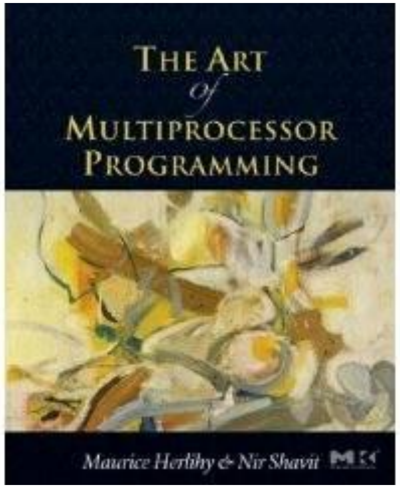
Programming Paradigms for Concurrency
Lecture 2 - Mutual Exclusion



Based on companion slides for
The Art of Multiprocessor Programming
by Maurice Herlihy & Nir Shavit

Modified by
Thomas Wies
New York University

Mutual Exclusion

- We will clarify our understanding of mutual exclusion
- We will also show you how to reason about various properties in an asynchronous concurrent setting

Mutual Exclusion

In his 1965 paper E. W. Dijkstra wrote:

"Given in this paper is a solution to a problem which, to the knowledge of the author, has been an open question since at least 1962, irrespective of the solvability. [...] Although the setting of the problem might seem somewhat academic at first, the author trusts that anyone familiar with the logical problems that arise in computer coupling will appreciate the significance of the fact that this problem indeed can be solved."

Mutual Exclusion

- Formal problem definitions
- Solutions for 2 threads
- Solutions for *n* threads
- Fair solutions
- Inherent costs

Warning

- You will never use these protocols
  - Get over it
- You are advised to understand them
  - The same issues show up everywhere
  - Except hidden and more complex

Why is Concurrent Programming so Hard?

- Try preparing a seven-course banquet
  - By yourself
  - With one friend
  - With twenty-seven friends …
- Before we can talk about programs
  - Need a language
  - Describing time and concurrency

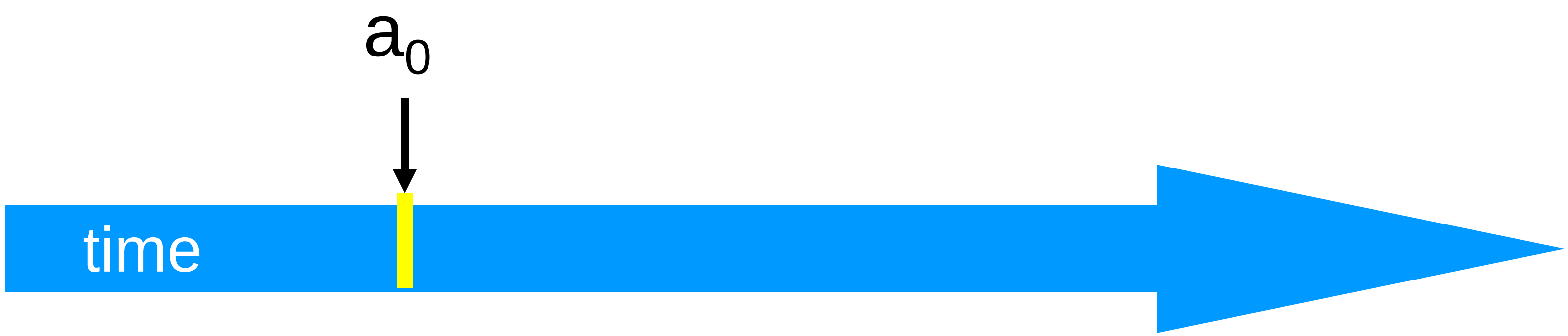
Time

- "Absolute, true and mathematical time, of itself and from its own nature, flows equably without relation to anything external." (I. Newton, 1689)

- "Time is, like, Nature's way of making sure that everything doesn't happen all at once." (Anonymous, circa 1968)

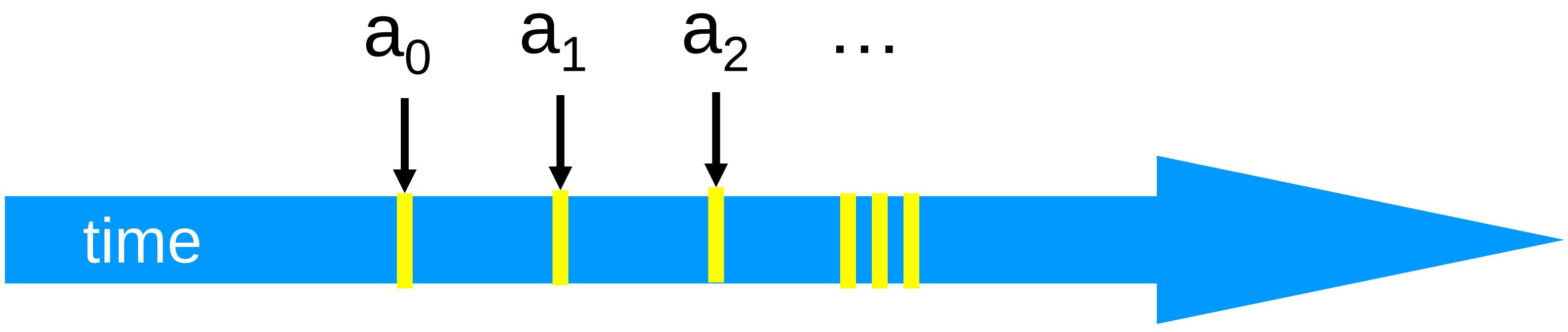
Events

- An *event* a_0 of thread A is
  - Instantaneous
  - No simultaneous events (break ties)



