Preconditioning and Boundary Conditions

by

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ABSTRACT

Consider the large systems of linear equations $A_h u_h = f_h$ which arise from the discretization of a second-order elliptic boundary value problem. Consider also the preconditioned systems (i) $B_h^{-1} A_h u_h = B_h^{-1} f_h$; and (ii) $A_h B_h^{-1} v_h = f_h$, $u_h = B_h^{-1} v_h$ where B_h is itself a matrix which arises from the discretization of another elliptic operator. We discuss the effect of boundary conditions (of A and B) on the L_2 and H_1 condition of $B_h^{-1} A_h$, $A_h B_h^{-1}$. In particular, in the case of H_2 regularity one finds that $\|B_h^{-1} A_h\|_{L_2}$ is uniformly bounded if and only if A^* and B^* have the same boundary conditions while $\|A_h B_h^{-1}\|_{L_2}$ is uniformly bounded if and only if A and B have the same boundary conditions. Similarly, $\|B_h^{-1} A_h\|_{H_1}$ is uniformly bounded only if A and B have homogeneous Dirichlet boundary conditions on the same portion of the boundary. This latter result does not depend upon H_2 regularity.
Key words. *preconditioning, elliptic operators, condition number, H_2 regularity, boundary conditions*

1. Introduction

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with a boundary $\partial\Omega$. Consider a uniformly elliptic operator A of the form

$$\begin{aligned} Au = &-[(a_{11} u_x)_x + (a_{12} u_y)_x + (a_{12} u_x)_y + (a_{22} u_y)_y] \\ &+ a_1 u_x + a_2 u_y + a_0 u \end{aligned} \tag{1.1a}$$

with boundary conditions of the form

$$u = 0 \quad , \quad \text{on} \quad \Gamma_0 = \Gamma_0(A) \tag{1.1b}$$

$$\frac{\partial u}{\partial \nu} = \alpha_0(\sigma) u(\sigma) + \alpha_1(\sigma) \frac{\partial u}{\partial \sigma} \quad , \quad \text{on} \quad \Gamma_1 = \Gamma_1(A) \tag{1.1c}$$

where $\Gamma_0(A), \Gamma_1(A)$ are a partition of $\partial\Omega$, i.e.,

$$\partial\Omega = \Gamma_0(A) \cup \Gamma_1(A)$$

and $\dfrac{\partial u}{\partial \nu}$ denotes the exterior normal derivative while $\dfrac{\partial u}{\partial \sigma}$ denotes the tangential derivative and σ denotes the tangential variable. The uniform ellipticity of A is expressed by two positive constants $0 < \lambda(A) \le \Lambda(A)$ such that, for all $(x,y) \in \overline{\Omega}$ we have the inequalities

$$\lambda(A)\, [\xi^2 + \eta^2] \le a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 \le \Lambda(A)[\xi^2 + \eta^2] \quad . \tag{1.2}$$

Consider a discretization A_h of this elliptic operator and the system of linear equations

$$A_h \underline{u} = \underline{f} \tag{1.3a}$$

which arises in the numerical solution of the boundary-value problem

$$Au = f \quad . \tag{1.3b}$$

Our concern is with the effective solution of (1.3a). In certain cases, e.g., the self-adjoint Laplace operator with Dirichlet boundary conditions ($\Gamma_0 = \partial\Omega, \Gamma_1 = \emptyset$) in a square, there are relatively fast and stable methods for the solution of this problem. However, in the general case of nonself-adjoint problems whose symmetric part may be indefinite, this can be an expensive and time consuming problem. Part of the difficulty arises because the condition number of the discrete approximations,

$$C_h(A_h) := ||A_h||_h \cdot ||A_h^{-1}||_h \quad , \tag{1.4}$$

is of order h^{-2}. Hence, one approach has been to ‘‘precondition’’ A_h to obtain a problem with a smaller condition number (cf [BP], [CoGu], [DoDu], [Dy], [ElSh], [Gu], and [Wi]). That is, given an appropriate B_h one considers either

$$B_h^{-1} A_h \underline{u} = B_h^{-1} \underline{f} \tag{1.5a}$$

or

$$A_h B_h^{-1} \underline{v} = \underline{f} \quad , \quad B_h^{-1} \underline{v} = \underline{u} \quad . \tag{1.5b}$$

In particular, it has been suggested that B_h be the discretization of another elliptic operator B which has been chosen so that, among other things, B_h^{-1} is relatively easy to obtain. In addition, it is frequently suggested that B be chosen as the ‘‘leading part’’ of A <u>with the same boundary conditions as A</u>. However, as we shall see, if one preconditions on the right, i.e., (1.5b), then this is a reasonable recipe while if one preconditions on the left, i.e., (1.5a) then it is better to choose B so that A^* and B^* have the same boundary conditions. The situation is further complicated by the question: which norm should be used in the computation of the condition number? While a careful look at the theory of iterative processes suggests that the L_2 (or l_2) norm is the correct choice, much of the literature (for finite-element equations) deals with the H_1 norm.

The goal of this paper is to clarify this situation. For this purpose, we will discuss both the L_2 condition numbers $C_{L_2}(B_h^{-1}A_h)$, $C_{L_2}(A_h B_h^{-1})$ and the H_1 condition numbers $C_{H_1}(B_h^{-1}A_h)$. However, in order to do this we must discuss the elliptic partial differential operator and the discrete approximation with greater care and precision.

Let $f \in L_2(\Omega)$ and consider the boundary-value problem

$$Au = f \quad . \tag{1.6}$$

Suppose that, in some sense, this problem has a unique solution for all $f \in L_2(\Omega)$. That is, we suppose that in some general sense A is invertible on $L_2(\Omega)$. If the coefficients of (1.1a), (1.1c) are smooth, $\partial\Omega$ is smooth and either $\Gamma_0 = \emptyset$ or $\Gamma_1 = \emptyset$ then (see [Gr]) A^{-1} is a bounded map from $L_2(\Omega)$ to $H_2(\Omega)$. In other words, there is a constant $K_1(A) > 0$ such that, for every given $f \in L_2$ the solution u of (1.6) is in $H_2(\Omega)$, satisfies the boundary conditions (1.1b), (1.1c) in the trace operator sense and

$$||u||_{H_2} \le K_1(A)\,||f||_{L_2} \quad . \tag{1.7}$$

Assuming these conditions are met we may describe the operator A as follows: the domain of A is given by

$$D(A) \equiv \{u \in H_2(\Omega):\ u \text{ satisfies } (1.1b),\ (1.1c)\} \tag{1.8}$$

Of course, A is a bounded map from its domain onto L_2. That is, there is a constant $K_2(A)$ such that

$$||Au||_{L_2} \le K_2(A)\,||u||_{H_2} \quad , \quad u \in D(A) \quad . \tag{1.9}$$

In the general case when neither $\Gamma_0 = \emptyset$ or $\Gamma_1 = \emptyset$ the situation is somewhat delicate. For example, let A be the Laplace operator. Then one can exhibit (polygonal) domains and boundary conditions for which (1.7) is false–(see [Gr]). In this latter case, some solutions (with $\Delta u = f \in L_2$) are not in $H_2(\Omega)$. Our discussion of the L_2 condition of $A_h B_h^{-1}$ and $B_h^{-1} A_h$ depends on the estimates (1.7), (1.9) and the constants $K_1(A), K_2(A)$, $K_1(A^*)$, $K_2(A^*)$, $K_1(B)$, $K_2(B)$, $K_1(B^*)$, $K_2(B^*)$. Hence, while our discussion is sometimes phrased in the generality of boundary conditions (1.1b), (1.1c), the reader is reminded that the H_2 regularity is usually essential to the discussion. Of course, this includes a large class of problems. On the other hand, our discussion of the H_1 condition does not require H_2 regularity and hence includes the case of the general boundary conditions (1.1b), (1.1c).

In discussing the discretizations, A_h, it is convenient to adopt the language of finite-element theory. However, our discussion is fairly general. Imagine a family of finite dimensional function spaces $\{S_h\}$ indexed by a discretization parameter $h \to 0$. Every function $v_h \in S_h$ is described by a finite vector $\underline{v} = \{v_j \ :\ j = 1,2,...,n_h\}$ depending on the choice of basis for S_h. The operator A_h of (1.3a) acts on this representing vector $\underline{v}$. The norm used in (1.4) is given by

$$<\underline{u},\underline{v}>_h \ = \ h^d \Sigma u_j v_j \tag{1.10a}$$

$$||\underline{u}||_h^2 \ = \ <\underline{u},\underline{u}>_h \quad , \tag{1.10b}$$

where d = dimension Ω ($d = 2$ in our current discussion). We will assume that these norms are uniformly equivalent to the L_2 norms of the functions v. That is, there is a constant $c > 0$ such that

$$\frac{1}{c}\,||v||_{L_2} \le ||\underline{v}||_h \le c\,||v||_{L_2} \quad .$$

For this reason, while we are concerned with $C_h(B_h^{-1}A_h)$ or $C_h(A_h B_h^{-1})$, it suffices to consider the L_2 condition numbers $C_{L_2}(B_h^{-1}A_h)$ or $C_{L_2}(A_h B_h^{-1})$. Of course, the L_2 norm is given by

$$(f,g) \ = \ \iint_\Omega fg\,dx\,dy \quad , \quad ||f||_{L_2}^2 \ = \ (f,f) \quad . \tag{1.10c}$$

Within the L_2 theory our basic results are easily stated.

I). If we seek an invertible elliptic operator B so that we obtain an estimate of the form

$$C_{L_2}(A_h B_h^{-1}) = ||A_h B_h^{-1}||_{L_2} \cdot ||B_h A_h^{-1}||_{L_2} \le K \quad , \quad \text{all } h \le h_0 \quad ; \qquad \textbf{(1.11a)}$$

then it is necessary (and essentially sufficient) that

$$D(A) = D(B) \quad . \qquad (1.11b)$$

II). If we seek an invertible elliptic operator B so that we obtain an estimate of the form

$$C_{L_2}(B_h^{-1} A_h) = ||B_h^{-1} A_h||_{L_2} \cdot ||A_h^{-1} B_h||_{L_2} \le K \quad , \quad \text{all } h \le h_0 \quad ; \qquad (1.12a)$$

then it is necessary (and essentially sufficient) that

$$D(A^*) = D(B^*) \quad . \qquad (1.12b)$$

Much of the previous analysis of these preconditioning strategies has been based on problems with Dirichlet boundary condition on all of $\partial\Omega$, i.e.,

$$\Gamma_0 = \partial\Omega \quad , \quad \Gamma_1 = \emptyset \quad .$$

Unfortunately, from the point-of-view of insight into the general case, the Dirichlet problem is misleading because, in that case

$$D(A) = D(A^*) \quad .$$

There is one notable exception, namely, the work of Bramble and Pasciak [BP] who proved a basic result about such preconditionings (see Theorem 5.1). Unfortunately they failed to appreciate (1.12a), (1.12b). Hence, in their example they used a preconditioner which satisfied (1.11b) not (1.12b). Thus, their one L_2 example is incorrect.

In fact, with the already noted exception of [BP], most of the earlier work on such elliptic preconditionings has not only been based on the Dirichlet problem but has been based on the H_1 condition number, not the L_2 condition number. Most of the analysis of the H_1 condition number is based on the concept of ''Spectral Equivalence'' introduced by D'Yakonov [DY]. Let $\{A_h\}$, $\{B_h\}$ be families of positive definite self-adjoint operators defined in $\{S_h\}$. These families are said to be uniformly ''spectrally equivalent'' if there exist constants c_1, c_2, independent of h such that

$$0 < c_1 \le \frac{<A_h \underline{u}, \underline{u}>_h}{<B_h \underline{u}, \underline{u}>_h} \le c_2 \quad , \quad u_h \in S_h \quad , \quad 0 < h \le h_0 \quad .$$

While D'Yakonov considered only the finite-dimensional case, it is easy to extend this concept to the infinite dimensional case (see [FMP]). Let $a(u,v)$, $b(u,v)$ be two positive definite bilinear forms defined on a Hilbert space V. We say that a and b are spectrally equivalent if

$$0 < c_1 \le \frac{a(u,u)}{b(u,u)} \le c_2 \quad , \quad u \in V \quad .$$

When dealing with the l_2 condition number the appropriate concept is L_2 norm equivalence. This concept was (implicitly) used in [BP]. A systematic study of norm equivalence is given in [FMP]. For our purposes we use the following definitions. Let A and B be two invertible elliptic operators. We say that A and B are L_2 norm equivalent on $L_2(\Omega)$ if there is a dense set $D \subset L_2(\Omega)$ and a constant $c > 0$ such that: for every $f \neq 0, f \in D$ we have

$$||AB^{-1} f||_{L_2} \leq c ||f||_{L_2}, \ ||BA^{-1} f||_{L_2} \leq c ||f||_{L_2} \quad . \tag{1.13}$$

We say that A^{-1} and B^{-1} are L_2 norm equivalent on $L_2(\Omega)$ if there is a dense set $D \subset L_2(\Omega)$ and a constant $c > 0$ such that: for every $u \in D$ we have

$$||A^{-1} Bu||_{L_2} \leq c ||u||_{L_2} \quad , \quad ||B^{-1} Au||_{L_2} \leq c ||u||_{L_2} \tag{1.14}$$

Similarly, let $\{A_h\}$ and $\{B_h\}$ be families of operators defined on $\{S_h\}$ (presumably A_h and B_h are discretizations of A and B respectively). We say that $\{A_h\}$, $\{B_h\}$ are <u>uniformly</u> L_2 norm equivalent if there is a constant $c > 0$ independent of h such that

$$||A_h B_h^{-1}||_{L_2} \leq c \quad , \quad ||B_h A_h^{-1}||_{L_2} \leq c \quad . \tag{1.15}$$

We say that A_h^{-1} and B_h^{-1} are uniformly L_2 norm equivalent if there is a constant $c > 0$ independent of h such that

$$||B_h^{-1} A_h||_{L_2} \leq c \quad , \quad ||A_h^{-1} B_h||_{L_2} \leq c \quad . \tag{1.16}$$

One sees immediately that (1.15) is equivalent to (1.11a) and (1.16) is equivalent to (1.12a).

The concepts of spectral equivalence and L_2 norm equivalence are different even for positive symmetric operators. For compact operators, norm equivalence implies spectral equivalence. The converse is not true [FMP]. In Section 4, Example 6 we give a simple example of two self-adjoint positive definite ordinary differential operators M and N which are spectrally equivalent while M^{-1} and N^{-1} are not L_2 norm equivalent and M and N are not L_2 norm equivalent. Similarly, if we let $\{M_h\}$ and $\{N_h\}$ denote the usual finite-difference approximations to M and N respectively, $\{M_h\}$ and $\{N_h\}$ are uniformly spectrally equivalent while neither $\{M_h^{-1}\}$, $\{N_h^{-1}\}$ nor $\{M_h\}$ $\{N_h\}$ are uniformly L_2 norm equivalent.

The H_1 norm equivalence A^{-1} and B^{-1} is closely related to spectral equivalence of the forms $<Au,v>_{L_2}$ and $<Bu,v>_{L_2}$ when A and B are positive definite and self-adjoint in $L_2(\Omega)$. For general operators we say that A^{-1} and B^{-1} are H_1 norm equivalent on $L_2(\Omega)$ if there exists a dense set $D \subseteq L_2(\Omega)$ and constants $o < c_1 \leq c_2$ such that for every $f \neq 0, f \in D$ we have

$$c_1 \leq \frac{||A^{-1} f||_{H_1}}{||B^{-1} f||_{H_1}} \leq c_2 \quad . \tag{1.17}$$

Although definition (1.17) will suffice for our purposes, we choose to reformulate it as a stronger statement about the weak forms of the operators A and B. In Section 2 A_w and B_w are defined on $H_1(\Omega,\Gamma_0(A)) \subseteq H_1(\Omega)$ and $H_1(\Omega,\Gamma_0(B)) \subseteq H(\Omega)$ respectively. We say that A_w^{-1} is H_1 norm equivalent to B_w^{-1} if there exists $D_1 \subseteq H_1(\Omega,\Gamma_0(B))$, D_1 dense in $H_1(\Omega,\Gamma_0(B))$, $D_2 \subseteq H_1(\Omega,\Gamma_0(A))$, D_2 dense in $H_1(\Omega,\Gamma_0(A))$, and constants $c_1, c_2 > 0$ such that for every $v \neq 0$, $v \in D_1$ we have

$$||A_w^{-1} B_w v||_{H_1} \le c_1 ||v||_{H_1} \tag{1.18a}$$

and for every $u \ne 0$, $u \in D_2$ we have

$$||B_w^{-1} A_w u||_{H_1} \le c_2 ||u||_{H_1} \quad . \tag{1.18b}$$

As stated earlier, bounds of the form (1.7) are not necessary for the H_1 results. Rather, we restrict ourselves to operators for which a special relationship exists between α_0 and α_1 in (1.1c). This will be explained in Section 2.

