \mid z = x + y\}$ and the graph of the multiplication function $\{\langle x, y, z \rangle \mid z = xy\}$ are elements of $\mathbb{O}_3$.*
- *The set $\{\langle x_1, \ldots, x_m \mid x_1 = x_m\}$ is an element of $\mathbb{O}_m$*
- *$\mathbb{O}_m$ is closed under pairwise union, intersection, and complementation.*
- *If $A \in \mathbb{O}_m$ then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ are in $\mathbb{O}_{m+1}$.*
- *If $A \in \mathbb{O}_{m+1}$ then the projection of A,onto the first m coordinates,*
  $\pi(A) = \{\langle x_1, \ldots, x_m \rangle \mid \exists_y \langle x_1, \ldots, x_m, y \rangle \in A\}$
  is in $\mathbb{O}_m$.

Examples of o-minimal structures include the class of semi-algebraic regions and the class of sub-analytic regions.

Definition 20 *Let $\mathbb{O}$ be an o-minimal collection over $\mathbb{R}$. A function Γ from $\mathbb{R}^m$ to $\mathbb{R}^n$ is* definable *with respect to $\mathbb{O}$ if the graph $\{\langle x, \Gamma(x) \rangle \mid x \in \mathbb{R}^m\}$ is an element of $\mathbb{O}_{m+n}$.*

For the remainder of this section, let $\mathbb{O}$ be an o-minimal domain. Let $\mathcal{A}$ be the class of homeomorphisms from $\mathbb{R}^k$ to itself that are definable relative to $\mathbb{O}$. Let $\mathcal{M}$ be the class of piecewise-linear homeomorphisms over $\mathbb{R}^k$. Let $UDef$ be the class of definable, topologically regular k-dimensional regions in $\mathcal{O}$, and let Def be the class of bounded regions in $UDef$.

Lemma 23 *For any finite set of definable regions $\mathcal{D} \subset UDef$ there exists a definable homeomorphism Ψ over their union such, for every $\mathbf{D} \in \mathcal{D}$, $\Psi(\mathbf{D})$ is the union of simplices in $\mathbb{R}^k$.*

Proof: This is the triangulation theorem for o-minimal structures. See [16] p. 130. ∎

Lemma 24 *Let $\langle \mathbf{X}_1 \ldots \mathbf{X}_m \rangle$ and $\langle \mathbf{Y}_1 \ldots \mathbf{Y}_m \rangle$ be sequences of polyhedra. If there is a defineable homemorphism Γ such that $\mathbf{Y}_i = \Gamma(\mathbf{X}_i)$ for $I = 1 \ldots m$ then there exist a PL homemophism Φ such that homeomorphic then they are PL homeomorphic. $\mathbf{Y}_i = \Phi(\mathbf{X}_i)$ for $I = 1 \ldots m$.*

That is, the space of definable homeomorphisms is rectifiable to the space of piecewise linear homeomorphisms over the domain of compact polyhedra.

Proof: See [15], p. 5, theorem 2.1.

Lemma 25 *Let $\mathcal{D}$ be a finite subset of UDef. Then there exists a definable homeomorphism Φ from $\mathbb{R}^k$ to $\mathbb{R}^k$ such that $\Phi(\mathcal{D}) \subset UPoly$. That is, UDef is embeddable in UPoly over $\mathcal{A}$.*

Likewise, Def is embeddable in Poly over $\mathcal{A}$.

Proof: Let $\mathcal{E} = \mathcal{D} \cup \{\mathbb{R}^k\}$; By lemma 23 there exists a homeomorphism Ψ_1 such that $\Psi_1(\mathbf{E})$ is the union of simplices for all $\mathbf{E} \in \mathcal{E}$. Note that $\Psi_1(\mathbf{E}) \subset \text{interior}(\Psi(\mathbb{R}^k))$ for each $\mathbf{E} \in \mathcal{E}$. Also, $\Psi_1(\mathbb{R}^k)$ is a bounded open polyhedron homeomorphic to $\mathbb{R}^k$ and therefore to the open box $(-1,1)^k$. By lemma 24 there exists a PL homeomorphism Ψ_2 from $\Phi_1(\mathbb{R}^k)$ to $(-1,1)^k$. Since Ψ_2 is piecewise linear, $\Psi_2(\Psi_1(\mathbf{D}))$ is still a regular polyhedron for each $\mathbf{D} \in \mathcal{D}$ and $\Psi_2(\Psi_1(\mathbf{D}))$ is inside $\mathbf{B}$. Now apply the piecewise projective transformation Θ and let Φ be the composition $\Phi = \Theta \circ \Psi_2 \circ \Psi_1$. Then for each $\mathbf{D} \in \mathcal{D}$, $\Phi(\mathbf{D})$ is a polytope.