Threads

- A *thread* A is (formally) a sequence a_0, a_1, ... of events
  - "Trace" model
  - Notation: $a_0 \rightarrow a_1$ indicates order

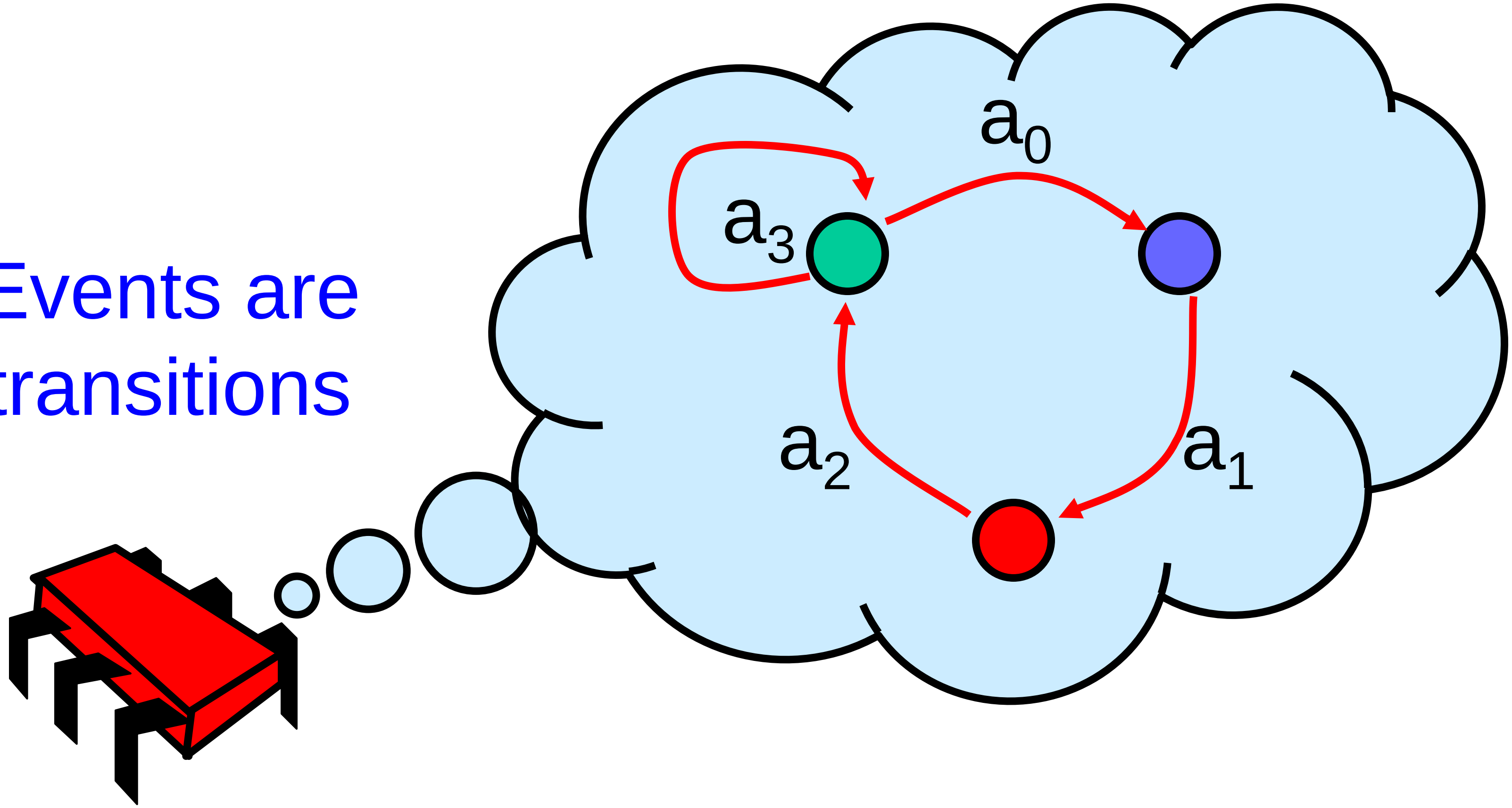


Example Thread Events

- Assign to shared variable
- Assign to local variable
- Invoke method
- Return from method
- Lots of other things …

Threads are State Machines

Events are transitions



States

- Thread State
  - Program counter
  - Local variables
- System state
  - Object fields (shared variables)
  - Union of thread states