In Section 2 we recall some basic facts about elliptic operators and their adjoints. We also discuss the ''weak'' forms of elliptic operators. In Section 3 we discuss norm equivalence of elliptic operators. The basic results are

Theorem 3.1. Let A and B be two invertible uniformly elliptic operators of the type described by (1.1). Assume the domain Ω is smooth and all invertible uniformly elliptic operators of the form (1.1) whose coefficients are smooth and whose boundary conditions are based on $\{\Gamma_0(A), \Gamma_1(A)\}$ or $\{\Gamma_0(B), \Gamma_1(B)\}$ yield estimates of the form (1.7), (1.9). (For example, assume (i) either $\Gamma_0(A) = \emptyset$ or $\Gamma_1(A) = \emptyset$, and (ii) either $\Gamma_0(B) = \emptyset$ or $\Gamma_1(B) = \emptyset$). Then,

I). A^{-1} and B^{-1} are L_2 norm equivalent on $L_2(\Omega)$ if and only if $D(A^*) = D(B^*)$,

II). A and B are L_2 norm equivalent on $L_2(\Omega)$ if and only if $D(A) = D(B)$. ■

Theorem 3.2. Let A and B be two invertible uniformly elliptic operators of Class N. Then, A_w^{-1} and B_w^{-1} are H_1 norm equivalent on $L_2(\Omega)$ if and only if $\Gamma_0(A) = \Gamma_0(B)$; that is, there is a constant $K_3(A{:}B)$ such that

$$||A_w^{-1} B_w v||_{H_1} \le K_3(A{:}B) \;\; ||v||_{H_1}$$

for every $v \in H_1(\Omega, \Gamma_0(B))$ and

$$||B_w^{-1} A_w u||_{H_1} \le K_3(A{:}B) \;\; ||u||_{H_1}$$

for every $v \in H_1(\Omega, \Gamma_0(A))$ if and only if $\Gamma_0(B) = \Gamma_0(A)$. ■

In Section 4 we discuss some examples which illustrate the results of Section 3 and provide insight into the results of Section 5. In Section 5 we discuss the uniform L_2 norm equivalence of $\{A_h^{-1}\}$ and $\{B_h^{-1}\}$, the uniform L_2 norm equivalence of $\{A_h\}$ and $\{B_h\}$ and the uniform H_1 norm equivalence of $\{A_h^{-1}\}$ and $\{B_h^{-1}\}$. In all cases, such uniform norm equivalence of the discrete operators can occur only if the corresponding norm equivalence holds for the operators A and B or A^{-1} and B^{-1} (see [FMP]). Moreover, under reasonable assumptions (including optimal order convergence) the L_2 norm equivalence of A^{-1} and B^{-1} implies the uniform L_2 norm equivalence of $\{A_h^{-1}\}$ and $\{B_h^{-1}\}$ and the L_2 norm equivalence of A and B implies the uniform L_2 norm equivalence of $\{A_h\}$ and $\{B_h\}$. The first of these results was stated in [BP]. Finally, for a particular class of finite-element methods, the H_1 norm equivalence of A^{-1} and B^{-1} implies the uniform H_1 norm equivalence of $\{A_h^{-1}\}$ and $\{B_h^{-1}\}$.

We close this introduction with the observation that one would usually combine preconditioning with a conjugate gradient type iteration. Bounds on $C_{L_2}(B_h^{-1} A_h)$, $C_{L_2}(A_h B_h^{-1})$ merely yield upper bounds for the rate of convergence of conjugate gradient methods. A thorough analysis of CG methods requires more information on the distribution of either the eigenvalues or the singular values of $B_h^{-1} A_h$ depending on the implementation. We discuss these questions, together with some detailed computational experiments in another paper.

2. Background

The discussion in this section recalls some basic facts about elliptic differential operators. This discussion clarifies the facts concerning ''natural boundary conditions,'' the role of tangential derivatives and the relationship to the boundary conditions of A^*. In addition we discuss some of the basic facts concerning A_w, the ''weak'' or H_1 form of the operator A. This discussion depends on the ''Green's identities'' for the differential operator A.

With the operator A described by (1.1a) and the boundary conditions (1.1b), (1.1c) we also have the operator $\tilde{A}$ given by (1.1a) without any boundary conditions.

Let $(n_x, n_y) = \vec{n}(x,y)$ be the unit outward normal to $\partial\Omega$ at a point $(x,y) \in \partial\Omega$. Then

$$\frac{\partial u}{\partial \nu} = u_x n_x + u_y n_y \quad , \tag{2.1a}$$

$$\frac{\partial u}{\partial \sigma} = -u_x n_y + u_y n_x \quad , \tag{2.1b}$$

The conormal derivative is given by

$$\frac{\partial u}{\partial \nu_A} := [a_{11} u_x + a_{12} u_y] n_x + [a_{12} u_x + a_{22} u_y] n_y \quad . \tag{2.1c}$$

This conormal derivative represents the ''flux'' associated with the operator $\tilde{A}$.

We assume that the divergence theorem holds in Ω and immediately obtain the first ''Green's Identity'': let u and v be smooth functions defined on $\overline{\Omega}$. Then

$$(\tilde{A}u, v) = a(u,v) - \int_{\partial\Omega} v \frac{\partial u}{\partial \nu_A} \, d\sigma \tag{2.2a}$$

where

$$a(u,v) = a_L(u,v) + \iint_{\Omega} v[a_1 u_x + a_2 u_y + a_0 u] \, dxdy \tag{2.2b}$$

and $a_L(u,v)$, the bilinear form associated with the ''leading part'' of $\tilde{A}$ is given by

$$a_L(u,v) = \iint_{\Omega} \{a_{11} u_x v_x + a_{12}(u_x v_y + u_y v_x) + a_{22} u_y v_y\} dxdy \quad . \tag{2.2c}$$

One more application of the divergence theorem yields

$$(\tilde{A}u,v) = a_L(u,v) - (u,[(a_1 v)_x + (a_2 v)_y]) + (a_0 u,v)$$
$$- \int_{\partial\Omega} [v \frac{\partial u}{\partial \nu_A} - (a_1 n_x + a_2 n_y)uv]\, d\sigma \quad . \tag{2.3}$$

Let $\tilde{A}^*$ denote the (formal) adjoint of $\tilde{A}$. That is, there are no boundary conditions associated with $\tilde{A}^*$ and

$$\tilde{A}^* v = -\{(a_{11} v_x)_x + (a_{12} v_y)_x + (a_{12} v_x)_y + (a_{22} v_y)_y\}$$
$$- (a_1 v)_x - (a_2 v)_y + a_0 v \quad . \tag{2.4}$$

The conormal derivative depends only on the leading part. Thus, the conormal associated with $\tilde{A}^*$ is the same as the conormal derivative associated with $\tilde{A}$. One more application of the divergence theorem yields ‘‘Green’s Second Identity.’’

$$(\tilde{A}u,v) - (u,\tilde{A}^* v) = \int_{\partial\Omega} \left[u \frac{\partial v}{\partial \nu_A} - v \frac{\partial u}{\partial \nu_A} \right] d\sigma$$
$$+ \int_{\partial\Omega} (a_1 n_x + a_2 n_y) uv\, d\sigma \quad . \tag{2.5}$$

The operator A^*, the L_2 adjoint of A, is obtained by using the boundary conditions (1.1b), (1.1c) and, if necessary, integration by parts along Γ_1 (converting tangential derivatives of u into tangential derivatives of v) to determine appropriate boundary conditions for $\tilde{A}^*$ so that

$$(Au,v) = (u,A^* v) \quad . \tag{2.6}$$

That is, A^* is given by

$$A^* v = \tilde{A}^* v \quad \text{in } \Omega \tag{2.7a}$$

and, on $\partial\Omega$, there are prescribed boundary conditions of the form

$$v = 0 \quad \text{on } \Gamma_0 \quad , \tag{2.7b}$$

$$\frac{\partial u}{\partial \nu} = \alpha_0^* v + \alpha_1^* \frac{\partial v}{\partial \sigma} \quad \text{on } \Gamma_1 \quad , \tag{2.7c}$$

where, α_0^* and α_1^* have been chosen so that (2.6) holds.

In order to find α_0^*, α_1^* it is convenient to rewrite $\frac{\partial u}{\partial \nu_A}$ as

$$\frac{\partial u}{\partial \nu_A} = N \frac{\partial u}{\partial \nu} + T \frac{\partial u}{\partial \sigma} \quad ,$$

where

$$N = a_{11} n_x^2 + 2a_{12} n_x n_y + a_{22} n_y^2 \geq \lambda(A) \quad ,$$

$$T = a_{12}(n_x^2 - n_y^2) + (a_{22} - a_{11}) n_x n_y \quad .$$

Observe that $\dfrac{\partial u}{\partial \nu_A}$ is always a nontangential derivative (by (1.2) $N \geq \lambda(A) > 0$). When the leading part of A is a diffusion operator with $a_{11} = a_{22}$ and $a_{12} = 0$ the conormal derivative is a positive multiple of the normal derivative. Moreover, unless one imposes artificial relationships between (n_x, n_y) and the coefficients $a_{11},\ a_{12},\ a_{22}$, this is the only class of operators for which this is so.

In many problems the boundary conditions associated with the operator A are of the special form

$$\frac{\partial u}{\partial \nu_A} = \alpha(\sigma) u \quad \text{on } \Gamma_1 \quad . \tag{2.8}$$

Indeed, the ''natural'' boundary condition associated with A are given by

$$\frac{\partial u}{\partial \nu_A} = 0 \quad \text{on } \Gamma_1 \quad . \tag{2.9}$$

When the boundary conditions for A are of the form (2.8) we may use (2.5) to see that the boundary conditions for A^* are given by

$$\frac{\partial v}{\partial \nu_A} = [\alpha - (a_1 n_x + a_2 n_y)] v := \alpha^* v \quad . \tag{2.10}$$

These formulae show why our discussion must generally include tangential derivatives and how the coefficients a_1, a_2 effect the boundary conditions of A^*.

In general, A is self-adjoint ($A = A^*$) if and only if

(i) $a_1 = a_2 = 0$

(ii) $\dfrac{\partial u}{\partial \nu_A} = \alpha u$ on Γ_1.

As we have written the operator $\tilde{A}$ in (1.1a) one expects to deal with a theory involving second derivatives, i.e., an H_2 theory. However, it is well known that there is an H_1 theory based on (2.2) (cf [Ci]). Indeed, this theory is the basis of much of the finite-element method. While there are some discussions of the finite-element method to general boundary conditions (see [BP]) much of that theory is limited to the case where the boundary conditions are given by (1.1b) and (2.8). We will denote such operators as operators of ''Class N'' to indicate that the boundary conditions involve only the conormal derivative. Our discussion of the H_1 theory and H_1 norm equivalence will be limited to this class of problems.

Such problems can be reformulated as follows. Let $\Gamma_0 = \Gamma_0(A)$ and let

$$H_1(\Omega, \Gamma_0) := \{\phi \in H_1(\Omega) : \gamma_0 \phi = 0 \quad \text{on } \Gamma_0\} \tag{2.11}$$

where γ_0 denotes the ''trace'' operator (see [Gr]). The function which satisfies (1.1b), (2.8) and

$$\tilde{A}u = f \quad \text{in } \Omega \tag{2.12}$$

also satisfies

$$a(u,v) - \int_{\Gamma_1(A)} \alpha uv d\sigma = (f,v) \quad , \quad \forall v \in H_1(\Omega,\Gamma_0(A)) \quad . \tag{2.13}$$

Equation (2.13) represents the ‘‘weak’’ form of the boundary-value problem (2.12), (1.1b), (2.8).

Let us reinterpret (2.13). Given a boundary value problem involving an operator of Class N we consider the bilinear form $\tilde{a}(u,v)$ on $H_1(\Omega,\Gamma_0) \times H_1(\Omega,\Gamma_0)$ defined by

$$\tilde{a}(u,v) = a(u,v) - \int_{\Gamma_1} \alpha uv d\sigma \quad . \tag{2.14}$$

For each fixed $u \in H_1(\Omega,\Gamma_0)$ the value $\tilde{a}(u,\ \cdot)$ is a bounded linear functional on $H_1(\Omega,\Gamma_0)$. The weak form A_w of A is the mapping taking u into $\tilde{a}(u,\ \cdot)$. That is,

$$(A_w u)(v) = \tilde{a}(u,v) \quad . \tag{2.15}$$

Thus, (2.13) is the statement that $(A_w u) \in L_2(\Omega)$ in the sense that $A_w u$ may be represented by f as

$$(A_w u)(v) = (f,v) \tag{2.16}$$

We have given this careful and explicit review of these ideas to emphasize the fact that in this H_1 formulation of boundary value problems; if A and B are two uniformly elliptic operators of Class N and $\Gamma_0(A) = \Gamma_0(B)$, then A_w and B_w have the same domain, $H_1(\Omega,\Gamma_0)$.

When working with $\tilde{a}(u,v)$ one must be able to bound boundary integrals. The basic estimate is: there is a constant $L_0 = L_0(\Omega)$, depending only on Ω, such that, for every $v \in H_1(\Omega)$,

$$\int_{\partial\Omega} v^2 d\sigma \le L_0 \left[||\nabla v||_{L_2} \cdot ||v||_{L_2} + ||v||_{L_2}^2 \right] \quad , \tag{2.17}$$

(see [Gr]). In fact, it is this estimate which enables one to assert that $A_w u$ is a bounded linear functional on $H_1(\Omega,\Gamma_0)$. That is, there is a constant $L_1(A) > 0$, such that, for all $u,v \in H_1(\Omega,\Gamma_0)$ we have

$$|\tilde{a}(u,v)| \le L_1(A) ||\ u\ ||_{H_1} \cdot ||v||_{H_1} \quad . \tag{2.18}$$

If $\tilde{a}(u,v)$ is symmetric, i.e., $\tilde{a}(u,v) = \tilde{a}(v,u)$, $(a_1 = a_2 = 0)$ we say A_w is positive definite if there is a constant $L_2 = L_2(A) > 0$ such that

$$L_2(A) ||v||_{H_1}^2 \le \tilde{a}(v,v) \quad , \quad \forall\ v \in H_1(\Omega,\Gamma_0) \quad . \tag{2.19}$$

Observe, that if A_w and B_w are two such positive definite operators with $\Gamma_0(A) = \Gamma_0(B)$, then, not only do A_w and B_w have the same domain but they are spectrally equivalent because

$$\frac{L_2(A)}{L_1(B)} \le \frac{\tilde{a}(v,v)}{\tilde{b}(v,v)} \le \frac{L_1(A)}{L_2(B)} \quad , \quad \forall\ v \in H_1(\Omega,\Gamma_0) \quad . \tag{2.20}$$

3. The Differential Operators

In this section we discuss the basic facts about L_2 norm equivalence of elliptic differential operators as well as some results on the H_1 norm equivalence of A_w^{-1} and B_w^{-1}. The basic results are Theorem 3.1 and Theorem 3.2.

Lemma 3.1. Let A and B be two invertible uniformly elliptic operators of the form (1.1a,b,c) such that (1.7) and (1.9) hold. Let $K_1(A)$ $K_1(B)$, $K_2(A)$, $K_2(B)$ be the constants associated with the bounds (1.7), (1.9). Assume

$$D(A) = D(B) \quad . \tag{3.1}$$

Then, AB^{-1} and BA^{-1} are defined on all of $L_2(\Omega)$ and

$$||AB^{-1}||_{L_2} \le K_2(A)K_1(B) \quad , \tag{3.2a}$$

$$||BA^{-1}||_{L_2} \le K_1(A)K_2(B) \quad . \tag{3.2b}$$

Thus, A and B are L_2 norm equivalent.

Proof. The proof is immediate. ■

Lemma 3.2. Let A and B be two invertible uniformly elliptic operators of the form (1.1a,b,c) such that (1.7) and (1.9) hold for A^* and B^*. Let $K_1(A^*)$, $K_2(A^*)$, $K_1(B^*)$, $K_2(B^*)$ be the constants associated with the bounds (1.7), (1.9). Assume

$$D(A^*) = D(B^*) \tag{3.3}$$

Let

$$D: = D(A) \cap D(B) \quad . \tag{3.4}$$

Then, D is dense in $L_2(\Omega)$ and the estimates (1.13) hold in the form

$$||B^{-1}Af||_{L_2} \le K_1(B^*)K_2(A^*)\,||f||_{L_2} \quad , \quad f \in D(A) \quad , \tag{3.5a}$$

$$||A^{-1}Bf||_{L_2} \le K_1(A^*)K_2(B^*)\,||f||_{L_2} \quad , \quad f \in D(B) \quad . \tag{3.5b}$$

Hence, A^{-1} and B^{-1} are L_2 norm equivalent.

Proof. The set D is dense because

$$C_0^\infty(\Omega) \subset D \quad .$$

Consider

$$||B^{-1}Af||_{L_2} = \sup_{\substack{v \in L_2 \\ v \ne 0}} \frac{(B^{-1}Af, v)}{||v||_{L_2}} = \sup_{\substack{v \in L_2 \\ v \ne 0}} \frac{(f, A^*(B^{-1})^* v)}{||v||_{L_2}} \le ||A^*(B^{-1})^*||_{L_2} \; ||f||_{L_2} \quad .$$

Hence, (3.5a) follows from Lemma 3.1. A similar argument yields (3.5b). ■

We now turn to the construction of special functions u_η, ϕ_η which enable us to prove the necessity of (3.1) and (3.3). These functions will be bounded in L_2 but become very large on a small segment of the boundary where the boundary conditions of A and B disagree. For the next two lemmas we assume that Ω has a particular shape (see Fig. 1).