If all of the $\mathbf{D}$ are bounded, then $\Phi(\mathbf{D})$ is bounded and thus in *Poly*. ∎.

Lemma 26 *The space of definable homeomorphisms $\mathcal{A}$ is rectifiable to the space of piecewise linear homeomorphisms $\mathcal{M}$ over UPoly.*

Proof: Let $\mathcal{D} = \mathbf{D}_1 \dots \mathbf{D}_m$ be a set of polytopes, and let Γ be a homeomorphism in $\mathcal{A}$ such that $\Gamma(\mathbf{D}_i)$ is a polytope for $i = 1 \dots m$. For $i = 1 \dots m$ let $\mathbf{P}_i = \text{closure}(\Theta^{-1}(\mathbf{D}_i))$; thus $\mathbf{P}_i \subset \bar{\mathbf{B}}$. Consider the composition $\Phi = \Theta^{-1} \circ \Gamma \circ \Theta$, which is a definable homeomorphism from $\mathbf{B}$ to itself. Note that for any $\mathbf{P}_i$, $\Phi(\mathbf{P}_i) = \Theta^{-1}(\Gamma(\mathbf{D}_i))$ is a polyhedron in $\mathbf{B}$; thus Φ is a definable homeomorphism mapping polyhedra in $\mathbf{B}$ to polyhedra in $\mathbf{B}$. By lemma 24, there exists a PL-mapping Ψ such that $\Psi(\mathbf{P}_i) = \Phi(\mathbf{P}_i)$ for $i = 1 \dots m$. Now let χ be the composition $\Theta \circ \Psi \circ \Theta^{-1}$. For any $\mathbf{D}_i$,

$$\chi(\mathbf{D}_i) = \Theta(\Psi(\mathbf{P}_i)) = \Theta(\Phi(\mathbf{P}_i)) = \Theta(\Theta^{-1}(\Gamma(\mathbf{P}_i))) = \Gamma(\mathbf{P}_i)$$

Then χ satisfies the conditions of lemma 19, so there exists a PL-mapping Φ such that $\Phi = \Gamma(\mathbf{P}_i)$. ∎.

Theorem 27 *Let $\mathcal{O}$ be an o-minimal structure. Let UDef be the class of regular regions in $\mathbb{R}^k$ definable in $\mathcal{O}$. Let $\mathcal{P}_1 \dots \mathcal{P}_m$ be topological predicates over regions in $\mathbb{R}^k$. Then the structures $\langle UDef, \mathcal{P}_1 \dots \mathcal{P}_m \rangle$ and $\langle UPoly, \mathcal{P}_1 \dots \mathcal{P}_m \rangle$ are elementarily equivalent.*

Let Def be the collection of bounded regions in UDef. Then $\langle Def, \mathcal{P}_1 \dots \mathcal{P}_m \rangle$ and $\langle Poly, \mathcal{P}_1 \dots \mathcal{P}_m \rangle$ are elementarily equivalent.

Proof: It is immediate that *UDef* is closed under $\mathcal{A}$ and that *UPoly* is closed under $\mathcal{M}$. Together with the results of lemma 23 and lemma 25, the conditions of lemma 9 are satisfied, so $\mathcal{C}$ and *Poly* are mutually extendible relative to $\mathcal{A}$. The result then follows theorem 8.

The same argument applies to *Def* and *Poly*. ∎

We have thus shown that the collections of k-dimensional regular regions corresponding to two o-minimal structures are elementarily equivalent. The o-minimal structures over Euclidean space are few but very important; they include the polyhedra, the semi-algebraic regions, and the sub-analytic regions.

5 Conclusion

We have demonstrated that the domain of rational polyhedra and the domain of definable regions relative to an o-minimal structure are each elementarily equivalent to the domain of polyhedra.

The most obvious problem left open is to find a theorem comparable to theorem 1, giving *geometric* criteria rather than algebraic or structural criteria sufficient to guarantee elementary equivalence to polyhedra that will apply in dimensions higher than 2. One would think it should be possible to find some reasonable set of geometric conditions that guarantee a simplicial structure that will support the kind of construction used in the proof of theorem 16, but I have not been able to work it through.