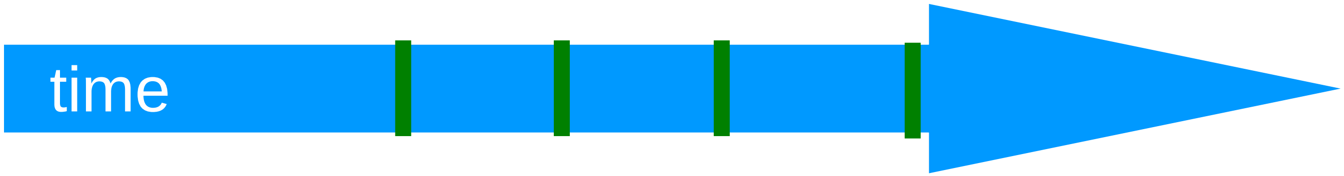
Concurrency

- Thread A

Concurrency

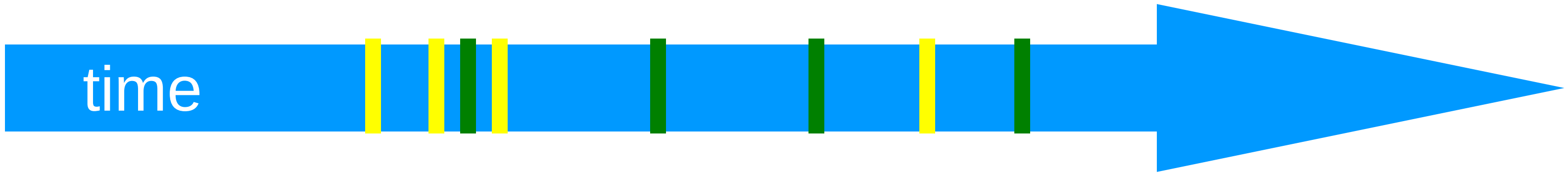
- Thread A

- Thread B



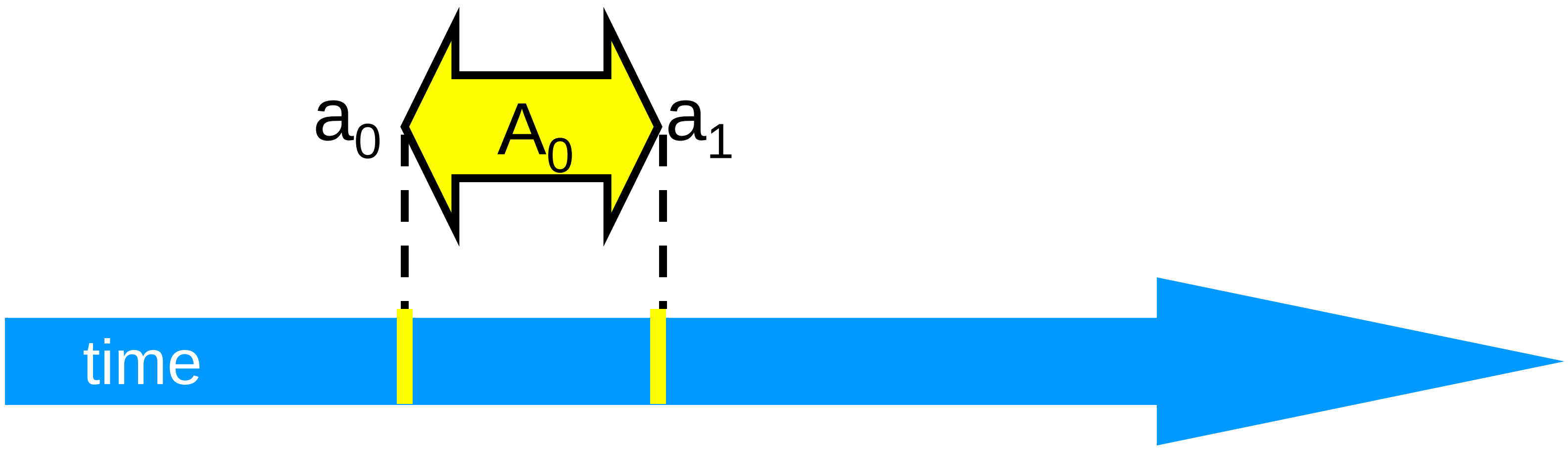
Interleavings

- Events of two or more threads
  - Interleaved
  - Not necessarily independent (why?)