Condition Ω. Ω is contained in the right-half-plane $x > 0$ and is convex. Moreover, there is an interval

$$\Gamma_2 := \{(0,y) : |y| < \delta\} \subset \partial\Omega$$

and the entire rectangle

$$R_1 := \{(x,y) : \ |y| \le \frac{\delta}{2} \ , \quad 0 \le x \le 1\}$$

is contained entirely in Ω with

$$\partial R_1 \cap \partial\Omega \subset \Gamma_2 \ . \ ■$$

Let $v(y)$ be a fixed function which satisfies

$$v \in C^\infty(-\infty,\infty) \tag{3.6a}$$

$$0 \le v(y) \le 1 \tag{3.6b}$$

$$v(y) = 1 \quad , \quad |y| \le \frac{\delta}{4} \tag{3.6c}$$

$$v(y) = 0 \quad , \quad |y| > \frac{\delta}{2} \quad . \tag{3.6d}$$

Lemma 3.3. Let B be an invertible uniformly elliptic operator given by

$$\begin{aligned} Bu = & -\{(b_{11}u_x)_x + (b_{12}u_y)_x + (b_{12}u_x)_y + (b_{22}u_y)_y\} \\ & + b_1 u_x + b_2 u_y + b_0 u \end{aligned} \tag{3.7a}$$

$$u = 0 \quad \text{on } \Gamma_0(B) \tag{3.7b}$$

$$\frac{\partial u}{\partial \nu} = \beta_0 u + \beta_1 \frac{\partial u}{\partial \sigma} \quad \text{on } \Gamma_1(B) \quad . \tag{3.7c}$$

Suppose Ω satisfies Condition Ω and

$$\Gamma_2 \subset \Gamma_1(B) \quad .$$

Then, for every η, $0 < \eta \le 1$, there are two functions $u_\eta, \phi_\eta \in C^2(\overline{\Omega})$ which satisfy

$$\textit{support } u_\eta \subset R_\eta \quad , \quad \textit{support } \phi_\eta \subset R_\eta \quad , \tag{3.8a}$$

where R_η is the rectangle

$$R_\eta := \{(x,y) : \ |y| \le \frac{\delta}{2} \quad , \quad 0 \le x \le \eta\} \quad . \tag{3.8b}$$

The function u_η satisfies

$$u_\eta \in D(B) \quad , \tag{3.9a}$$

$$u_\eta(0,y) = \eta^{-1/2} v(y) \quad ; \quad (0,y) \in \partial\Omega \quad , \tag{3.9b}$$

$$\|u_\eta\|_{L_2} \le c \quad , \quad \|u_\eta\|_{H_1} \le c\eta^{-1} \quad , \quad \|u_\eta\|_{H_2} \le c\eta^{-2} \quad , \tag{3.9c}$$

for some constant $c > 0$, independent of η. The function ϕ_η satisfies

$$\phi_\eta = 0 \quad \text{on } \partial\Omega \quad , \tag{3.10a}$$

$$\frac{\partial \phi_\eta}{\partial x}(0,y) = \eta^{-3/2} v(y) \quad ; \quad (0,y) \in \partial\Omega \quad , \tag{3.10b}$$

$$\|\phi_\eta\|_{L_2} \le c,\ \|\phi_\eta\|_{H_1} \le c\eta^{-1} \quad , \quad \|\phi_\eta\|_{H_2} \le c\eta^{-2} \quad , \tag{3.10c}$$

for come constant c independent of η. *Proof.* Define

$$g(x:\eta) = \begin{cases} (1 - \dfrac{x}{\eta})^3 (1 + \dfrac{3x}{\eta}) \quad , & 0 \le x \le \eta \quad , \\ 0 \quad , & \eta < x \quad . \end{cases} \tag{3.11}$$

Observe that $g(x:\eta) \in C^2[0,\infty)$ and

$$g(0:\eta) = 1 \quad , \quad \frac{d}{dx}\, g(0:\eta) = 0 \quad . \tag{3.12}$$

Set

$$u_\eta(x,y) = \eta^{-1/2}\ \{v(y) - x[\beta_0(y)v(y) + \beta_1(y)v'(y)]\}g(x:\eta) \quad , \tag{3.13a}$$

$$\phi_\eta(x,y) = \eta^{-3/2}\ v(y)xg(x:\eta) \quad . \tag{3.13b}$$

The desired estimates all follow from elementary computations. ■

Lemma 3.4. Let Ω and B be as in Lemma 3.3. Let A be an invertible uniformly elliptic operator which differs from B only in the boundary conditions. That is,

$$\tilde{A} = \tilde{B} \quad , \quad D(A) \ne D(B) \quad . \tag{3.14a}$$

Specifically, let A satisfy the boundary conditions

$$u = 0 \quad , \quad \text{on } \Gamma_0(A) \quad , \tag{3.14b}$$

$$\frac{\partial u}{\partial \nu} = \alpha_0 u + \alpha_1 \frac{\partial u}{\partial \sigma} \quad , \quad \text{on } \Gamma_1(A) \quad . \tag{3.14c}$$

We consider several cases.

Case 1. $\Gamma_2 \subset \Gamma_1(A)$, but

$$|\alpha_0 - \beta_0| + |\alpha_1 - \beta_1| > 0 \quad \text{on } \Gamma_2 \quad . \tag{3.15}$$

Let u_η be the function given by (3.13a). Then, there is a constant $c_1 > 0$ such that

$$\|A^{-1}Bu_\eta\|_{L_2} \ge c_1\eta^{-1/2}\|u_\eta\|_{L_2} \quad . \tag{3.16}$$

Case 2. $\Gamma_2 \subset \Gamma_0(A)$.

Then (3.16) holds. Moreover, let ϕ_η be the function given by (3.13b). Then

$$\phi_\eta \in D(A)$$

and there is a constant $c_2 > 0$ such that

$$\|B^{-1}A\phi_\eta\|_{L_2} \ge c_2\eta^{-3/2}\ \|\phi_\eta\|_{L_2} \quad . \tag{3.17}$$

Proof. Let

$$A^{-1} B u_\eta = \psi_\eta \quad , \tag{3.18a}$$

$$w_\eta = u_\eta - \psi_\eta \quad . \tag{3.18b}$$

Then

$$A \psi_\eta = B u_\eta$$

and w_η satisfies

$$\tilde{A} w_\eta = \tilde{B} w_\eta = 0 \quad . \tag{3.19a}$$

and since $\psi_\eta = (w_\eta - u_\eta) \in D(A)$, the boundary conditions of A applied to w_η is equal to the boundary conditions of A applied to u_η.

This yields

$$w_\eta = 0 \quad \text{on } \Gamma_0(A) \quad , \tag{3.19b}$$

$$\frac{\partial w_\eta}{\partial \nu} = \alpha_0 w_\eta + \alpha_1 \frac{\partial w_\eta}{\partial \sigma} \quad \text{on } \Gamma_1(A)/\Gamma_2 \quad . \tag{3.19c}$$

Case 1. On Γ_2 we have

$$(w_\eta)_x + \alpha_0 w_\eta + \alpha_1 (w_\eta)_y = (u_\eta)_x + \alpha_0 u_\eta + \alpha_1 (u_\eta)_y = \eta^{-1/2} [s(y)] \tag{3.20a}$$

where

$$s(y) = (\alpha_0 - \beta_0) v(y) + (\alpha_1 - \beta_1) v'(y) \quad . \tag{3.20b}$$

Observe that $v(y)$ can be chosen so that $s(y)$ is <u>not</u> identically zero.
Case 2. On Γ_2 we have

$$w_\eta = u_\eta = \eta^{-1/2} v \quad . \tag{3.21}$$

In both cases the fact that A is invertible means that w_η is uniquely determined by (3.19), (3.20), and (3.21). However, a simple computation shows that $\eta^{-1/2} w_1$ is a solution. Hence

$$w_\eta = \eta^{-1/2} w_1$$

and the estimates (3.16) follows from (3.9c) and the triangle inequality.

It is clear from (3.10a) that in Case 2, $\phi_\eta \in D(A)$. In order to prove (3.17), let

$$B^{-1} A \phi_\eta = \psi_\eta \quad , \tag{3.22a}$$

$$w_\eta = \phi_\eta - \psi_\eta \quad . \tag{3.22b}$$

Then

$$\tilde{A}w_\eta = \tilde{B}w_\eta = 0 \quad . \tag{3.23}$$

Now, w_η satisfies the boundary conditions of B on $\partial\Omega/\Gamma_2$, and on Γ_2 we have

$$(w_\eta)_x + \beta_0 w_\eta + \beta_1 (w_\eta)_y = (\phi_\eta)_x + \beta_0 \phi_\eta + \beta_1 (\phi_\eta)_y = \eta^{-3/2} v(y) \quad . \tag{3.24}$$

Once more, the invertibility of B implies w_η is the unique solution of (3.23) and (3.24). Again

$$w_\eta = \eta^{-3/2} w_1$$

and we obtain (3.17). ■

Theorem 3.1. Let A and B be two invertible uniformly elliptic operators of the type described by (1.1). Assume the domain Ω is smooth and all invertible uniformly elliptic operators of the form (1.1) whose coefficients are smooth and whose boundary conditions are based on these partitions yield estimates of the form (1.7), (1.9). (For example, assume (i) either $\Gamma_0(A) = \emptyset$ or $\Gamma_1(A) = \emptyset$, and (ii) either $\Gamma_0(B) = \emptyset$ or $\Gamma_1(B) = \emptyset$). Then

I) A^{-1} and B^{-1} are L_2 norm equivalent if and only if $D(A^*) = D(B^*)$,

II) A and B are L_2 norm equivalent if and only if $D(A) = D(B)$.

Proof. Since the ''if'' parts of these statements are proven in Lemma 3.2 and Lemma 3.1, respectively, we need only prove the ''only if'' part. We start with statement (I).

Because L_2 norm equivalence is transitive we use Lemma 3.2 to conclude that we need only consider the case where

$$\tilde{A} = \tilde{B} \quad , \tag{3.25a}$$

and

$$D(A^*) \neq D(B^*) \quad . \tag{3.25b}$$

However, (3.25a) and (3.25b) imply that

$$D(A) \neq D(B) \quad . \tag{3.26}$$

Let $\Gamma_3 \subset \partial\Omega$ be a portion or the boundary on which the boundary conditions are different. By a smooth mapping we may map Ω onto a domain Ω' which satisfies *Condition* Ω and $\Gamma_2 \subset \Gamma_3'$, the image of Γ_3. Using Lemma 3.4 we see that $A^{-1}B$ and $B^{-1}A$ are unbounded on their domains. Hence they cannot be bounded on any dense set. Thus, we have proven statement (I).

Consider statement (II). Once more it suffices to consider the case where (3.25a) and (3.25b) hold. The proof proceeds by contraposition. Assume there is a dense set D for which (1.13) holds. For every $f \in D$ we have

$$||AB^{-1}f||_{L_2} = sup \left\{ \frac{(AB^{-1}f, v)}{||v||_{L_2}} \quad , \quad v \in L_2, v \neq 0 \right\} \quad . \tag{3.27a}$$

Since $D(A^*)$ is dense in $L_2(\Omega)$ we may restrict the function v of (3.28a) to be in $D(A^*)$; that is,

$$||AB^{-1}f||_{L_2} = sup \left\{ \frac{(AB^{-1}f,v)}{||v||_{L_2}} \quad , \quad v \in D(A^*), v \neq 0 \right\} \quad . \tag{3.27b}$$

Let $0 < \varepsilon < 1$ be given. Since statement (I) holds and (3.25b) holds, there is a $v_\varepsilon \in D(A^*)$ such that

$$||v_\varepsilon||_{L_2} = 1 \quad , \tag{3.28a}$$

$$||(B^*)^{-1}A^*v_\varepsilon||_{L_2} > \frac{1}{\varepsilon} \quad . \tag{3.28b}$$

Since D is dense, there is an $f_\varepsilon \in D$ such that

$$||(B^*)^{-1}A^*v_\varepsilon - f_\varepsilon||_{L_2} \leq \varepsilon^2 \; ||(B^*)^{-1}A^*v_\varepsilon||_{L_2} \quad . \tag{3.29}$$

$$||AB^{-1}f_\varepsilon||_{L_2} \geq \frac{|(f_\varepsilon,(B^*)^{-1}A^*v_\varepsilon)|}{||v_\varepsilon||_{L_2}} = |(f_\varepsilon,(B^*)^{-1}A^*v_\varepsilon)| \quad . \tag{3.30}$$

Using (3.28b) and (3.29) we have

$$||AB^{-1}f_\varepsilon||_{L_2} \geq ||(B^*)^{-1}A^*v_\varepsilon||_{L_2^2} \; [1 - \varepsilon^2] \quad . \tag{3.31}$$

Using the triangle inequality we have

$$||f_\varepsilon|| \leq (1 + \varepsilon^2) \; ||(B^*)^{-1}A^*v_\varepsilon|| \quad .$$

Hence, (3.31) yields

$$||AB^{-1}f_\varepsilon||_{L_2} \geq \frac{1}{\varepsilon} \; \frac{(1 - \varepsilon^2)}{(1 + \varepsilon^2)} \; ||f_\varepsilon||_{L_2} \quad . \tag{3.32}$$

Thus, (1.13) cannot hold and the Theorem is proven. ■

Remark. Observe that in the proof of (I) we exhibit a sequence on which $(B^{-1}A)$ is unbounded while in the proof of (II) we prove that

$$F: = \{f \in L_2 : B^{-1}f \in D(A)\}$$

is not dense. Both AB^{-1} and BA^{-1} are bounded on F. To see this we merely observe that inequalities (1.7) and (1.9) yield

$$||AB^{-1}f||_{L_2} \leq K_2(A)K_1(B)\,||f||_{L_2} \quad , \quad f \in F \quad .$$

Thus, AB^{-1} is bounded wherever it is defined, but it is not defined on a dense set unless $D(A) = D(B)$. ■

We now turn to the H_1 norm equivalence of A_w^{-1} and B_w^{-1} where both A and B are of Class N. We begin with a basic lemma on the H_1 closure of $D(A)$. We assume $A_w u = f$ possesses a unique solution u for all $f \in L_2(\Omega)$. In this case we define $D(A) := \{u = A_w^{-1}f ; f \in L_2(\Omega)\}$. It may happen that $D(A) \not\subseteq H_2(\Omega)$. However we always have (i) $D(A) \subset H_1(\Omega,\Gamma_0(A))$, and (ii) there is a $\delta > 0$ and $D(A) \subseteq H_{1+\delta}(\Omega)$.

Lemma 3.5. Let V denote the H_1 closure of $D(A)$.

Then

$$V = H_1(\Omega, \Gamma_0(A)) \quad . \tag{3.33}$$

Proof. Since $D(A) \subset H_1(\Omega, \Gamma_0(A))$ (and since $H_1(\Omega, \Gamma_0(A))$ is closed in H_1) we have

$$V \subset H_1(\Omega, \Gamma_0(A)) \quad .$$

Suppose that there is a $\phi \in H_1(\Omega, \Gamma_0(A))$ and $\phi \notin V$.

Consider the invertible uniformly elliptic operator B of Class N given by

$$\tilde{B}u = -[(a_{11}u_x)_x + (a_{12}u_y)_x + (a_{12}u_x)_y + (a_{22}u_y)_y\} + Ku \tag{3.34a}$$

$$u = 0 \quad \text{on} \quad \Gamma_0(A) \quad , \tag{3.34b}$$

$$\frac{\partial u}{\partial \nu_B} = \alpha u \quad \text{on } \Gamma_1(A) \quad , \tag{3.34c}$$

where α is the coefficient in the definition of the boundary conditions for A. Since $\dfrac{\partial u}{\partial \nu_B} = \dfrac{\partial u}{\partial \nu_A}$ we have

$$D(A) = D(B) \quad . \tag{3.35}$$

The constant K is chosen positive and large so as to ensure that B is invertible and that the bilinear form

$$\begin{aligned} \tilde{b}(u,v) = \int\!\!\int_\Omega & (a_{11}u_x v_x + a_{12}(u_x v_y + u_y v_x) + a_{22}u_y v_y)\,dxdy \\ & + K(u,v) - \int_{\Gamma_1(A)} \alpha u v d\sigma \end{aligned} \tag{3.36}$$

is an $H_1(\Omega, \Gamma_0(A))$ inner-product. That is, there are constants $0 < c_1 \le c_2$ such that

$$c_1 \; ||u||_{H_1} \le \tilde{b}(u,u)^{1/2} \le c_2 ||u||_{H_1} \quad , \quad \forall \; u \in H_1(\Omega, \Gamma_0(A)) \quad .$$

The estimate (2.17) can be used to show that such constants exist for all K sufficiently large. Thus, $\tilde{b}$ induces a norm that is equivalent to the H_1 norm. We may consider V to be the $\tilde{b}$ closure of $D(A)$.

Since $\phi \notin V$ there is a $\psi \in H_1(\Omega, \Gamma_0(A))$ which is ‘‘orthogonal’’ to V in the $\tilde{b}(\,\cdot\,,\,\cdot\,)$ inner product. That is, for all $u \in D(A)$

$$\tilde{b}(u, \psi) = 0 \quad .$$

However, for each $f \in L_2(\Omega)$ we may solve

$$Bu = f \quad . \tag{3.37}$$

Clearly, the solution $u \in D(B) = D(A)$. But, the discussion of Section 2 shows that

$$0 = \tilde{b}(u,\psi) = (f,\psi) \qquad \textbf{(3.38)}$$

Hence, $\psi \in H_1(\Omega,\Gamma_0(A))$ is L_2 orthogonal to $L_2(\Omega)$, which implies

$$\psi = 0$$

and the lemma is proven. ■

The main result which follows, namely, Theorem 3.2, can now be proven by a rather general argument. However, we proceed with the following more detailed discussion as it immediately generalizes to the discrete case.

