Proof of the Cauchy-Schwarz inequality. To replace the proof given on p. 25.

I cannot now imagine why I included a proof of the Cauchy-Schwarz inequality from the triangle inequality, left unproven in the text, since a complete proof is not very difficult. Anyway, here is a proof, albeit not a very insightful one, since the value of q below is pulled out of a hat.

Let $\vec{x}$ and $\vec{y}$ be two vectors; we want to prove that $|\vec{x} \bullet \vec{y}| \leq |\vec{x}| \cdot |\vec{y}|$. Let $q = (\vec{x} \bullet \vec{y})/|\vec{y}|^2$. Let $\vec{w} = \vec{x} - q \cdot \vec{y}$. Then $0 \leq |\vec{w}|^2 = \vec{w} \bullet \vec{w}$, since that is true for all vectors. But

$$\vec{w} \bullet \vec{w} = (\vec{x} - q\vec{y}) \bullet (\vec{x} - q\vec{y}) = \vec{x} \bullet \vec{x} - 2q \cdot (\vec{x} \bullet \vec{y}) + q^2 \cdot (\vec{y} \bullet \vec{y}) =$$

$$|\vec{x}|^2 - 2\frac{\vec{x} \bullet \vec{y}}{|\vec{y}|^2} \cdot \vec{x} \bullet \vec{y} + \left(\frac{\vec{x} \bullet \vec{y}}{|\vec{y}^2|}\right)^2 \cdot |\vec{y}|^2 = |\vec{x}|^2 - \frac{(\vec{x} \bullet \vec{y})^2}{|\vec{y}|^2}$$

Thus we have

$$|\vec{x}|^2 - \frac{(\vec{x} \bullet \vec{y})^2}{|\vec{y}|^2} \geq 0$$

So $|\vec{x}|^2 \cdot |\vec{y}|^2 \geq (\vec{x} \bullet \vec{y})^2$, so $|\vec{x}| \cdot |\vec{y}| \geq |\vec{x} \bullet \vec{y}|$.

Once we have proven the Cauchy-Schwarz inequality, then it is easy to prove the triangle inequality:

$|\vec{x} + \vec{y}|^2 = (\vec{x} + \vec{y}) \bullet (\vec{x} + \vec{y}) = \vec{x} \bullet \vec{x} + 2\vec{x} \bullet \vec{y} + \vec{y} \bullet \vec{y} = |\vec{x}|^2 + 2\vec{x} \bullet \vec{y} + |\vec{y}|^2 \leq$ (by Cauchy-Schwarz) $|\vec{x}|^2 + 2|\vec{x}| \cdot |\vec{y}| + \vec{y}^2 = (|\vec{x}| + |\vec{y}|)^2$.

Hence $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$.