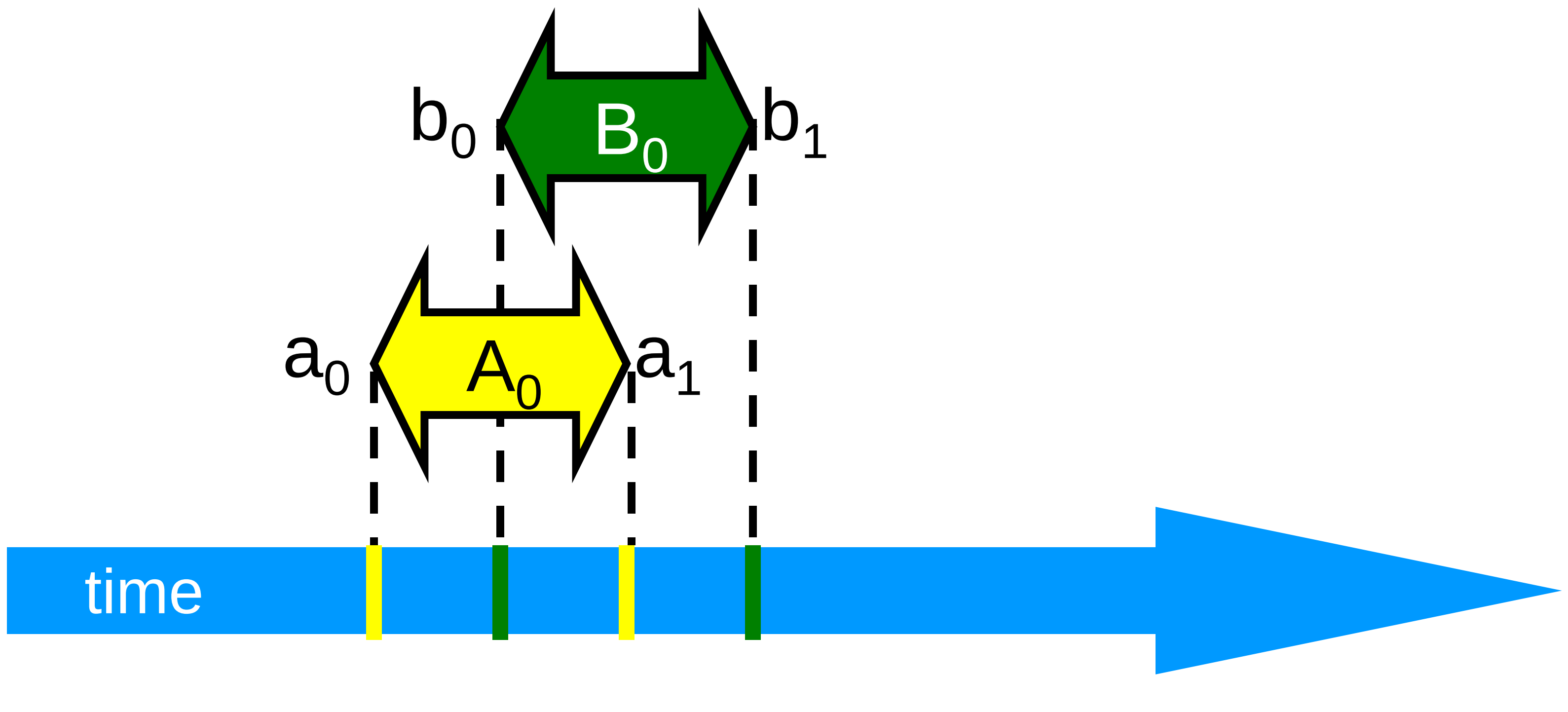


Intervals

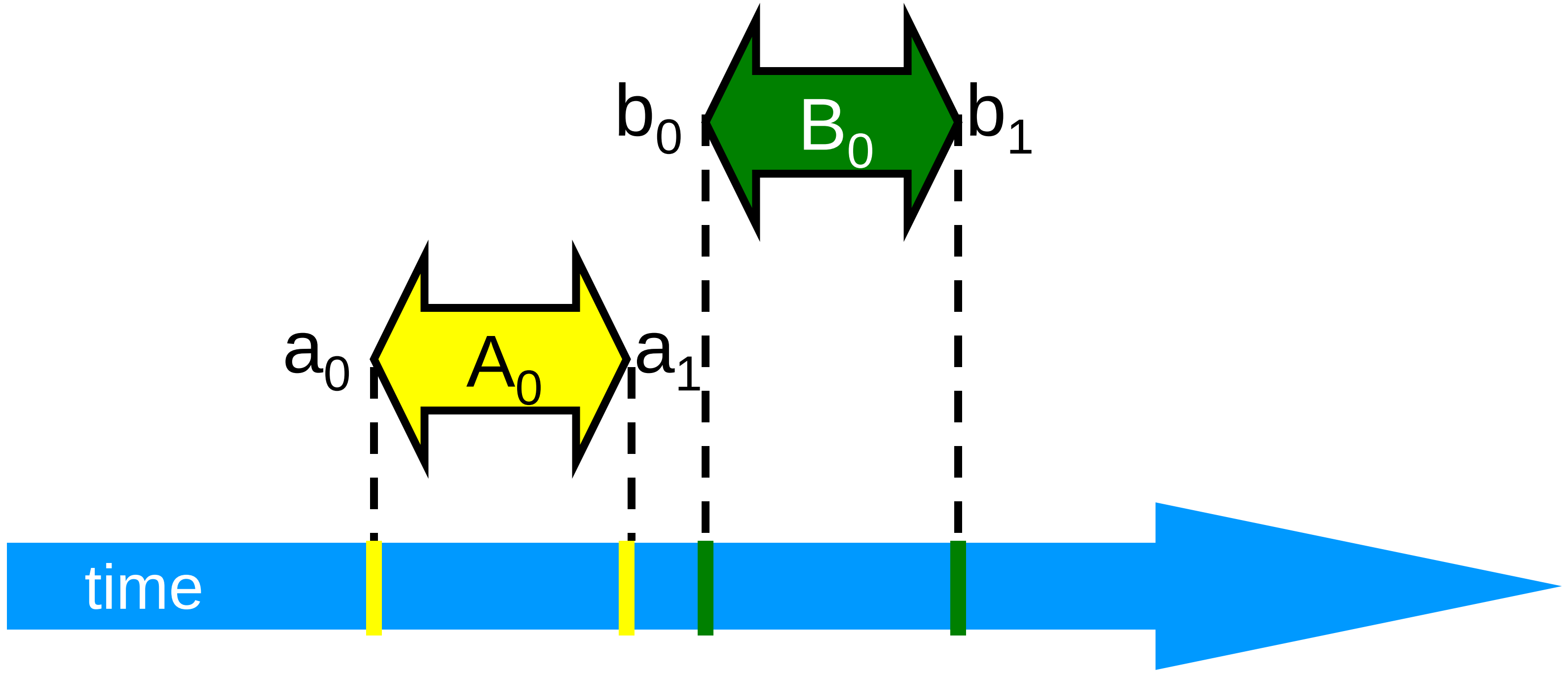
- An *interval* $A_0 = (a_0, a_1)$ is
  - Time between events a_0 and a_1