Lemma 3.6. Let A and B be two invertible uniformly elliptic operators of Class N which satisfy

$$\tilde{A}u = \tilde{B}u + c_o u \qquad (3.39a)$$

where $c_o \neq 0$ is a constant, and

$$D(A) = D(B) \quad . \qquad (3.39b)$$

Let $u,v \in D(A) = D(B)$ satisfy

$$A_w u = B_w v \qquad (3.40)$$

Then, there is a constant $K_3(A{:}B)$ such that

$$||u||_{H_1} \leq K_3(A{:}B)\ ||v||_{H_1} \quad , \qquad (3.41a)$$

$$||v||_{H_1} \leq K_3(A{:}B)\ ||u||_{H_1} \quad . \qquad (3.41b)$$

Proof. Let

$$w = u - v$$

then

$$\tilde{A}w = \tilde{A}u - [\tilde{B}v + c_o v] = -c_o v \quad .$$

That is

$$Aw = -c_o v \quad .$$

Since $v \in D(A), v \in L_2(\Omega)$ and the invertibility of A on $L_2(\Omega)$ implies that A_w is invertible on $L_2(\Omega)$ and

$$||w||_{H_1} \leq K_1 |c_0|\ ||v||_{L_2} \leq K_1 |c_0|\ ||v||_{H_1}$$

for some appropriate constant $K_1 > 0$. Similarly

$$\tilde{B}w = [\tilde{A}u - c_0 u] - \tilde{B}v = -c_0 u$$

and for some $K_2 > 0$

$$||w||_{H_1} \le K_2 \ |c_0| \ ||u||_{H_1} \quad .$$

The estimates (3.41a), (3.41b) now follow from the triangle inequality. ∎

Corollary. Let A_w and B_w be as in Lemma 3.6. If $u,v \in H_1(\Omega,\Gamma_0(A))$ satisfy

$$A_w u = B_w v \quad .$$

Then (3.41a) and (3.41b) hold.

Proof. From Lemma 3.5 we know that $D(A) = D(B)$ is dense in $H_1 \ (\Omega,\Gamma_0(A))$. Thus, there exists a sequence $u_n \in D(A)$ such that $||u_n - u||_{H_1} \to 0$. Since B is invertible, there exists $v_n \in D(B)$ such that

$$A_w u_n = Au_n = Bv_n = B_w v_n$$

for every n. Now, Lemma 3.6 implies that $||v_n - \hat{v}||_{H_1} \to 0$ for some $\hat{v} \in H_1(\Omega,\Gamma_0(A))$. The boundedness and invertibility of A_w and B_w yield $\hat{v} = v$. Applying (3.41a,b) to u_n, v_n and taking limits yields the result. ∎

Theorem 3.2. Let A and B be two invertible uniformly elliptic operators of Class N. Then, A_w^{-1} and B_w^{-1} are H_1 norm equivalent on $L_2(\Omega)$ if and only if $\Gamma_0(A) = \Gamma_0(B)$; that is, there is a constant $K_3(A{:}B)$ such that

$$||A_w^{-1} B_w v||_{H_1} \le K_3(A{:}B) \ ||v||_{H_1} \tag{3.42a}$$

for every $v \in H_1(\Omega,\Gamma_0(B))$ and

$$||B_w^{-1} A_w u||_{H_1} \le K_3(A{:}B) \ ||u||_{H_1} \tag{3.42b}$$

for every $v \in H_1(\Omega,\Gamma_0(A))$ if and only if $\Gamma_0(B) = \Gamma_0(A)$.

Proof. We will prove (3.42a). Since norm equivalence is transitive we may use Lemma 3.6 and assume that the coefficient a_0 of the operator A satisfies

$$a_0 \ge K \tag{3.43}$$

for any suitably chosen $K > 0$. We shall choose K later.

Assume $A_w u = B_w v$ and $\Gamma_0(A) = \Gamma_0(B)$. Using the representations (2.14), (2.15) we have for every $\phi \in H_1 \ (\Omega,\Gamma_0(A))$

$$a(u,\phi) - \int_{\Gamma_1} \alpha u \phi d\sigma = b(v,\phi) - \int_{\Gamma_1} \beta v \phi d\sigma \quad . \tag{3.44}$$

Setting $\phi = u$ and using the form of A and B this equation yields an estimate

$$\lambda(A)\,||\nabla u||_{L_2^2} + K||u||_{L_2^2} \le \Lambda(B)\,||\nabla u||_{L_2}\,||\nabla v||_{L_2}$$
$$+ c_1 ||u||_{L_2} \left[||\nabla u||_{L_2} + ||\nabla v||_{L_2} + ||v||_{L_2} \right] \tag{3.45}$$
$$+ c_2 \left[\int_{\Gamma_1} u^2 d\sigma + \int_{\Gamma_1} v^2 d\sigma \right] ,$$

where the constant c_1, depends on $a_1,\ a_2,\ b_1,\ b_2,\ b_0$ but not on a_0 while the constant c_2 depends on α and β.

Using the inequality

$$ab \le \frac{\varepsilon}{2}\, a^2 + \frac{1}{2\varepsilon}\, b^2 \tag{3.46}$$

several times in (3.45) we obtain the estimate

$$\frac{1}{2}\,\lambda(A)\,||\nabla u||_{L_2^2} + \frac{1}{2}\,K||u||_{L_2^2} \le \bar{c}\,\left[||\nabla v||_{L_2^2} + ||v||_{L_2^2} \right]$$
$$+ c_1 ||u||_{L_2} \cdot ||\nabla u||_{L_2} \tag{3.47}$$
$$+ c_2 \left[\int_{\Gamma_1} u^2 d\sigma + \int_{\Gamma_1} v^2 d\sigma \right] ,$$

where $\bar{c}$ is a large constant. Using the estimate (2.17) we may omit the two boundary integrals and use larger values of $\bar{c}$ and c_1. That is, we now have

$$\frac{1}{2}\,\lambda(A)\ ||\nabla u||_{L_2^2} + \frac{1}{2}\,K||u||_{L_2^2} \le \bar{c}\ ||v||^2_{H_1} + c_1\ ||u||_{L_2} \cdot ||\nabla u||_{L_2} \tag{3.48}$$

where $\bar{c}$ depends on $a_1,\ a_2,\ b_1,\ b_2,\ b_0$ and $L_0 = L_0(\Omega)$, but $\bar{c}$ does not depend on K. We now choose

$$K = \frac{4c_1^2}{\lambda(A)} \quad , \quad \varepsilon = \frac{1}{2}\,\frac{\lambda(A)}{c_1}$$

and apply (3.46) once more. This yields

$$\frac{1}{4}\,\lambda(A)\ ||\nabla u||_{L_2^2} + \frac{3K}{4}\ ||u||_{L_2^2} \le \bar{c}\ ||v||^2_{H_1} \tag{3.49}$$

and (3.42a) follows with $K_3(A{:}B) = \bar{c}/\min\,(\frac{1}{4}\,\lambda(A),\,\frac{3K}{4})$.

Now suppose $\Gamma_0(A) \cap \Gamma_1(B) \ne 0$. By transitivity we may assume $\tilde{A} = \tilde{B}$. Let us map Ω onto a domain with *Condition* Ω so that $\Gamma_2 \subset \Gamma_0(A) \cap \Gamma_0(B)$. Consider the function $q_\eta = \eta\phi_\eta$ where ϕ_η is defined in (3.13b). By (3.10c) we have $||q_\eta||_{H_1} \le c$.

Following the development in the proof of Lemma 3.4 yields

$$|| B^{-1} A q_\eta ||_{H_1} \geq \eta^{-1/2} c \; || q_\eta ||_{H_1} \tag{3.50}$$

for some $c > 0$. This completes the proof. ■

4. Examples

In this section we look at simple examples which illustrate the results of Section 3 and give insight into the results of Section 5. The operators here are all ordinary differential operators defined on $L_2[0,1]$. In each case we carry out explicit calculations, based on the appropriate Green's function, to exhibit the relevant results of Section 3.

Let

$$\begin{aligned}
Lu &= -u'' , & u(0) &= u(1) = 0 , \\
Mu &= -u'' , & u(0) &= u'(1) = 0 , \\
Nu &= -u'' , & u(0) &= au(1) + u'(1) = 0 , \\
Ru &= -u'' + au' , & u(0) &= u'(1) = 0 ,
\end{aligned} \tag{4.1a}$$

where $a \geq 0$. Notice that

$$R^* u = -u'' - au' , \quad u(0) = au(1) + u'(1) = 0 . \tag{4.1e}$$

We will examine the equivalence of these operators in the L_2 norm and the equivalence of their inverses in both the L_2 and H_1 norms. The Green's function for N is given by

$$G_N(x,s) = \begin{cases} s\left(1 - \dfrac{a}{1+a} x\right) , & 0 \leq s \leq x \\ x\left(1 - \dfrac{a}{1+a} s\right) , & x \leq s \leq 1 . \end{cases} \tag{4.2}$$

The Green's functions for L and M are found by setting $a = \infty$ and $a = 0$ in (4.2) respectively. The Green's function for R is given by

$$G_R(x,s) = \begin{cases} \dfrac{1}{a}\,(1 - e^{-as}) , & 0 \leq s \leq x \\ \dfrac{1}{a}\,(e^{ax} - 1) e^{-as} , & x \leq s \leq 1 . \end{cases} \tag{4.3}$$

The Green's function for R^* is found by reversing the roles of x and s in (4.3).

Example 1). In this example we consider M and R. Since $D(M) = D(R)$, Theorem 3.1 implies that M and R are L_2 norm equivalent. Since $D(M^*) \neq D(R^*)$, M^{-1} is not L_2 norm equivalent to R^{-1}. However, $\Gamma_0(M) = \Gamma_0(R)$ implies that M^{-1} and R^{-1} are H_1 norm equivalent.

Example 1.a). L_2 norm equivalence of M and R. We have $D(M) = D(R)$.

A straightforward calculation yields

$$MR^{-1} f = M \int_0^1 G_R(x,s) f(s) ds = f - ae^{ax} \int_x^1 e^{-as} f(s) ds \tag{4.4a}$$

and

$$RM^{-1} f = R \int_0^1 G_M(x,s) f(s) ds = f + a \int_x^1 f(s) ds \quad . \tag{4.4b}$$

Since $D(M) = D(R)$, (4.4a,b) are defined for all $f \in L_2[0,1]$. Write

$$MR^{-1} f = f - \int_0^1 k_1(x,s) f(s) ds \quad , \tag{4.5a}$$

where

$$k_1(x,s) = \begin{cases} 0 \ , & 0 \le s \le x \\ ae^{a(x-s)} \ , & x \le s \le 1 \end{cases} \quad . \tag{4.5b}$$

Then, we see

$$\left[\int_0^1 \int_0^1 k_1^2(x,s) dxds \right]^{½} \le \sqrt{\frac{a}{2}} \quad . \tag{4.6}$$

This yields

$$|| MR^{-1} f ||_{L_2} \le \left[1 + \sqrt{\frac{a}{2}} \right] || f ||_{L_2} \tag{4.7}$$

A similar calculation yields

$$|| RM^{-1} f ||_{L_2} \le \left[1 + \frac{a}{\sqrt{2}} \right] || f ||_{L_2} \tag{4.8}$$

Thus, as predicted by Theorem 3.1, we have shown that M and R are L_2 norm equivalent. Notice the norms grow with the parameter a.

Example 1.b) L_2 norm equivalence of M^{-1} and R^{-1}. Now consider M^{-1} and R^{-1}. We have

$$\begin{aligned} R^{-1} Mu &= \int_0^1 G_R(x,s)(-u''(s)) ds \\ &= u - u(0) + \frac{1}{a} e^{-a}(e^{ax} - 1)(au(1) + u'(1)) - \int_0^1 k_2(x,s) u(s) ds \quad , \end{aligned} \tag{4.9a}$$

where

$$k_2(x,s) = \begin{cases} -ae^{-as} , & 0 \le s \le x , \\ a(e^{ax} - 1)e^{-as} , & x \le s \le 1 . \end{cases} \tag{4.9b}$$

For $u \in D(M)$ this becomes

$$R^{-1}Mu = u + ae^{-a}(e^{ax} - 1)u(1) - \int_0^1 k_2(x,s)u(s)ds \quad . \tag{4.10}$$

Likewise, we have

$$M^{-1}Ru = u - u(0) + x(au(1) - u'(1)) - a\int_0^x u(s)ds \quad . \tag{4.11}$$

For $u \in D(R)$ this becomes

$$M^{-1}Ru = u + axu(1) - a\int_0^x u(s)ds \quad . \tag{4.12}$$

Both (4.10) and (4.12) contain terms involving $u(1)$. This quantity is not bounded on $D(M) = D(R)$.

To see this let

$$p(z) = (4z^3 - 3z^4) \tag{4.13}$$

The polynomial $p(z)$ has the properties

$$p(0) = p'(0) = p''(0) = 0$$

$$p(1) = 1 \quad , \quad p'(1) = 0$$

$$\max_{0 \le x \le 1} [p(z)] = 1$$

If we let

$$u_\eta(x) = \begin{cases} 0 \quad , & 0 \le x \le 1 - \eta \quad , \\ \eta^{-1/2} p\left(\dfrac{x - 1 + \eta}{\eta}\right) \quad , & 1 - \eta \le x \le 1 \quad , \end{cases} \tag{4.14}$$

Then, $u_\eta \in D(M) = D(R)$ and

$$||u_\eta||_{L_2} \le 1 \quad , \quad |u_\eta(1)| = \eta^{-1/2} \quad .$$

Thus, there exist K_1 K_2, $\eta_0 > 0$ such that for $0 < \eta \le \eta_0$

$$||R^{-1}Mu_\eta||_{L_2} \ge K_1\eta^{-1/2} \; ||u_\eta||_{L_2} \quad , \tag{4.15a}$$

$$||M^{-1}Ru_\eta||_{L_2} \ge K_2\eta^{-1/2} \; ||u_\eta||_{L_2} \quad . \tag{4.15b}$$

Example 1.c). H_1 norm equivalence of M^{-1} and R^{-1}. On the other hand, we see that $\Gamma_0(M) = \Gamma_0(R)$. Theorem 3.2 predicts that $||R^{-1}M||_{H_1}$ and $||M^{-1}R||_{H_1}$ are both bounded. We will use the following lemma.

Lemma 4.1. Let $u \in H_1[0,1]$. Then

$$|u(1)| \le \sqrt{2}\ \|u\|_{H_1} \quad . \tag{4.16}$$

Proof. We have

$$u(1) = u(x) + \int_x^1 u'(x)\,ds \quad .$$

Thus,

$$|u(1)| \le |u(x)| + \int_0^1 |u'(s)|\ ds$$

$$\le |u(x)| + \|u'\|_{L_2} \quad .$$

Integrating both sides yields

$$|u(1)| \le \int_0^1 |u(s)|\,ds + \|u'\|_{L_2}$$

$$\le \|u\|_{L_2} + \|u'\|_{L_2} \le \sqrt{2}\ \|u\|_{H_1} \quad . \quad \blacksquare$$

Also, we note that the integral operators in (4.10) and (4.12) are bounded in the H_1 norm. Taking the H_1 norm of both sides of (4.10) and (4.12) and using Lemma 4.1 establishes the bounds.

Example 2). In this example we consider N and R. Since $D(N) \ne D(R)$, N is not L_2 norm equivalent to R. However, since $D(N^*) = D(R^*)$, N^{-1} is L_2 norm equivalent to R^{-1}. Also, $\Gamma_0(N) = \Gamma_0(R)$, so N^{-1} is H_1 norm equivalent to R^{-1}.

Example 2.a). L_2 norm equivalence of N and R. Similar to (4.4a,b) we have

$$NR^{-1}f = f - ae^{ax} \int_x^1 e^{-as} f(s)\,ds \tag{4.17a}$$

and

$$RN^{-1}f = f + a \int_x^1 f(s)\,ds - \frac{a^2}{1+a} \int_0^1 s f(s)\,ds \quad . \tag{4.17b}$$

As the bounds (1.7) and (1.9) imply, these operators are bounded whenever they apply. However, NR^{-1} and $R^{-1}N$ are not defined on a dense subset of $L_2[0,1]$. We shall see in Example 5 that centered finite difference approximations to these operators yield discrete analogues to (4.17a,b) that are not bounded as the mesh is refined.