Another open problem: The results in this paper apply to first-order structures with *any* topological relations over the space of regions. The more common object of study, however, is the first-order language with the single predicate `C(x,y)` (regions `x` and `y` are connected). Quite a bit is known about this more limited language. The computational complexity has been characterized quite fully: it is undecidable over the domain of polyhedra and even over the domain of unions of rectangles [6]; over the domain of regular regions, the language has the undecidability of the second-order theory of the integers [14]. Some strong positive results about expressivity have been proven: a variety of individual relations have been shown to be expressible in this language [12, 5], and Pratt-Hartmann [10] has proved that, in $\mathbb{R}^2$ and $\mathbb{R}^3$, any specific topological layout of regions that can be attained by polyhedra can be characterized by a formula in the language. However, the limits of expressivity are not known; for instance, it is not known whether the binary relation "Regions **P** and **Q** are homeomorphic" can be expressed in the language.

By contrast, the expressive power of languages over regions that include some notion of distance or of convexity, such as that language with the predicate `Convex(r)` (`r` is a convex region) or `Closer(x,y,z)` (region `x` is closer to `y` than to `z`), is known to be very great [3]. In fact, the only domain first-order equivalent to *Poly* for such languages is *Poly* itself; one can in effect assert the statement, "all regions are polygons and all polygons are regions" in either of these languages.

References

[1] Beynon, W.M. 1977. "On Rational Subdivisions of Polyhedra with Rational Vertices." *Canadian Journal of Mathematics,* **19**(2), 238-242.

[2] Cohn, A. and Renz, J. 2008. "Qualitative Spatial Representation and Reasoning." In *The Handbook of Knowledge Representation,* F. van Harmelen, V. Lifschitz, and B. Porter (eds.), Elsevier, Oxford, 2008.

[3] Davis, E. 2006. "The Expressivity of Quantifying over Regions." *Journal of Logic and Computation,* **16**, 891-916.

[4] Egenhofer, M. and Franzosa, R. 1991. "Point-Set Topological Spatial Relations." *International Journal of Geographical Information Systems,* **5**(2), 161-174.

[5] Gotts, N.M. 1994, "How far can we 'C'? Defining a doughnut using connection alone." *Fourth International Conference on Principles of Knowledge Representation and Reasoning.* 246-257.

[6] Grzegorczyk, A. 1951. "Undecidability of some topological theories." *Fundamenta Mathematicae.* **38**, 137-152.

[7] Kontchakov, R., Pratt-Hartmann, I., and Zakharyaschev, M. 2010. "Interpreting Topological Logics over Euclidean Space." *Twelfth International Conference on Principles of Knowledge Representation and Reasoning (KR-10).*

[8] Kontchakov, R., Pratt-Hartmann, I., Wolter, F. and Zakharyaschev, M. 2010. "Spatial logics with connectedness predicates." *Logical Methods in Computer Science,* to appear.

[9] Munkres, J. 1984. *Elements of Algebraic Topology.* Menlo Park, CA: Addison Wesley.

[10] Pratt-Hartmann, I. 2007. "First-Order Mereotopology". In M. Aiello, I. Pratt-Hartmann, and J. van Benthem (eds.) *Handbook of Spatial Logics.* New York:Springer-Verlag

[11] Randell, D.A, Cui, Z., and Cohn, A.G. 1992. "Modelling topological and metrical properties of physical properties," *First International Conference on Principles of Knowledge Representation and Reasoning (KR-89).* 55-66.

[12] Randell, D.A, Cui, Z., and Cohn, A.G. 1992. "A spatial logic based on regions and connection." *Third International Conference on Principles of Knowledge Representation and Reasoning (KR-92).* 165-176.

[13] Rourke, C.P. and Sanderson, B.J. 1982. *Introduction to Piecewise-Linear Topology.* New York:Springer-Verlag.

[14] Schaefer, M. and Stefanovic, D. 2004. "Decidability of string graphs." *Journal of Computer and System Sciences.* **68**, 319-334.

[15] Shiota, M. 2010. "PL and differential topology in o-minimal structure" http://arxiv.org/abs/1002.1508v3

[16] van den Dries, L. 1998. *Tame Topology and O-minimal Structures.* Cambridge: Cambridge University Press.