Intervals may Overlap

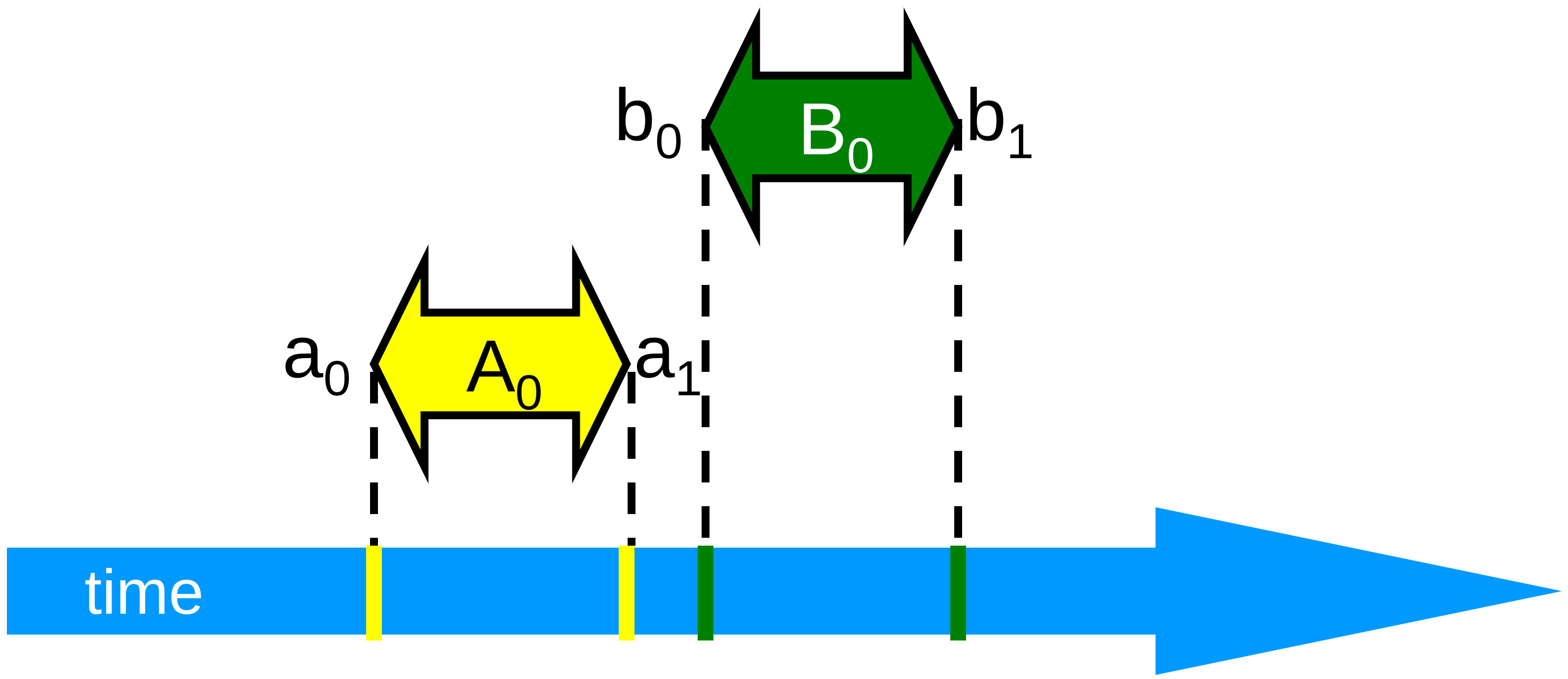


Intervals may be Disjoint

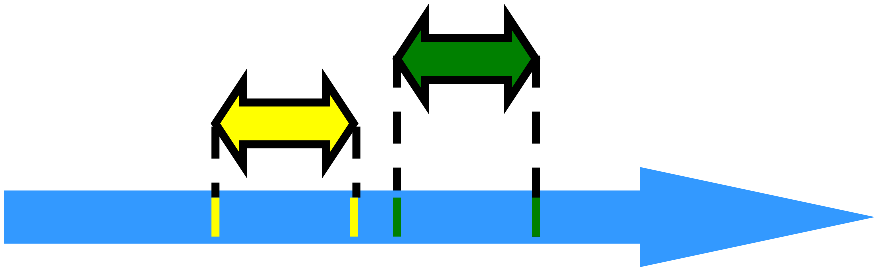


Precedence

Interval A_0 precedes interval B_0

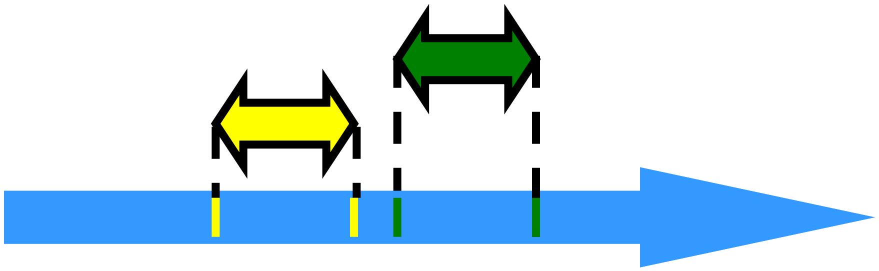


Precedence



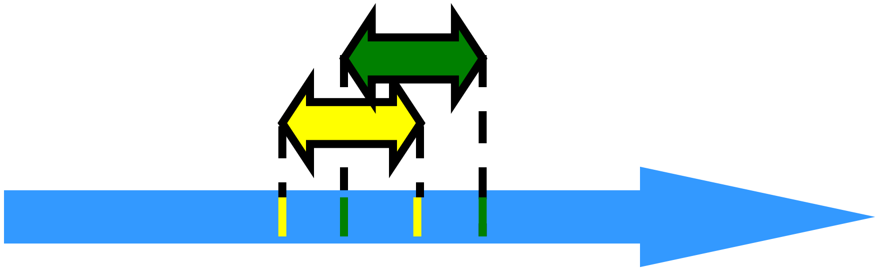
- Notation: $A_0 \rightarrow B_0$
- Formally,
  - End event of A_0 before start event of B_0
  - Also called "happens before" or "precedes"

Precedence Ordering



- Remark: $A_0 \rightarrow B_0$ is just like saying
  – 1066 AD ➔ 1492 AD,
  – Middle Ages ➔ Renaissance,
- Oh wait,
  – what about this week vs this month?

Precedence Ordering



- Never true that A $\rightarrow$ A
- If A $\rightarrow$B then not true that B $\rightarrow$A
- If A $\rightarrow$B & B $\rightarrow$C then A $\rightarrow$C
- Funny thing: A $\rightarrow$B & B $\rightarrow$A might both be false!

Strict Partial Orders

(review)

- Irreflexive:
  - Never true that A ➔ A
- Antisymmetric:
  - If A ➔ B then not true that B ➔ A
- Transitive:
  - If A ➔ B & B ➔ C then A ➔ C

Strict Total Orders
(review)

- Also
  - Irreflexive
  - Antisymmetric
  - Transitive
- Except that for every distinct A, B,
  - Either A ➔ B or B ➔ A

Repeated Events

```
while (mumble) {
   a0; a1;
}
```

$\mathbf{a_0^k}$

k-th occurrence of event a_0

$\mathbf{A_0^k}$

k-th occurrence of interval $A_0 = (a_0, a_1)$

Implementing a Counter

```
public class Counter {
  private long value;

  public long getAndIncrement() {