Example 2.b). L_2 norm equivalence of N^{-1} and R^{-1}. Since $\tilde{M} = \tilde{N}$, that is, M differs from N only in boundary conditions, we have $R^{-1}\tilde{N} = R^{-1}\tilde{M}$ and (4.9a,b) apply. For $u \in D(N)$ (4.9a,b) become

$$R^{-1}Nu = u - \int_0^1 k_2(x,s)u(s)ds \quad . \tag{4.18}$$

Similar to (4.11) we have

$$N^{-1}Ru = u - (1 - \frac{a}{1+a}x)\ u(0) - \frac{1}{1+a}xu'(1) - \int_0^1 k_3(x,s)u(s)ds \tag{4.19a}$$

where

$$k_3(x,s) = \begin{cases} \dfrac{a^2}{1+a}x - a \ , & 0 \le s \le x \ , \\ \dfrac{a^2}{1+a}x \ , & x \le s \le 1 \ . \end{cases} \tag{4.19b}$$

For $u \in D(R)$ this becomes

$$N^{-1}Ru = u - \int_0^1 k_3(x,s)u(s)ds \quad . \tag{4.20}$$

Clearly, $R^{-1}N$ and $N^{-1}R$ are bounded in L_2 norm.
Example 2.c). H_1 norm equivalence of N^{-1} and R^{-1}. From (4.18) and (4.20) we see that $||R^{-1}N||_{H_1}$ and $||N^{-1}R||_{H_1}$ are bounded. This agrees with Theorem 3.2 since $\Gamma_0(N) = \Gamma_0(R)$.
Example 3. In this example we consider L, M, and N. Notice that $\tilde{L} = \tilde{M} = \tilde{N}$ and that each is a self-adjoint positive definite operator. Also notice that $D(L) \ne D(M) \ne D(N) \ne D(L)$ so that no two are L_2 norm equivalent, nor are any two of their inverses. However, $\Gamma_0(L) \ne \Gamma_0(M) = \Gamma_0(N)$. Thus, L^{-1} and M^{-1} are not H_1 norm equivalent while M^{-1} and N^{-1} are H_1 norm equivalent. Finally, the bilinear forms associated with these operators, $l(u,v)$, $m(u,v)$ and $n(u,v)$, are defined on $H_1(\Omega,\Gamma_0(L)) \ne H_1(\Omega,\Gamma_0(M)) = H_1(\Omega,\Gamma_0(N))$, respectively. Since $D = H_1(\Omega,\Gamma_0(L)) \cap H_1(\Omega,\Gamma_0(M))$ is not dense in either space, L and M are not spectrally equivalent. On the other hand, we will see that $m(u,v) = n(u,v)$ on $H_1(\Omega,\Gamma_0(M))$.
Example 3.a). L_2 norm and H_1 norm equivalence of M^{-1} and N^{-1}. Consider

$$\begin{aligned} N^{-1}Mu &= \int_0^1 G_N(x,s)(-u''(s))ds \\ &= u - (1 - \frac{a}{1+a}x)\ u(0) - \frac{1}{1+a}x(au(1) + u'(1)) \quad . \end{aligned} \tag{4.21}$$

For $u \in D(M)$ we have

$$N^{-1}Mu = u - \frac{a}{1+a}xu(1) \quad . \tag{4.22}$$

Recall that $G_M(x,s)$ is found by setting $a = 0$ in $G_N(x,s)$. Thus, $M^{-1}N$ is found by setting

$a = 0$ in (4.21) to yield

$$M^{-1}Nu = u - u(0) - xu'(1) \quad . \tag{4.23}$$

For $u \in D(N)$ we have $u'(1) = -au(1)$, which yields

$$M^{-1}Nu = u + axu(1) \quad . \tag{4.24}$$

Both $M^{-1}N$ and $N^{-1}M$ contain a boundary operator that is unbounded in L_2 norm but bounded in H_1 norm (see lemma 4.1). Thus, $||M^{-1}N||_{L_2}$ and $||N^{-1}M||_{L_2}$ are unbounded while $||M^{-1}N||_{H_1}$ and $||N^{-1}M||_{H_1}$ are bounded. Also note that $MN^{-1} = (N^{-1}M)^*$ and $NM^{-1} = (M^{-1}N)^*$. Thus, $||MN^{-1}||_{L_2}$ and $||NM^{-1}||_{L_2}$ are also unbounded.

Example 3.b). L_2 norm and H_1 norm equivalence of M^{-1} and L^{-1}. Equation (4.23) can be used to find $M^{-1}L$, since $\tilde{N} = \tilde{L}$.

Here, however, $u \in D(L)$ yields

$$M^{-1}Lu = u - xu'(1) \quad . \tag{4.25}$$

Similarly, $L^{-1}M$ can be found from (4.21) by setting $a = \infty$ to yield

$$L^{-1}Mu = u - (1 - x)u(0) - xu(1) \quad .$$

For $u \in D(M)$ this is

$$L^{-1}Mu = u - xu(1) \quad . \tag{4.26}$$

Again, both $L^{-1}M$ and $M^{-1}L$ contain an unbounded boundary term. Thus, $||L^{-1}M||_{L_2} = ||ML^{-1}||_{L_2}$ and $||M^{-1}L||_{L_2} = ||LM^{-1}||_{L_2}$ are unbounded. On the other hand, Lemma 4.1 shows that $||L^{-1}M||_{H_1}$ is bounded on $D(M)$. However, $||M^{-1}L||_{H_1}$ is not bounded on $D(L)$. To see this let

$$q(z) = (z^4 - z^3) \quad . \tag{4.27}$$

The polynomial $q(z)$ has the properties

$$q(0) = q'(0) = q''(0) = 0 \quad ,$$

$$q(1) = 0 \quad , \quad q'(1) = 1 \quad ,$$

$$\max_{0 \le x \le 1} |q(z)| = \frac{27}{256} \quad ,$$

$$\max_{0 \le x \le 1} |q'(z)| = \frac{1}{4} \quad .$$

If we let

$$u_\eta(x) = \begin{cases} 0 \quad , & 0 \le x \le 1 - \eta \quad , \\ \eta^{1/2} q\left[\dfrac{x - 1 + \eta}{\eta}\right] \quad , & 1 - \eta \le x \le 1 \quad , \end{cases} \tag{4.28}$$

then $u_\eta \in D(L)$ and

$$\| u_\eta \|_{H_1}^2 \le \frac{\eta}{9} + \frac{1}{4} \quad , \quad | u'_\eta(1) | = \eta^{-1/2} \quad .$$

Thus, there exists $K, \eta_0 > 0$ such that for $\eta < \eta_0$ we have

$$\| M^{-1} L u_\eta \|_{H_1} \ge K \eta^{-1/2} \ \| u_\eta \|_{H_1} \quad . \tag{4.29}$$

Example 3.c). Spectral equivalence of $l(u,v)$, $m(u,v)$ and $n(u,v)$. The bilinear forms associated with the weak forms of *L*, *M*, and *N* are

$$l(u,v) = \int_0^1 u'(s) v'(s)\, ds \tag{4.30a}$$

for $u,v \in H_1([0,1], \Gamma_0(L))$ and

$$m(u,v) = \int_0^1 u'(s) v'(s)\, ds \quad , \tag{4.30b}$$

$$n(u,v) = \int_0^1 u'(s) v'(s)\, ds + a u^2(1) \quad , \tag{4.30c}$$

for $u,v \in H_1([0,1], \Gamma_0(M)) = H_1([0,1], \Gamma_0(N))$. Since $D = H_1([0,1], \Gamma_0(L)) \cap H_1([0,1], \Gamma_0(M))$ is not dense in either space, we cannot say $l(u,v)$ is spectrally equivalent to either $m(u,v)$ or $n(u,v)$. Example 6.c will show that the discrete analogues are not uniformly spectrally equivalent. Now Lemma 4.1 yields

$$u(1)^2 \le 2 m(u,u) \quad ,$$

which in turn yields

$$1 \le \frac{n(u,u)}{m(u,u)} \le 1 + 2a \quad . \tag{4.31}$$

Thus, $m(u,u)$ and $n(u,u)$ are spectrally equivalent on $H_1([0,1], \Gamma_0(M))$.

We now turn to the discrete analogues of the examples above. First we make some definitions. Let $n \ge 2$ be a positive integer and let

$$h = \frac{1}{n+1} \quad . \tag{4.32}$$

Define the uniform grid

$$x_j = jh \quad , \quad j = 0,1,...,n+1 \quad ,$$

on the interval $[0,1]$ and consider the mesh functions, $\underline{u} = \{u_i\}_{i=0}^{n+1}$, that is, vectors of values associated with these points. We define the discrete L_2 inner product between mesh functions

$$<\underline{u},\underline{v}>_h = h\sum_{j=1}^{n} u_j v_j \tag{4.33a}$$

and discrete L_2 norm

$$||u||_{0,h}^2 = <\underline{u},\underline{u}>_h \quad . \tag{4.33b}$$

Observe that this norm depends only upon the inner points $\{x_j\}_{j=1}^{n}$, and differs from the l_2 norm on these points by the factor $h^{1/2}$. We also define the semi-norm

$$|u|_{1,h}^2 = \frac{1}{h}\sum_{j=0}^{n} (u_{j+1} - u_j)^2 \tag{4.34a}$$

and the discrete H_1 norm

$$||\underline{u}||_{1,h}^2 = ||\underline{u}||_{0,h}^2 + |\underline{u}|_{1,h}^2 \quad . \tag{4.34b}$$

Similar to Lemma 4.1 we have

Lemma 4.2. For any mesh function and any $0 \le l \le n+1$

$$|u_l| \le \sqrt{2}\ ||\underline{u}||_{1,h} \quad .$$

Further, if $u_0 = 0$, then

$$|u_l| \le |\underline{u}|_{1,h}$$

$$||\underline{u}||_{0,h} \le \frac{1}{\sqrt{2}}\ |\underline{u}|_{1,h}$$

Proof. The first part is directly analogous to the proof of Lemma 4.1. To see the second part,

$$\begin{aligned} |u_l|^2 &= |\sum_{j=0}^{l-1} (u_{j+1} - u_j)|^2 \le (h\sum_{j=0}^{l-1} 1)(\frac{1}{h}\sum_{j=0}^{l-1} (u_{j+1} - u_j)^2) \\ &\le x_l\ |\underline{u}|_{1,h}^2 \end{aligned} \tag{4.35}$$

Since $x_l < 1$ the first bound is proven. Now using (4.35) we have

$$||\underline{u}||_{0,h}^2 = \sum_{j=1}^{n} |u_j|^2 h \le |\underline{u}|_{1,h}^2\ h^2 \sum_{j=1}^{n} j = h^2\ \frac{n(n+1)}{2}\ |\underline{u}|_{1,h}^2 \quad .$$

Since $h = \dfrac{1}{n+1}$ the lemma is proven. ■

Now consider a centered finite difference approximation to the equation

$$Su = -u'' + \alpha u' = f \quad , \quad u(0) = \gamma u(1) + u'(1) = 0 \quad . \tag{4.36}$$

We have

$$u_0 = 0 \tag{4.37a}$$

$$-(1+\frac{\alpha h}{2})\,u_{j-1} + 2u_j - (1-\frac{\alpha h}{2})u_{j+1} = h^2 f_j \quad , \quad j = 1,\ldots,n \tag{4.37b}$$

$$-(1-\frac{\gamma h}{2})u_n + (1+\frac{\gamma h}{2})u_{n+1} = 0 \quad . \tag{4.37c}$$

Eliminating (4.37a) and (4.37c) yields the $n \times n$ matrix equation for the interior mesh values, $\underline{\tilde{u}} = (u_1, \ldots, u_n)^T$,

$$S_h \underline{\tilde{u}} = h^2 \underline{f} \quad , \tag{4.38}$$

where S_h is a tridiagonal matrix

$$S_h = \begin{bmatrix} 2, \; -(1-\frac{\alpha h}{2}) & & \\ \cdots \; -(1+\frac{\alpha h}{2}) \;\; , \;\; 2 \;\; , \;\; -(1-\frac{\alpha h}{2}) \; \cdots & & \\ & & -(1+\frac{\alpha h}{2}), \; 1+\beta \end{bmatrix} \tag{4.39a}$$

and

$$\beta = \left[1 - \frac{(1-\frac{\alpha h}{2})(1-\frac{\gamma h}{2})}{(1+\frac{\gamma h}{2})} \right] \quad . \tag{4.39b}$$

All of the operators (4.1a-e) fit this pattern. For example, R has $\alpha = a$, $\gamma = 0$ which yields $\beta = \frac{ah}{2}$. On the other hand, R^* has $\alpha = -a$, $\gamma = a$ which also yields $\beta = \frac{ah}{2}$. Thus, $(R^*)_h = (R_h)^T$ as one would expect. The operators L_h, M_h and N_h can be similarly extracted from S_h. To find their inverses one need only consider

$$M_h^{-1} = \begin{bmatrix} 1 & 1 & 1 & & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & & 3 \\ & \cdot & & & \cdot \\ & \cdot & & & \cdot \\ & \cdot & & & \cdot \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} , \tag{4.40}$$

and the following Lemma.

Lemma 4.3. Suppose $A = (B + \beta \underline{vw}^T)$ where A and B are both invertible $n \times n$ matrices. Then

$$A^{-1} = (B^{-1} - \gamma \underline{\hat{v}}\underline{\hat{w}}^T)$$

where

$$\underline{\hat{v}} = B^{-1}\underline{v}$$

$$\underline{\hat{w}}^T = \underline{w}^T B^{-1}$$

$$\gamma = \beta/(1 + \beta(\underline{w}^T B^{-1}\underline{v}))$$

Proof. The proof is by direct calculation. ■ Notice that

$$L_h = M_h + \beta \underline{e}_n \underline{e}_n^T \quad , \qquad \beta = 1 \quad , \tag{4.41a}$$

$$N_h = M_h + \beta \underline{e}_n e_n^T \quad , \qquad \beta = \frac{ah}{1 + \frac{ah}{2}} \quad , \tag{4.41b}$$

where $\underline{e}_n^T = (0, \ldots, 0, 1)$. Using Lemma 4.3, we see

$$L_h^{-1} = M_h^{-1} - \gamma \underline{vv}^T \quad , \qquad \gamma = h \quad , \tag{4.42a}$$

$$N_h^{-1} = M_h^{-1} - \gamma \underline{vv}^T \quad , \qquad \gamma = \frac{ah}{1 + a - \frac{ah}{2}} \quad , \tag{4.42b}$$

where $\underline{v}^T = (1,2,3, \ldots, n)$.

Finally, we remark that

$$||S_h||_{0,h} = ||S_h||_{l_2} \le ||S_h||_F \tag{4.43}$$

where $||S_h||_{l_2}$ is the l_2 norm and $||S_h||_F$ is the Frobenius norm;

$$||A||_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \quad .$$

With these results we are ready to examine the discrete analogues to the above examples.

Example 4. This example shows that M_h and R_h are uniformly equivalent in the discrete L_2 norm while M_h^{-1} and R_h^{-1} are not, that is, $||M_h R_h^{-1}||_{0,h}$ and $||R_h M_h^{-1}||_{0,h}$ are uniformly bounded while $||M_h^{-1} R_h||_{0,h}$ and $||R_h^{-1} M_h||_{0,h}$ are not. It also shows that $||M_h^{-1} R_h||_{l,h}$ and $||R_h^{-1} M_h||_{1,h}$ are uniformly bounded.

Example 4.a). Uniform discrete L_2 norm equivalence of M_h and R_h. Consider M_h and R_h. From (4.36) and (4.39a,b) we have

$$R_h = \begin{bmatrix} 2 \ , -(1 - \frac{ah}{2}) & & & & \\ \cdots \ -(1 + \frac{ah}{2}) \ , \ 2 \ , \ -(1 - \frac{ah}{2}) \ \cdots & & & & \\ & & & -(1 + \frac{ah}{2}), \ (1 + \frac{ah}{2}) \end{bmatrix} . \tag{4.44}$$

Multiplying (4.40) by (4.44) yields

$$R_h M_h^{-1} = (1 + \frac{ah}{2}) \ I_h + ahU_h \tag{4.45a}$$

where

$$U_h = \begin{bmatrix} 0 & 1 & . & . & . & 1 \\ & \cdot & \cdot & & & . \\ & & \cdot & \cdot & & . \\ & & & \cdot & \cdot & . \\ & & & & \cdot & 1 \\ & & & & & 0 \end{bmatrix} \tag{4.45b}$$

Notice the similarity to (4.4b). Since

$$||hU_h||_F = \sqrt{h^2 \frac{n(n-1)}{2}} \leq \frac{1}{\sqrt{2}} \quad ,$$

we have

$$||R_h M_h^{-1}||_{0,1} \leq (1 + \frac{a}{\sqrt{2}} + \frac{ah}{2}) \tag{4.46}$$

which resembles (4.8). Since U is strictly upper triangular it is easy to compute $M_h R_h^{-1}$.
We have

$$M_h R_h^{-1} = \frac{1}{1 + \frac{ah}{2}} (I_h - \delta \Delta_h) \quad , \quad \delta = \frac{ah}{1 + \frac{ah}{2}} \tag{4.47a}$$

and

$$(\Delta_h)_{ij} = \begin{cases} 0 \quad , & j \leq i \quad , \\ (1 - \delta)^{(j - i - 1)} \quad , & j > i \quad . \end{cases} \tag{4.47b}$$

Since $|1 - \delta| < 1$, we have by direct computation

$$||\delta \Delta_h||_F \leq \sqrt{\frac{a}{2} + \frac{1}{4} + \frac{ah}{4}} \quad , \tag{4.48}$$

which yields

$$||M_h R_h^{-1}||_{0,1} \leq 1 + \sqrt{\frac{a}{2} + \frac{1}{4} + \frac{ah}{4}} \quad , \tag{4.49}$$

which resembles (4.7).

Example 4.b). Uniform discrete L_2 norm equivalence of M_h^{-1} and R_h^{-1}. Now consider the inverses. Multiplying (4.44) by (4.40) yields

$$M_h^{-1} R_h = (1 - \frac{ah}{2}) \, I_h - ahU_h^T + ah\underline{v} \, \underline{e}_n^T \tag{4.50}$$

where $\underline{e}_n$ is as in (4.41) and v is as in (4.42). Notice the similarity to (4.12). Here a rank one matrix takes the place of the boundary operator. The first two terms in the right-hand side are bounded as $h \rightarrow 0$, but the third is not.

We have

$$\frac{\| ah\underline{v}\underline{e}_n^T\underline{e}_n \|_{0,h}}{\| \underline{e}_n \|_{0,h}} = \frac{|a| \ \| h\underline{v} \|_{0,h}}{\| \underline{e}_n \|_{0,h}} \quad . \tag{4.51a}$$

Now

$$\| h\underline{v} \|_{0,h}^2 = h^3 \sum_{j=1}^{n} j^2 = \frac{1}{3} + 0(h) \quad , \tag{4.51b}$$

and

$$\| \underline{e}_n \|_{0,h} = h^{1/2} \quad . \tag{4.51c}$$

Thus, there exists $K, h_0 > 0$ such that for $h < h_o$

$$\| M_h^{-1} R_h \|_{0,h} \geq K h^{-1/2} \quad . \tag{4.52}$$

The matrix $R_h^{-1} M_h$ is messy to compute so we only remark that it can be computed from $M_h^{-1} R_h$ by finding the inverse of the first two terms in (4.50) and applying Lemma 4.3. The result is a bounded matrix plus an unbounded rank one matrix. The net result yields

$$C_{0,h}(M_h^{-1} R_h) = \| M_h^{-1} R_h \|_{0,h} \ \| R_h^{-1} M_h \|_{0,h} \geq K h^{-1} \quad . \tag{4.53}$$

Example 4.c). Uniform discrete H_1 norm equivalence of M_h^{-1} and R_h^{-1}. We now consider $\| M_h^{-1} R_h \|_{1,h}$. Since $\| \underline{u} \|_{0,h}$ only required the interior points we could ignore the effect of the boundary points. Here, however, we must include (4.37a,c). If $\underline{u}$ is a grid function satisfying (4.37a,b,c), then

$$| \underline{u} |_{1,h}^2 = \frac{1}{h} (u_1)^2 + | \underline{\tilde{u}} |_{1,h}^2 + \frac{1}{h} \left(\frac{\gamma h}{2 + \gamma h}\right)^2 (u_n)^2 \tag{4.54}$$

where $\underline{\tilde{u}}$ represents the interior values. Now suppose $\underline{u}$ satisfies (4.37a,b,c) for R (i.e., $\alpha = a, \gamma = 0$) and $\underline{w}$ satisfies (4.37a,b,c) for M (i.e., $\alpha = 0$, $\gamma = 0$). Then we have

$$w_0 = 0 \quad , \tag{4.55a}$$

$$\underline{\tilde{w}} = M_h^{-1} R_h \underline{\tilde{u}} \quad , \tag{4.55b}$$

$$w_{n+1} = w_n \quad . \tag{4.55c}$$

Then, from (4.54) we have

$$| \underline{w} |_{1,h}^2 = \frac{1}{h} \ | w_1 |^2 + | M_h^{-1} R_h \underline{\tilde{u}} |_{1,h}^2 \quad . \tag{4.56}$$

From (4.50) we have

$$w_1 = \left(1 - \frac{ah}{2}\right) u_1 + ahu_n \quad .$$

Lemma 4.2 yields

$$|w_1| \le |u_1| + ah|\underline{u}|_{1,h} \quad ,$$

and

$$\frac{1}{h} \; |w_1|^2 \le \frac{1}{h} \; |u_1|^2 + (2a + a^2 h) \; |\underline{u}|^2_{1,h} \tag{4.57}$$

From (4.50) and Lemma 4.2 we also have

$$\begin{aligned} |M_h^{-1} R_h \underline{\tilde{u}}|_{1,h} &\le |\underline{\tilde{u}}|_{1,h} + a|hU^T \underline{\tilde{u}}|_{1,h} + a|u_n| \; |h\underline{v}|_{1,h} \\ &\le |\underline{\tilde{u}}|_{1,h} + a||\underline{\tilde{u}}||_{0,h} + a|u_n| \\ &\le (1 + a(1 + \frac{1}{\sqrt{2}})) \; |\underline{\tilde{u}}|_{1,h} \quad . \end{aligned} \tag{4.58}$$

Combining (4.57) and (4.58) in (4.56) yields

$$|\underline{w}|_{1,h} \le K_1 \; |\underline{u}|_{1,h} \tag{4.59}$$

for some $K_1 > 0$. Finally using (4.59) and Lemma 4.2 we have

$$\begin{aligned} ||\underline{w}||^2_{1,h} &= ||\underline{w}||^2_{0,h} + |\underline{w}|^2_{1,h} \\ &\le (1 + \frac{1}{\sqrt{2}}) \; |\underline{w}|^2_{1,h} \le (1 + \frac{1}{\sqrt{2}}) K_1^2 \; |\underline{u}|^2_{1,h} \\ &\le K_2 \; ||\underline{u}||^2_{1,h} \quad , \end{aligned} \tag{4.60}$$

for some $K_2 > 0$. Thus, $||M_h^{-1} R_h||_{1,h}$ is bounded. Bounds on $||R_h^{-1} M_h||_{1,h}$ can be found in the same fashion.

Example 5. This example shows that $||N_h R_h^{-1}||_{0,h}$ is not uniformly bounded, but $||N_h^{-1} R_h||_{0,h}$ and $||R_h^{-1} N_h||_{0,h}$ are uniformly bounded. It also shows that $||N_h^{-1} R_h||_{1,h}$ is uniformly bounded.

Example 5.a). Uniform discrete L_2 norm equivalence of N_h and R_h. Multiplying (4.42b) by R_h we have

$$R_h N_h^{-1} = R_h M_h^{-1} - \gamma R_h \underline{v}\underline{v}^T \quad , \quad \gamma = \frac{ah}{1 + a - \frac{ah}{2}} \quad . \tag{4.61}$$

Using (4.44) we see

$$R_h \underline{v} = ah\underline{1} + (1 - \frac{ah}{2})\underline{e}_n \quad , \tag{4.62}$$

where $\underline{1} = (1,1, \ldots, 1)^T$. Combining (4.45) and (4.62) with (4.61) yields

$$R_h N_h^{-1} = (1 + \frac{ah}{2}) I_h + ahU_h - \frac{a^2 h^2}{1 + a + \frac{ah}{2}} \; \underline{1} \; \underline{v}^T - \frac{a(1 - \frac{ah}{2})h}{1 + a - \frac{ah}{2}} \; \underline{e}_n \underline{v}^T \tag{4.63}$$

Notice the similarity to (4.17b). The first three terms on the right-hand side of (4.63) correspond to the terms in (4.17b). We have previously shown that the first two are bounded.

Consider

$$||h^2 \underline{1}\ \underline{v}^T||_F^2 \le \frac{2h^4}{3}\ n^4 \le \frac{2}{3}\quad . \tag{4.64}$$

The final term is unbounded. Consider

$$\frac{||h\underline{e}_n \underline{v}^T \underline{v}||_{0,h}}{||\underline{v}||_{0,h}} = ||\underline{e}_n||_{0,h}\ ||\underline{v}||_{0,h} \sim \frac{1}{\sqrt{3}}\ h^{-1/2}\quad . \tag{4.65}$$

Thus, RN^{-1} is bounded wherever it is defined, but the discrete analogue $R_h N_h^{-1}$ is not uniformly bounded on $\mathbf{R}^n$.

Example 5.b). Uniform discrete L_2 norm equivalence of N_h^{-1} and R_h^{-1}.

Multiplying (4.42b) on the right by R_h yields

$$N_h^{-1} R_h = M_h^{-1} R_h - \gamma \underline{v}\ \underline{v}^T R_h \quad , \quad \gamma = \frac{ah}{1 + a - \frac{ah}{2}}\quad , \tag{4.66}$$

where

$$\underline{v}^T R_h = -ah\underline{1}^T + (1 + a - \frac{ah}{2})\ \underline{e}_n^T \tag{4.67}$$

Combining (4.67) and (4.50) into (4.66) we see the final terms miraculously cancel to yield.

$$N_h^{-1} R_h = (1 - \frac{ah}{2})\ \underline{I}_h - ahU_h^T + \frac{a^2 h^2}{1 + a - \frac{ah}{2}}\ \underline{v}\underline{1}^T \tag{4.68}$$

Again, notice the similarity to (4.20). To show that the last term is bounded we have as in (4.64)

$$||h^2 \underline{v}\underline{1}^T||_F^2 = ||h^2 \underline{1}\ \underline{v}^T||_F^2 \le \frac{2}{3}\quad . \tag{4.69}$$

Thus, combining (4.46) and (4.69) yields

$$||N_h^{-1} R_h||_{0,h} \le (1 + \frac{a}{\sqrt{2}} + \frac{ah}{2}) + \sqrt{\frac{2}{3}}\ (\frac{a^2}{1 + a - \frac{ah}{2}}) \tag{4.70}$$

The uniform boundedness of $||R_h^{-1} N_h||_{0,h}$ can be obtained from $N_h^{-1} R_h$ by inverting the first two terms of (4.68) and applying lemma 4.3. The result is uniformly a bounded operator much like (4.47a,b) and a uniformly bounded rank one operator.

Example 5.c). Uniform discrete H_1 norm equivalence of N_h^{-1} and R_h^{-1}. As in Example 4.c the first two terms of (4.68) are bounded in discrete H_1 norm. Since $|h\underline{v}|_{1,h} = 1$, we have by lemma 4.2

$$|h^2 \underline{v}\underline{1}^T\underline{u}|_{1,h} \le |h\underline{v}|_{1,h} \sum_{j=1}^{n} |u_j| h \le |h\underline{v}|_{1,h} \; ||\underline{u}||_{0,h} \le \frac{1}{\sqrt{2}} \; |\underline{u}|_{1,h} \quad . \tag{4.71}$$

Thus, using (4.69), (4.71), and Lemma 4.2 we have

$$\begin{aligned} ||h^2 \underline{v}\ \underline{1}^T\underline{u}||_{1,h}^2 &= ||h^2 \underline{v}\ \underline{1}^T\underline{u}||_{0,h}^2 + |h^2 \underline{v}\ \underline{1}^T\underline{u}|_{1,h}^2 \\ &\le \frac{2}{3}\ ||u||_{0,h}^2 + \frac{1}{2}\ |u|_{1,h}^2 \\ &\le ||u||_{1,h}^2 \quad . \end{aligned} \tag{4.72}$$

The boundedness of $||R_h^{-1}N_h||_{1,h}$ could be obtained in a similar manner as $R_h^{-1}N_h$. We omit this lengthy calculation.

Example 6. The matrices L_h, M_h, and N_h are all symmetric positive definite with domains $D(L) \ne D(M) \ne D(N) \ne D(L)$. Thus, no two are uniformly equivalent in discrete L_2 norm, nor are their inverses. However, we have $\Gamma_0(L) \ne \Gamma_0(M) = \Gamma_0(N)$. This example will show that M_h and N_h are uniformly equivalent in discrete H_1 norm while L_h and M_h are not. Finally, we will show that M_h and N_h are uniformly spectrally equivalent while L_h and M_h are not.

Example 6.a). Uniform discrete H_1 norm equivalence of N_h^{-1} and M_h^{-1}. Equation (4.41b) yields

$$M_h^{-1}N_h = I + \beta \underline{v}\ \underline{e}_n^T \quad , \quad \beta = \frac{ah}{1 + \frac{ah}{2}} \quad , \tag{4.73a}$$

and (4.42b) yields

$$N_h^{-1}M_h = I - \gamma \underline{v}\ \underline{e}_n^T \quad , \quad \gamma = \frac{ah}{(1 + a - \frac{ah}{2})} \quad . \tag{4.73b}$$

Equations (4.73a,b) mimic (4.23) and (4.22) respectively. We have

$$||M_h^{-1}N_h||_{0,h} \ge \frac{||M_h^{-1}N_h\underline{e}_n||_{0,n}}{||\underline{e}_n||_{0,h}} \ge |\beta| \ \frac{||\underline{v}||_{0,h}}{||\underline{e}_n||_{0,h}} - 1 \ \cong \frac{a}{\sqrt{3}}\ h^{-½} \tag{4.74a}$$

$$||N_h^{-1}M_h||_{0,h} \ge \frac{||N_h^{-1}M_h\underline{e}_n||_{0,h}}{||\underline{e}_n||_{0,h}} \ge |\gamma| \ \frac{||\underline{v}||_{0,h}}{||\underline{e}_n||_{0,n}} - 1 \ \cong \frac{a}{\sqrt{3}(1 + a)}\ h^{-½} \quad . \tag{4.74b}$$

Since $M_h N_h^{-1} = (N_h^{-1}M_h)^*$, $N_h M_h^{-1} = (M_h^{-1}N_h)^*$ we also have

$$||M_h N_h^{-1}||_{0,h} \cong \frac{a}{\sqrt{3}(1 + a)} h^{-½} \quad , \tag{4.75a}$$

$$||N_h M_h^{-1}||_{0,h} \cong \frac{a}{\sqrt{3}} h^{-1/2} \quad . \tag{4.75b}$$

To find bounds in the discrete H_1 norm notice that

$$|h\underline{v}\underline{e}_n^T\underline{u}|_{1,h} = |h\underline{v}|_{1,h} \ |u_n| \le |\underline{u}|_{1,h} \tag{4.76}$$

and we see both $||M_h^{-1}N||_{1,h}$ and $||N_h^{-1}M_h||_{1,h}$ are bounded.
Example 6.b). Uniform discrete L_2 norm and H_1 norm equivalence of M_h^{-1} and L_h^{-1}.

Equation (4.41a) and (4.42a) yield

$$M_h^{-1}L_h = I + \underline{v}\ \underline{e}_n^T \quad , \tag{4.77a}$$

$$L_h^{-1}M_h = I - h\underline{v}\ \underline{e}_n^T \quad . \tag{4.77b}$$

As in the previous examples, we see that

$$||M_h^{-1}L_h||_{0,h} = ||L_h M_h^{-1}||_{0,h} \ge \frac{||\underline{v}||_{0,h}}{||\underline{e}_n||_{0,h}} - 1 \cong \frac{1}{\sqrt{3}} h^{-3/2} \quad , \tag{4.78a}$$

$$||L_h^{-1}M_h||_{0,h} = ||M_h L_h^{-1}||_{0,h} \ge \frac{||h\underline{v}||_{0,h}}{||\underline{e}_n||_{0,h}} - 1 \cong \frac{1}{\sqrt{3}} h^{-1/2} \quad , \tag{4.78b}$$

Equation (4.76) implies that $||L_h^{-1}M_h||_{1,h}$ is bounded. However, consider (4.77a). The rank one term satisfies

$$\frac{|\underline{v}\underline{e}_n^T\underline{e}_n|_{1,h}}{|\underline{e}_n|_{1,h}} = \frac{|\underline{v}|_{1,h}}{|\underline{e}_n|_{1,h}} = h^{-1/2} \quad .$$

Thus, there exists $K, h_0 > 0$ such that for $h < h_0$

$$||M_h^{-1}L_h\underline{e}_n||_{1,h} \ge Kh^{-1/2}\,||\underline{e}_n||_{1,h} \quad . \tag{4.79}$$

Example 6.c). Uniform Spectral equivalence of L, M and N. Consider Eqs. (4.73a) and (4.77a). The eigenvalues of $M_h^{-1}N_h$ are 1 and

$$(1 + \beta\underline{e}_n^T\underline{v}) = \left[1 + \frac{a}{1 + \frac{ah}{2}} \ \frac{n}{n+1}\right] \quad ,$$

while the eigenvalues of $M_h^{-1}L_h$ are 1 and $(1 + \underline{e}_n^T\underline{v}) = (1 + n) = h^{-1}$. Thus,

$$1 \le \frac{<N_h\underline{u},\underline{u}>}{<M_h\underline{u},\underline{u}>} \le 1 + a \tag{4.79a}$$

while

$$1 \le \frac{<L_h \underline{u}, \underline{u}>}{<M_h \underline{u}, \underline{u}>} \le h^{-1} \quad , \qquad \textbf{(4.79b)}$$

and the upper bound is achieved.

Special notice should be taken of $M_h^{-1} N_h$. Its spectrum is uniformly bounded but its discrete L_2 norm is not.

5. Discretizations

In this section we discuss the implications of the results of Section 3 for the families of finite-dimensional operators $\{A_h\}$, $\{B_h\}$ which arise from the discretization of the invertible, uniformly elliptic operators *A* and *B*. While most of our results apply to general discretization processes, it is convenient to present our discussion within the finite-element framework.

We suppose we have a family of finite-dimensional function spaces $\{S_h\}$ indexed by a discretization parameter $h \to 0$ and invertible operators $\{A_h\}$ and $\{B_h\}$. That is, for every $f \in L_2(\Omega)$ we have $u, v \in H_2(\Omega)$ the solutions of

$$Au = f \quad , \quad Bv = f \qquad (5.1a)$$

and u_h and v_h the solutions of the

$$A_h u_h = \mathbf{P}_h f \quad , \quad B_h u_h = \mathbf{P}_h f \qquad (5.1b)$$

where $\mathbf{P}_h$ is a projection of $L_2(\Omega)$ onto S_h. With a slight abuse of notation we write

$$u_h = A_h^{-1} f \quad , \quad v_h = B_h^{-1} f \quad . \qquad (5.1c)$$

We begin with the theorem of Bramble and Pasciak [BP] which deals with the uniform L_2 norm equivalence of $\{A_h^{-1}\}$ and $\{B_h^{-1}\}$, given that A^{-1} and B^{-1} are L_2 norm equivalent. Then, we deal with the corresponding result for the uniform L_2 norm equivalence of $\{A_h\}$ and $\{B_h\}$. For both of these results we assume that the discrete operators yield optimal order convergence estimates and satisfy a uniform inverse condition.