    temp  = value;
    value = temp + 1;
    return temp;
  }
}
```

Make these steps *indivisible* using locks

Locks (Mutual Exclusion)

```
public interface Lock {

 public void lock();

 public void unlock();
}
```

Locks (Mutual Exclusion)

```
public interface Lock {

 public void lock();

 public void unlock();
}
```

acquire lock

Locks (Mutual Exclusion)

```
public interface Lock {

 public void lock();
 public void unlock();
}
```

acquire lock

release lock

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

acquire Lock

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

Release lock
(no matter what)

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

critical section

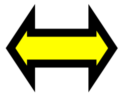
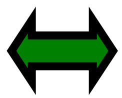
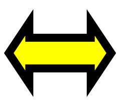
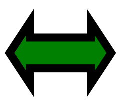
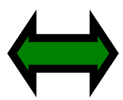
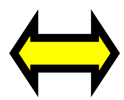
Mutual Exclusion

- Let $CS_i^k \Leftrightarrow$ be thread i's k-th critical section execution

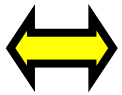
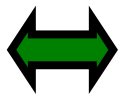
Mutual Exclusion

- Let CS_i^k $\Leftrightarrow$ be thread i's k-th critical section execution
- And CS_j^m $\Leftrightarrow$ be thread j's m-th critical section execution

Mutual Exclusion

- Let CS_i^k  be thread i's k-th critical section execution
- And CS_j^m  be j's m-th execution
- Then either
  -   or  