Definition: the family $\{A_h\}$ of discrete operators approximating the invertible uniformly elliptic operator *A* satisfies *Condition OP* if there exists a constant $M_1(A)$, depending on *A* but not *h*, such that; for every $f \in L_2(\Omega)$ we have

$$|| A_h^{-1} f - A^{-1} f ||_{L_2} \le h^2 M_1(A) \, || f ||_{L_2} \quad . \qquad (5.2)$$

Definition: The family $\{A_h\}$ satisfies a uniform inverse condition, *Condition INV* if there exists a constant $M_2(A)$, depending only on *A* and not on *h*, such that; for every $v_h \in S_h$ we have

$$|| A_h v_h ||_{L_2} \le M_2(A) h^{-2} \; || v_h ||_{L_2} \quad . \qquad (5.3)$$

Theorem 5.1. (Bramble and Pasciak): Let *A* and *B* be two invertible, uniformly elliptic operators and let A^{-1} and B^{-1} be L_2 norm equivalent. That is, (1.14) holds. Specifically, we assume that (3.5a,b) holds.

Let $\{A_h\}$ and $\{B_h\}$ be the associated discretizations. Let $\{A_h\}$ and $\{B_h\}$ both satisfy condition *OP* and condition *INV*. Then, there is a constant $M_3(A{:}B)$ such that; if $\{u_h\}$ and $\{v_n\}$. satisfy

$$A_h u_h = B_h v_h \tag{5.4a}$$

then they also satisfy

$$||u_h||_{L_2} \le M_3(A{:}B)\ ||v_h||_{L_2} \quad , \tag{5.4b}$$

$$||v_h||_{L_2} \le M_3(A{:}B)\ ||u_h||_{L_2} \quad , \tag{5.4c}$$

Proof. Since this result is stated but not proven in [BP]; we present an elementary proof. Let

$$A_h u_h = B_h v_h = f_h \tag{5.5a}$$

$$\hat{u}_h = A^{-1} f_h \quad , \quad \hat{v}_h = B^{-1} f_h \tag{5.5b}$$

Then from (5.2) we have

$$||u_h||_{L_2} \le ||\hat{u}_h||_{L_2} + h^2 M_1(A)\ ||f_h||_{L_2} \quad .$$

From (3.5a) and (5.3) we have

$$||u_h||_{L_2} \le K_1(A^*) K_2(B^*)\ ||\hat{v}_h||_{L_2} + h^2 M_1(A) M_2(B) h^{-2}\ ||v_h||_{L_2} \quad . \tag{5.6}$$

However, we also have from (5.2)

$$||\hat{v}_h||_{L_2} \le ||v_h||_{L_2} + h^2 M_1(B)\ ||f_h||_{L_2} \quad ,$$

which becomes

$$||\hat{v}_h||_{L_2} \le ||v_h||_{L_2} + h^2 M_1(B) M_2(B) h^{-2}\ ||v_h||_{L_2} \quad . \tag{5.7}$$

substituting (5.7) into (5.6) we obtain

$$||u_h||_{L_2} \le [K_1(A^*) K_2(B^*)(1 + M_1(B) M_2(B)) + M_1(A) M_2(B)]\ ||v_h||_{L_2} \quad . \tag{5.8}$$

Thus we have obtained (5.4b). Interchanging the roles of *u* and *v* yields (5.4c) and the theorem is proven. ■

Theorem 5.2: Let *A* and *B* be invertible, uniformly elliptic operators which are L_2 norm equivalent, that is, (3.2a) and (3.2b) hold. Assume that the approximating discrete operators $\{A_h\}$ and $\{B_h\}$ satisfy condition *OP* and condition *INV*. There is a constant $M_4(A{:}B)$ such that, for every $f_h \in S_h$ we have

$$||A_h B_h^{-1} f_h||_{L_2} \le M_4(A{:}B)\ ||f_h||_{L_2} \quad , \tag{5.9a}$$

$$||B_h A_h^{-1} f_h||_{L_2} \le M_4(A{:}B)\ ||f_h||_{L_2} \quad . \tag{5.9b}$$

Thus, the families $\{A_h\}$ and $\{B_h\}$ are uniformly L_2 norm equivalent.

Proof. Let $||f_h||_{L_2} = 1$. Set

$$B_h A_h^{-1} f_h = g_h \quad . \tag{5.10}$$

We will show

$$||g_h||_{L_2} \leq K_2(B) K_1(A) \; [1 + M_2(B) M_1(B)] + M_1(A) M_2(B) \quad . \tag{5.11}$$

Let

$$BA^{-1} f_h = \overline{g} \quad , \tag{5.12a}$$

$$A^{-1} f_h = \psi_h \quad , \tag{5.12b}$$

$$A_h^{-1} f_h = \phi_h \quad . \tag{5.12c}$$

Then

$$A\psi_h = f_h \quad , \quad B\psi_h = \overline{g} \quad , \tag{5.13a}$$

$$A_h \phi_h = f_h \quad , \quad B_h \phi_h = g_h \quad , \tag{5.13b}$$

Let $w_h \in S_h$ be the solution of

$$B_h w_h = \overline{g} \tag{5.14}$$

Then, (3.2b) and condition *OP* yields

$$||\psi_h - \phi_h||_{L_2} \leq h^2 M_1(A) \; ||f_h||_{L_2} = h^2 M_1(A) \tag{5.15a}$$

$$||\psi_h - w_h||_{L_2} \leq h^2 M_1(B) ||\overline{g}||_{L_2} \leq h^2 M_1(B) K_2(B) K_1(A) \quad . \tag{5.15b}$$

Using (5.3) we have

$$||\overline{g} - g_h||_{L_2} = ||B_h(w_h - \phi_h)||_{L_2} \leq h^{-2} M_2(B) \; ||w_h - \phi_h||_{L_2} \quad .$$

Using (5.15a,b) we have

$$||\overline{g} - g_h||_{L_2} \leq M_1(B) M_2(B) K_2(B) K_1(A) + M_1(A) M_2(B)$$

Hence

$$||g_h||_{L_2} \leq ||\overline{g}||_{L_2} + ||\overline{g} - g_h||_{L_2} \leq K_2(B) K_1(A)[1 + M_1(B) M_2(B)] + M_1(A) M_2(B)$$

Thus, (5.11) has been proved and (5.9b) is established. Reversing the roles of A_h and B_h completes the proof. ∎

Theorem 3.1 states that the operators A^{-1} and B^{-1} are L_2 norm equivalent if and only if $D(A^*) = D(B^*)$. If this condition is not satisfied then A_h^{-1} and B_h^{-1} cannot be uniformly equivalent in L_2 norm (see Theorem 2.9 of [FMP]). Our next result deals with the rate of growth of $C_{L_2}(A_h^{-1} B_h)$ in the case

$$D(A^*) \neq D(B^*) \quad . \tag{5.16}$$

Theorem 5.3. Let A and B be two invertible, uniformly elliptic operators which satisfy (5.16). That is, they do not satisfy the hypothesis of Theorem 5.1. Let $\{A_h\}$ and $\{B_h\}$ satisfy condition *OP*. We consider two cases.

Case 1. There is a segment $\Gamma_3 \subset \Gamma_1(A) = \Gamma_1(B)$ and on this segment the boundary coefficients $\alpha_0^*,\ \beta_0^*, \alpha_1^*,\ \beta_1^*$ are not equal;

$$|\alpha_0^* - \beta_0^*| + |\alpha_1^* - \beta_1^*| > 0 \quad \text{on } \Gamma_3 \quad . \tag{5.17}$$

Then, there is a constant $c_1 > 0$ such that

$$\| B_h^{-1} A_h \|_{L_2} \geq c_1 h^{-1/2} \quad , \tag{5.18a}$$

$$\| A_h^{-1} B_h \|_{L_2} \geq c_1 h^{-1/2} \quad , \tag{5.18b}$$

and

$$C_{L_2}(A_h^{-1} B_h) \geq c_1^2 h^{-1} \quad . \tag{5.18c}$$

Case 2. There is a segment $\Gamma_3 \subset \Gamma_0(A) \cap \Gamma_1(B)$. Then, there are constants $c_1, c_2 > 0$ such that

$$\| A_h^{-1} B_h \|_{L_2} \geq c_1 h^{-1/2} \quad , \tag{5.19a}$$

$$\| B_h^{-1} A_h \|_{L_2} \geq c_2 h^{-3/2} \quad , \tag{5.19b}$$

and

$$C_{L_2}(A_h^{-1} B_h) \geq c_2 c_1 h^{-2} \quad . \tag{5.19c}$$

Proof. Consider $A_h^{-1} B_h$ in both Case 1 and Case 2. The construction of Lemma 3.3 and the discussion in Lemma 3.4 show that, after a smooth mapping of Ω onto Ω', a domain which satisfies Condition Ω, there is a function $\hat{u}_h \in D(B)$ and two constants $\overline{c} > 0,\ \overline{c}_1 > 0$ such that

$$\| \hat{u}_h \|_{L_2} = 1 \quad , \quad \| B\hat{u}_h \|_{L_2} \leq \overline{c} h^{-2} \quad . \tag{5.20a}$$

And, if $\hat{v}_h = A^{-1} B\hat{u}_h$ then

$$\| \hat{v}_h \|_{L_2} = \| A^{-1} B\hat{u}_h \|_{L_2} \geq \overline{c}_1 h^{-1/2} \quad . \tag{5.20b}$$

Let $u_h \in S_h$ be the solution of

$$B_h u_h = \mathbf{P}_h B\hat{u}_h \tag{5.21}$$

while $v_h \in S_h$ the solution of

$$A_h v_h = \mathbf{P}_h B\hat{u}_h = B_h u_h \quad . \tag{5.22a}$$

That is

$$v_h = A_h^{-1} B_h u_h \quad . \tag{5.22b}$$

Then using Condition *OP* (5.2)

$$\|u_h\|_{L_2} \le \|\hat{u}_h\|_{L_2} + \|\hat{u}_h - u_h\|_{L_2} \le 1 + M_1(B)h^2 \|B\hat{u}_h\|_{L_2} \quad .$$

Using (5.20a) we obtain

$$\|u_h\|_{L_2} \le [1 + M_1(B)\overline{c}] \quad . \tag{5.23}$$

On the other hand

$$\overline{c}_1 h^{-1/2} \le \|\hat{v}_h\|_{L_2} \le \|v_h\|_{L_2} + \|\hat{v}_h - v_h\|_{L_2} \le \|v_h\|_{L_2} + h^2 M_1(A)\|A\hat{v}_h\|_{L_2} \quad .$$

Since $A\hat{v}_h = B\hat{u}_h$ we have

$$\overline{c}_1 h^{-1/2} \le \|v_h\|_{L_2} + M_1(A)\overline{c} \quad . \tag{5.24}$$

Comparing (5.23) and (5.24) yields (5.18b). Reversing the roles of A_h and B_h yields (5.18a) and (5.19a).

A similar argument based on the function ϕ_h constructed in Lemma 3.3 and the discussion in Lemma 3.4 yields (5.19b). The Theorem now follows. ■

We now examine the rate of growth of $C_{L_2}(A_h B_h^{-1})$ in the case $D(A) \ne D(B)$.

Theorem 5.4. Let A and B be two invertible uniformly elliptic operations which are not L_2 norm equivalent.

That is

$$D(A) \ne D(B) \tag{5.25}$$

Let $\{A_h^*\}$ and $\{B_h^*\}$ satisfy Condition *OP* with respect to A^* and B^*. As in Theorem 5.3, we consider two cases.

Case 1. There is a segment $\Gamma_3 \subset \Gamma_1(A) = \Gamma_1(B)$ and on this segment the boundary coefficients $\alpha_0,\ \beta_0,\ \alpha_1,\ \beta_1$ are not equal. That is

$$|\alpha_0 - \beta_0| + |\alpha_1 - \beta_1| > 0 \text{ on } \Gamma_3 \quad . \tag{5.26}$$

Then, there is a constant $c_1 > 0$ such that

$$\|A_h B_h^{-1}\|_{L_2} \ge c_1 h^{-1/2} \quad , \tag{5.27a}$$

$$\|B_h A_h^{-1}\|_{L_2} \ge c_1 h^{-1/2} \quad , \tag{5.27b}$$

and

$$C_{L_2}(A_h B_h^{-1}) \geq c_1^2 h^{-1} \quad . \tag{5.27c}$$

Case 2. There is a segment $\Gamma_3 \subset \Gamma_0(A) \cap \Gamma_1(B)$. Then, there are constants $c_1, c_2 > 0$ such that

$$|| B_h A_h^{-1} ||_{L_2} \geq c_1 h^{-½} \tag{5.28a}$$

$$|| A_h B_h^{-1} ||_{L_2} \geq c_2 h^{-3/2} \tag{5.28b}$$

and

$$C_{L_2}(A_h B_h^{-1}) \geq c_1 c_2 h^{-2} \quad . \tag{5.28c}$$

Proof. We observe that $\Gamma_0(A) = \Gamma_0(A^*)$ and $\Gamma_0(B) = \Gamma_0(B^*)$. The theorem then follows from Theorem 5.3 and the observation

$$|| (B_h^*)^{-1} A_h^* ||_{L_2} = || A_h B_h^{-1} ||_{L_2} \quad , \quad || (A_h^*)^{-1} B_h^* ||_{L_2} = || B_h A_h^{-1} ||_{L_2} \quad . \quad \blacksquare$$

These negative results immediately raise the question of upper bounds on the rate at which $C_{L_2}(B_h^{-1} A_h) \to \infty$. The general case is not resolved. However, there is an important class of problems for which we can show that (5.18) and (5.27) provide the exact asymptotic rate.

As we have said earlier, the theory of numerical methods for elliptic boundary-value problem for general boundary conditions (1.1b), (1.1c) is rather sparse. On the other hand, the finite-element theory for operators of Class N rests on a solid basis and is well understood. Unfortunately, even when comparing two operators of Class N whose conormal derivatives are essentially different, our method of proof requires that we deal with the general case (1.1c). Since we are unwilling to make assumptions (which may be unrealistic) about optimal order convergence for such problems--and hence complete transitivity of uniform discrete L_2 norm equivalence of A_h^{-1} and B_h^{-1}, we limit outselves to a special class of problems. Fortunately, this class is rich enough to include some cases of great interest, e.g., preconditioning by the discretizations B_h of an operator B *with the same leading part as A, and* the case where the leading parts of both A and B are diffusion operators.

Let A be an invertible uniformly elliptic operator of Class N with coefficients $a_{11},\ a_{12}\ a_{22},\ a_1,\ a_2,\ a_0$ and boundary conditions

$$u = 0 \text{ on } \Gamma_0(A) \quad , \quad \frac{\partial u}{\partial \nu_A} = \alpha u \text{ on } \Gamma_1(A) \tag{5.29}$$

Let $\gamma(x,y)$ be a smooth function which satisfies

$$0 < \gamma_0 \le \gamma(x,y) \le \gamma_1 \quad , \quad (x,y) \in \overline{\Omega} \tag{5.30a}$$

with certain constants γ_0, γ_1. Let B be an invertible uniformly elliptic operator of Class N with coefficients b_{11}, b_{12}, b_{22}, b_1, b_2, b_0 which satisfy

$$b_{11} = \gamma a_{11} \quad , \quad b_{12} = \gamma a_{12} \quad , \quad b_{22} = \gamma a_{22} \quad . \tag{5.30b}$$

Let $\Gamma_0(A) = \Gamma_0(B)$ and let the boundary conditions for B be

$$u = 0 \text{ on } \Gamma_0(B) = \Gamma_0(A) \quad , \quad \frac{\partial u}{\partial \nu_B} = \beta u \text{ on } \Gamma_1(B) = \Gamma_1(A) \quad . \tag{5.31}$$

Let $K > 0$ and define the uniformly elliptic operators A^K, B^K as follow. Let

$$A^K u = -\{(a_{11}u_x)_x + (a_{12}u_x)_y + (a_{12}u_y)_x + (a_{22}u_y)_y\} + Ku \quad , \tag{5.32a}$$

$$u = 0 \text{ on } \Gamma_0(A) \quad , \tag{5.32b}$$

$$\frac{\partial u}{\partial \nu_A} = \alpha^* u \text{ on } \Gamma_1(A) \quad , \tag{3.32c}$$

where α^* is the function associated with the boundary conditions for A^*. Let

$$B^K v = -\{(a_{11}v_x)_x + (a_{12}v_x)_y + (a_{12}v_y)_x + (a_{22}v_y)_y\} + Kv \quad , \tag{5.33a}$$

$$u = 0 \text{ on } \Gamma_0(A) = \Gamma_0(B) \quad , \tag{5.33b}$$

$$\frac{\partial u}{\partial \nu_B} = \beta^* u \text{ on } \Gamma_1(B) \quad , \tag{5.33c}$$

where β^* is the function associated with the boundary conditions for B^*. Observe that

$$\frac{\partial u}{\partial \nu_A} = \frac{\partial u}{\partial \nu_{A^K}} \quad , \quad \frac{\partial u}{\partial \nu_B} = \gamma \, \frac{\partial u}{\partial \nu_{B^K}} \quad .$$

So that A^K and B^K are self-adjoint elliptic operators of Class N with

$$\tilde{A}^K = \tilde{B}^K$$

$$D(A^K) = D(A^*) \quad , \quad D(B^K) = D(B^*) \quad .$$

Let $\{S_h\}$ be a family of finite-element spaces (say piecewise polynomial spaces). Let the discrete operators, A_h, B_h be determined by the bilinear forms $\tilde{\alpha}(\cdot,\cdot)$, $\tilde{b}(\cdot,\cdot)$. That is,

A) Consider the problem $Au = f$, $f \in L_2(\Omega)$. A function $u_h \in S_h \cap H_1(\Omega,\Gamma_0)$ is the finite-element approximant of the solution, u, if; for every $w_h \in S_h \cap H_1(\Omega,\Gamma_0)$, u_h

satisfies

$$a_L(u_h,w_h) + \iint_\Omega w_h[a_1(u_h)_x + a_2(u_h)_y + a_0 u_h]\,dxdy - \int_{\Gamma_1} \alpha u_h w_h d\sigma = (f,w_h) \quad . \tag{5.34}$$

where $a_L(\cdot,\cdot)$ is given by (2.2c).

B) Consider the problem $Bv = f, f \in L_2(\Omega)$. A function $v_h \in S_h \cap H_1(\Omega,\Gamma_0)$ is the finite-element approximant of the solution, v; if for every $w_h \in S_h \cap H_1(\Omega,\Gamma_0), v_h$ satisfies

$$b_L(v_h,w_h) + \iint_\Omega w_h[b_1(v_h)_x + b_2(v_h)_y + b_0 v_h]\,dxdy - \int_{\Gamma_1} \beta v_h w_h d\sigma = (f,w_h) \quad . \tag{5.35}$$

The operators A_h^K and B_h^K are defined in a similar fashion.

A.K) Consider the problem $A^K u^K = f,\ f \in L_2(\Omega)$. A function $u_h^K \in S_h \cap H_1(\Omega,\Gamma_0)$ is the finite-element approximant of the solution, u^K, if; for every $w_h \in S_h \cap H_1(\Omega,\Gamma_0), u_h^K$ satisfies

$$a_L(u_h^K,w_h) + K(u_h^K,w_h) - \int_{\Gamma_1} \alpha^* u_h^K w_h = (f,w_h) \quad .$$

B.K) Consider the problem $B^K v^K = f,\ f \in L_2(\Omega)$. A function $v_h^K \in S_h \cap H_1(\Omega,\Gamma_0)$, is the finite-element approximant of the solution, v^k, if; for every $w_h \in S_h \cap H_1(\Omega,\Gamma_0), v_h^k$ satisfies

$$a_L(v_h^K,w_h) + K(v_h^K,w_h) - \int_{\Gamma_1} \frac{1}{\gamma} \beta^* v_h^K w_h) = (f,w_h) \quad .$$

Theorem 5.5. Let A, B, A^K, B^K, A_h, $B_{h,}$ A_h^K, B_h^K be as described above and assume the hypotheses of Theorem 3.1. Assume there is an $h_0 > 0$ and a $K_0 > 0$ such that for all $0 < h \le h_0$ and all $K \ge K_0$ the operators A^K, B^K are invertible and the discrete operators $\{A_h\}$, $\{B_h\}$, $\{A_h^K\}$, $\{B_h^K\}$ satisfy Condition *OP* and Condition *INV*. Further, assume that the finite-element spaces S_h satisfy a uniform inverse condition: there is a constant $M > 0$ such that, for all $w_h \in S_h$ we have

$$||\nabla w_h||_{L_2} \le Mh^{-1} ||w_h||_{L_2} \quad . \tag{5.36}$$

Suppose there is a segment $\Gamma_3 \subset \Gamma_1(A)$ and

$$\alpha^* = \frac{1}{\gamma} \beta^* \text{on } \Gamma_1(A)/\Gamma_3 \quad , \tag{5.37a}$$

$$|\alpha^* - \frac{1}{\gamma} \beta^*| > 0 \text{ on } \Gamma_3 \quad . \tag{5.37b}$$

Then there are constants $c_1, c_2 > 0$ such that

$$c_1 h^{-½} \le ||A_h^{-1} B_h||_{L_2} \le c_2 h^{-½} \quad , \quad h \le h_0 \quad , \tag{5.38a}$$

$$c_1 h^{-½} \le ||B_h^{-1} A_h||_{L_2} \le c_2 h^{-½} \quad , \quad h \le h_0 \quad . \tag{5.38b}$$

Proof. The lower bounds of (5.38a), (5.38b) were established in Theorem 5.3. In fact, using Theorem 5.1 and the fact that we may interchange the roles of A and B, it suffices to prove a bound of the form

$$||(B_h^K)^{-1} A_h^K||_{L_2} \le c_2 h^{-½}$$

for some $K > K_0$. Let $K > K_0$ and assume that

$$u_h, v_h \in S_h \cap H_1(\Omega, \Gamma_0)$$

satisfy

$$A_h^K u_h = B_h^K v_h \quad , \tag{5.39a}$$

$$||u_h||_{L_2} = 1 \quad . \tag{5.39b}$$

Let $w_h = u_h - v_h$. Then

$$\begin{aligned} a_L(w_h, w_h) + K||w_h||_{L_2^2} &\le ||\frac{1}{\gamma} \beta^*||_\infty \int_{\partial\Omega} |w_h|^2 d\sigma \\ &\quad + ||\alpha^* - \frac{1}{\gamma} \beta^*||_\infty \int_{\partial\Omega} |w_h| \, |u_h| \, d\sigma \quad . \end{aligned} \tag{5.40}$$

Let $\bar{c} = 2(||\alpha^*||_\infty + ||\frac{1}{\gamma} \beta^*||_\infty)$. Then using (2.1) on the left and (2.17) on the right, we have

$$\begin{aligned} \lambda(A) \; ||\nabla w_h||_{L_2}^2 + K \; ||w_h||_{L_2}^2 &\le \bar{c} L_0 \; [||\nabla w_h||_{L_2} \cdot ||w_h||_{L_2} + ||w_h||_{L_2}^2] \\ &\quad + \bar{c} L_0 \; [||\nabla u_h||_{L_2} \cdot ||u_h||_{L_2} + ||u_h||_{L_2}^2] \quad . \end{aligned} \tag{5.41}$$

Apply (3.46) with

$$\varepsilon = \frac{\lambda(A)}{\bar{c} L_0} \quad , \quad K = \frac{\bar{c} L_0}{\lambda(A)} \quad .$$

To obtain

$$||w_h||^2_{H_1} \le \overline{c}_1 \; [\, ||\nabla u_h||_{L_2} \cdot ||u_h||_{L_2} + ||u_h||^2_{L_2}] \quad ,$$

with $\overline{c}_1 = \overline{c}L_0/\min(\dfrac{\lambda(A)}{2} \;\; , \;\; \dfrac{1}{2} K)$. Using (5.36) we have

$$||w_h||^2_{H_1} \le \overline{c}_1 \; [Mh^{-1} + 1] \, ||u_h||^2_{L_2} \le \overline{c}_2 h^{-1} \;\; ||u_h||^2_{L_2} \quad .$$

Thus, using the triangle inequality we have

$$||(B_h^K)^{-1} A_h^K u_h||_{L_2} = ||v_h||_{L_2} \le ||u_h||_{L_2} + ||u_h - v_h||_{L_2} \quad .$$

That is

$$||(B_h^K)^{-1} A_h^K u_h||_{L_2} = ||v_h||_{L_2} \le ||u_h||_{L_2} + ||w_h||_{H_1} \le c_2 h^{-½} \;\; ||u_h||_{L_2} \quad .$$

The theorem now follows. ■

Observe that this proof remains valid if we only require the equalities (5.30) to hold on $\Gamma_1(A)$.

Remark: While this theorem seems to allow the case $\Gamma_0(A) \ne \varnothing$ and $\Gamma_1(A) \ne \varnothing$, H_2 estimates are essential to (i) reduce the general problem to a discussion of A^K and B^K, (ii) apply Theorem 5.3 in order to obtain the lower bounds of (5.38). However, consider the special case where the leading part of $\tilde{A} =$ leading part of $\tilde{B}$. In this case, the argument of Lemma 3.6 enables us to handle the case where $a_0 \ge K$ or $b_0 \ge K$. The argument of Theorem 5.5 then proves the upper bounds of (5.38).

We close this section with two results on H_1-norm equivalence of $\{A_h^{-1}\}$ and $\{B_h^{-1}\}$.

Theorem 5.6. Let A and B be two invertible uniformly elliptic operators of Class N with

$$\Gamma_0(A) = \Gamma_0(B) \quad .$$

Let $\{A_h\}$ and $\{B_h\}$ be the associated discretizations. Suppose these discretizations are of the form (5.34), (5.35); that is, the discretization is obtained by merely restricting (2.14) to the subspace $S_h \cap H_1(\Omega,\Gamma_0(A))$. Let there be an $h_0 > 0$ such that, for all $0 < h \le h_0$, A_h and B_h are uniformly invertible on $H_1^*(\Omega,\Gamma_0)$, the space of bounded linear functionals, on $H_1(\Omega,\Gamma_0)$. Then

$$||(A_h^{-1} B_h)||_{H_1} \text{ and } ||B_h^{-1} A_h||_{H_1}$$

are uniformly bounded for $0 < h \le h_0$.

Proof. The arguments in the proof of Theorem 3.2 go over essentially word-for-word. ■

Theorem 5.7. Let A and B be two invertible uniformly elliptic operators. Let the discrete operators satisfy the H_1 analog of condition *OP*. That is, there are constants $M_5(A)$, $M_5(B)$ such that, for all $f \in L_2$

$$||A_h^{-1} f - A^{-1} f||_{H_1} \le h M_5(A) \, ||f||_{L_2} \tag{5.42}$$

with a similar estimate for B^{-1} and B_h^{-1}. Suppose $\Gamma_0(A) \cap \Gamma_1(B) \ne \emptyset$. Then there is a constant $\overline{c} > 0$ such that

$$||B_h^{-1} A_h||_{H_1} \ge \overline{c} h^{-½} \quad . \tag{5.43}$$

Proof. We will prove the theorem in the case described in Lemma 3.4, Case 2. We recall that proof. Using the function ϕ_η given by (3.13b) we construct ψ_η so that

$$A\phi_\eta = B\psi_\eta$$

and

$$w_\eta = \psi_\eta - \phi_\eta = \eta^{-3/2} w_1 \quad , \quad 0 < n \le 1 \quad ,$$

where w_1 is a fixed function.

Unfortunately, $||\phi_\eta||_{H_1}$ is of order η^{-1}. However, we may consider q_η

$$q_\eta = \eta\phi_\eta \quad . \tag{5.44}$$

Then, (3.10c) asserts that

$$||q_\eta||_{H_1} \le c \quad .$$

Let

$$f_\eta = \eta \; \psi_\eta \quad .$$

Then

$$Aq_\eta = Bf_\eta$$

and

$$f_\eta = q_\eta + \eta^{-1/2} w_1 \quad , \tag{5.45}$$

where w_1 is a fixed function. Let $n = h$ in (5.45) and let u_h, v_h satisfy

$$A_h u_h = Aq_h \tag{5.46a}$$

$$B_h v_h = Aq_h = Bf_h \quad . \tag{5.46b}$$

Then,

$$||u_h||_{H_1} \le ||q_h||_{H_1} + ||q_h - u_h||_{H_1} \tag{5.47a}$$

$$||f_h||_{H_1} \le ||v_h||_{H_1} + ||v_h - f_h||_{H_1} \quad . \tag{5.47b}$$

Using (5.42), (1.9), (5.44)and (3.10c) we see that

$$||q_h - u_h||_{H_1} \le hM_5(A) \; ||Aq_h||_{L_2} \le hM_5(A)||q_h||_{H_2} \le M_5(A)c \quad , \tag{5.48a}$$

$$||v_h - f_h||_{H_1} \le hM_5(B) \; ||Aq_h||_{L_2} \le hM_5(B)||q_h||_{H_2} \le M_5(B)c \quad , \tag{5.48b}$$

Therefore, using (5.47a) we see that

$$||u_h||_{H_1} \le \tilde{c} \quad , \quad \text{some constant} \quad ,$$

while (5.47b), (5.48b), and (5.45) imply that

$$h^{-1/2} \; \| w_1 \|_{H_1} - c \le \| f_h \|_{H_1} \le \| v_h \|_{H_1} + M_5(B)\,c$$

and the theorem follows. ∎

6. Summary

In this paper, we have discussed the effect of boundary conditions on the norm equivalence of uniformly elliptic operators and their discretizations. The motivation for examining equivalent operators stems from the desire to construct uniformly equivalent families of discrete approximations. It was shown in [FMP] that the discretizations cannot be uniformly equivalent unless their limit operators are equivalent.

We have examined the L_2 norm equivalence of the forward operators, A and B and the L_2 norm and H_1 norm equivalence of the inverse operators A^{-1} and B^{-1}. The results on L_2 norm equivalence depend upon the H_2 regularity bounds (1.7) and (1.9). Equivalence in L_2 norm in the absence of H_2 regularity is an open question. The results on H_1 norm equivalence do not depend on H_2 regularity but are restricted to operators of Class N. Both L_2 and H_1 results apply to a wide class of operators.

In brief, in the presence of H_2 regularity for all relevant uniformly elliptic operators we have shown that

I. A^{-1} and B^{-1} are L_2 norm equivalent on $L_2(\Omega)$ if and only if $D(A^*) = D(B^*)$.

II. A and B are L_2 norm equivalent on $L_2(\Omega)$ if and only if $D(A) = D(B)$.

These results show that the conventional wisdom of choosing B to be the leading part of A with the same boundary conditions as A is only appropriate if post conditioning is being used, that is, if $C_{L_2}(AB^{-1})$ is to be bounded. If $C_{L_2}(B^{-1}A)$ is to be bounded, which is most often the case when preconditioning is used, then we must choose the boundary conditions of B so that $D(A^*) = D(B^*)$.

The result on H_1 norm equivalence can be simply stated.

III. A^{-1} and B^{-1} are H_1 norm equivalent on $L_2(\Omega)$ if and only if $\Gamma_0(A) = \Gamma_0(B)$.

Since H_2 regularity is not required for this result it is appropriate to discuss general boundary conditions for operators of Class N. Also, this result is not based upon the requirement that A and B yield positive definite bilinear forms. However, in this case little theory exists on how to construct discrete iterations that are based on a discrete H_1 norm.

In Section 5, we examined families of discretizations. We established some rates of growth for the various discrete conditions when the limit operators did not satisfy I, II, or III above. We also showed that if discretizations satisfy Condition *OP* and Condition *INV* then

IV. $\{A_h^{-1}\}$ and $\{B_h^{-1}\}$ are L_2 norm equivalent on $L_2(\Omega)$ if and only if A^{-1} and B^{-1} are L_2 norm equivalent on $L_2(\Omega)$.

V. $\{A_h\}$ and $\{B_h\}$ are L_2 norm equivalent on $L_2(\Omega)$ if and only if A and B are L_2 norm equivalent on $L_2(\Omega)$.

The first result is an adaptation of a result by [BP].

We also show that A and B are in Class N and if $\{A_h\}$ and $\{B_h\}$ were derived from restricting the weak forms of A and B to the subspace S_h and are uniformly invertible then

VI. $\{A_h^{-1}\}$ and $\{B_h^{-1}\}$ are H_1 norm equivalent on $L_2(\Omega)$ if and only if A^{-1} and B^{-1} are H_1 norm equivalent.

Thus uniform discrete L_2 equivalence can be easily established if Condition *OP* and Condition *INV* hold. The latter is always almost present. If Condition *OP* does not hold then our techniques are inadequate. Some specific examples of finite difference operators are examined in [FMP].

The discrete H_1 norm result is rather general. It applies to a wide class of finite element discretizations using the weak form of the operators. Again we remark that the theory for discrete iterations based on the discrete H_1 norm is limited for nonself-adjoint positive definite operators.

We also remark that two different discretizations of the same operator which act on the same subspace will yield uniformly L_2 norm equivalent families if they both possess Condition *OP* and Condition *INV*.

Finally, we emphasize that, while condition numbers play an important role in convergence bounds for iterative methods, they do not tell the whole story. Firstly, a condition may be bounded with respect to *h*, but very large. Thus, some equivalent operators may provide a poor preconditioning. Secondly, preconditioning by a nonequivalent operator may produce a bounded operator plus an unbounded operator of low rank. For example, notice that in each of the examples in Section 4, the unbounded portion was a rank one operator. In the proper context, conjugate gradient methods would converge much faster than bounds using the condition number might imply. In fact, in Example (6.b) a conjugate gradient iteration would converge in two steps despite the unbounded condition number. In more than one dimension the situation is more complicated. The rank of the unbounded portion appears to increase in proportion to the square root of the number of grid points on a boundary segment with conflicting boundary conditions. An analysis of this phenomenon together with other numerical issues will be discussed in a future report.

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