Mutual Exclusion

- Let CS_i^k  be thread i's k-th critical section execution
- And CS_j^m  be j's m-th execution
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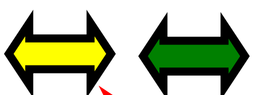
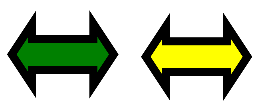
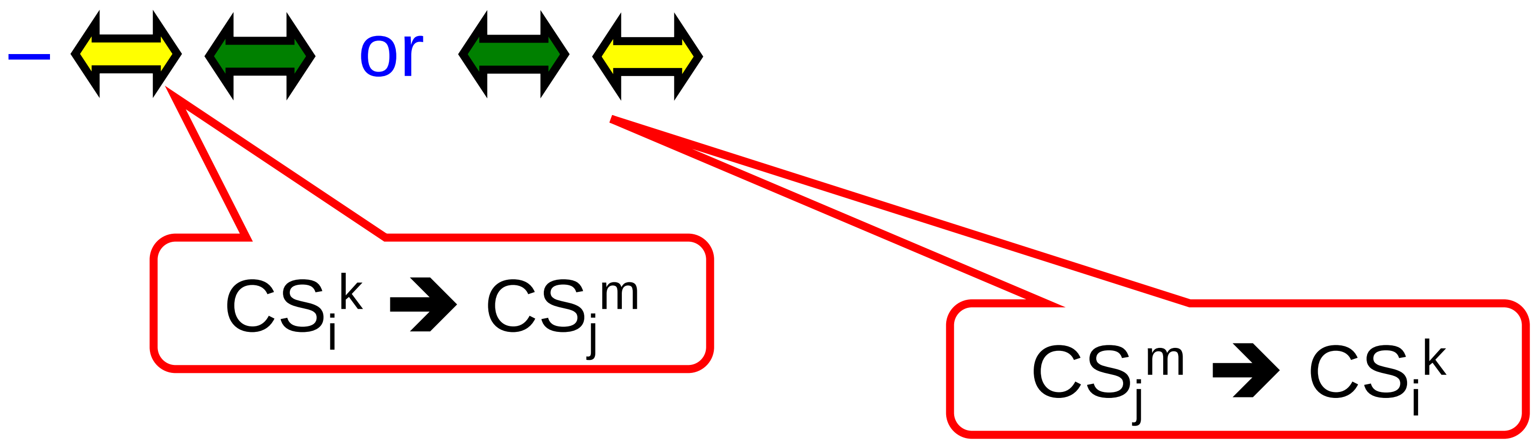
  -  or 

$CS_i^k \rightarrow CS_j^m$

Mutual Exclusion

- Let CS_i^k ⇔ be thread i’s k-th critical section execution
- And CS_j^m ⇔ be j’s m-th execution
- Then either



Deadlock-Free

- If some thread calls **lock()**
  - And never returns
  - Then other threads must complete **lock()** and **unlock()** calls infinitely often
- System as a whole makes progress
  - Even if individuals starve

Starvation-Free

- If some thread calls lock()
  - It will eventually return
- Individual threads make progress

Two-Thread vs *n*-Thread Solutions

- 2-thread solutions first
  - Illustrate most basic ideas
  - Fits on one slide
- Then *n*-thread solutions

Two-Thread Conventions

```
class … implements Lock {
  …
  // thread-local index, 0 or 1
  public void lock() {
    int i = ThreadID.get();
    int j = 1 - i;
  …
  }
}
```

Two-Thread Conventions

```
class … implements Lock {
  …
  // thread-local index, 0 or 1
  public void lock() {
    int i = ThreadID.get();
    int j = 1 - i;
  …
  }
}
```

Henceforth: i is current thread, j is other thread

LockOne

```
class LockOne implements Lock {
private boolean[] flag = new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

LockOne

```
class LockOne implements Lock {
private boolean[] flag = new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

Each thread has flag

LockOne

```
class LockOne implements Lock {
private boolean[] flag = new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

Set my flag

LockOne

```
class LockOne implements Lock {
private boolean[] flag = new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

Wait for other flag to become false

LockOne Satisfies Mutual Exclusion

- Assume CS_A^j overlaps CS_B^k
- Consider each thread's last (j-th and k-th) read and write in the lock() method before entering
- Derive a contradiction

From the Code

- **$\text{write}_A(\text{flag[A]=true}) \rightarrow \text{read}_A(\text{flag[B]==false}) \rightarrow \text{CS}_A$**
- **$\text{write}_B(\text{flag[B]=true}) \rightarrow \text{read}_B(\text{flag[A]==false}) \rightarrow \text{CS}_B$**

```
class LockOne implements Lock {
…
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

From the Assumption

- **read_A(flag[B]==false) → write_B(flag[B]=true)**

- **read_B(flag[A]==false) → write_A(flag[A]=true)**

Combining

- Assumptions:
  - **$read_A(flag[B]==false) \rightarrow write_B(flag[B]=true)$**
  - **$read_B(flag[A]==false) \rightarrow write_A(flag[A]=true)$**
- From the code
  - **$write_A(flag[A]=true) \rightarrow read_A(flag[B]==false)$**
  - **$write_B(flag[B]=true) \rightarrow read_B(flag[A]==false)$**

Combining

- Assumptions:
  - **$read_A$(flag[B]==false) → $write_B$(flag[B]=true)**
  - $read_B$(flag[A]==false) → $write_A$(flag[A]=true)
- From the code
  - $write_A$(flag[A]=true) → $read_A$(flag[B]==false)
  - **$write_B$(flag[B]=true) → $read_B$(flag[A]==false)**

Combining

- Assumptions:
  - $read_A(flag[B]==false) \rightarrow write_B(flag[B]=true)$
  - $read_B(flag[A]==false) \rightarrow write_A(flag[A]=true)$
- From the code
  - $write_A(flag[A]=true) \rightarrow read_A(flag[B]==false)$
  - $write_B(flag[B]=true) \rightarrow read_B(flag[A]==false)$

Combining

- Assumptions:
  - $read_A(flag[B]==false) \rightarrow write_B(flag[B]=true)$
  - $read_B(flag[A]==false) \rightarrow write_A(flag[A]=true)$
- From the code
  - $write_A(flag[A]=true) \rightarrow read_A(flag[B]==false)$
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- Assumptions:
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Combining

- Assumptions:
  - $read_A(flag[B]==false) \rightarrow write_B(flag[B]=true)$
  - $read_B(flag[A]==false) \rightarrow write_A(flag[A]=true)$
- From the code
  - $write_A(flag[A]=true) \rightarrow read_A(flag[B]==false)$
  - $write_B(flag[B]=true) \rightarrow read_B(flag[A]==false)$

Cycle!

Deadlock Freedom

- LockOne Fails deadlock-freedom
  - Concurrent execution can deadlock

```
flag[i] = true;      flag[j] = true;
while (flag[j]){}    while (flag[i]){}
```

  - Sequential executions OK

LockTwo

```
public class LockTwo implements Lock {
 private int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

LockTwo

```
public class LockTwo implements Lock {
 private int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

Let other go first

LockTwo

```
public class LockTwo implements Lock {
 private int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

Wait for permission

LockTwo

```
public class Lock2 implements Lock {
 private int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

Nothing to do

LockTwo Claims

- Satisfies mutual exclusion
  - If thread **`i`** in CS
  - Then **`victim == j`**
  - Cannot be both 0 and 1
- Not deadlock free
  - Sequential execution deadlocks
  - Concurrent execution does not

```
public void LockTwo() {
  victim = i;
  while (victim == i) {};
 }
```

Peterson’s Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Peterson's Algorithm

Announce I'm interested

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Peterson's Algorithm

Announce I'm interested

Defer to other

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Announce I'm interested

Defer to other

Wait while other interested & I'm the victim

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Announce I'm interested

Defer to other

Wait while other interested & I'm the victim

No longer interested

Mutual Exclusion

(1) $\text{write}_B(\text{Flag[B]=true}) \rightarrow \text{write}_B(\text{victim=B})$

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
```

From the Code

Also from the Code

(2) $\text{write}_A(\text{victim}=A) \rightarrow \text{read}_A(\text{flag}[B]) \rightarrow \text{read}_A(\text{victim})$

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
```

Assumption

(3) $\text{write}_B(\text{victim}=B) \rightarrow \text{write}_A(\text{victim}=A)$

W.L.O.G. assume A is the last thread to write **victim**

Combining Observations

(1) $\text{write}_B(\text{flag[B]=true}) \rightarrow \text{write}_B(\text{victim=B})$

(3) $\text{write}_B(\text{victim=B}) \rightarrow \text{write}_A(\text{victim=A})$

(2) $\text{write}_A(\text{victim=A}) \rightarrow \text{read}_A(\text{flag[B]})$
$\rightarrow \text{read}_A(\text{victim})$

Combining Observations

(1) $\text{write}_B(\text{flag[B]=true}) \rightarrow$

(3) $\text{write}_B(\text{victim=B}) \rightarrow$

(2) $\text{write}_A(\text{victim=A}) \rightarrow \text{read}_A(\text{flag[B]})$
$\rightarrow \text{read}_A(\text{victim})$

Combining Observations

(1) $\text{write}_B(\text{flag[B]=true}) \rightarrow$

(3) $\text{write}_B(\text{victim=B}) \rightarrow$

(2) $\text{write}_A(\text{victim=A}) \rightarrow \text{read}_A(\text{flag[B]})$
$\rightarrow \text{read}_A(\text{victim})$

A read flag[B] == true and victim == A, so it could not have entered the CS **(QED)**

Deadlock Free

```
public void lock() {
   …
   while (flag[j] && victim == i) {};
```

- Thread blocked
  - only at **`while`** loop
  - only if other's flag is true
  - only if it is the **victim**
- Solo: other's flag is false
- Both: one or the other not the victim

Starvation Free

- Thread `i` blocked only if `j` repeatedly re-enters so that

  `flag[j] == true` and `victim == i`
- When `j` re-enters
  - it sets `victim` to `j`.
  - So `i` gets in

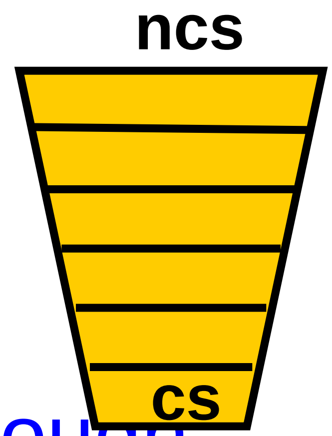
```
public void lock() {
  flag[i] = true;
  victim    = i;
  while (flag[j] && victim == i) {};
}

public void unlock() {
  flag[i] = false;
}
```

The Filter Algorithm for *n* Threads

There are n "waiting rooms" called levels

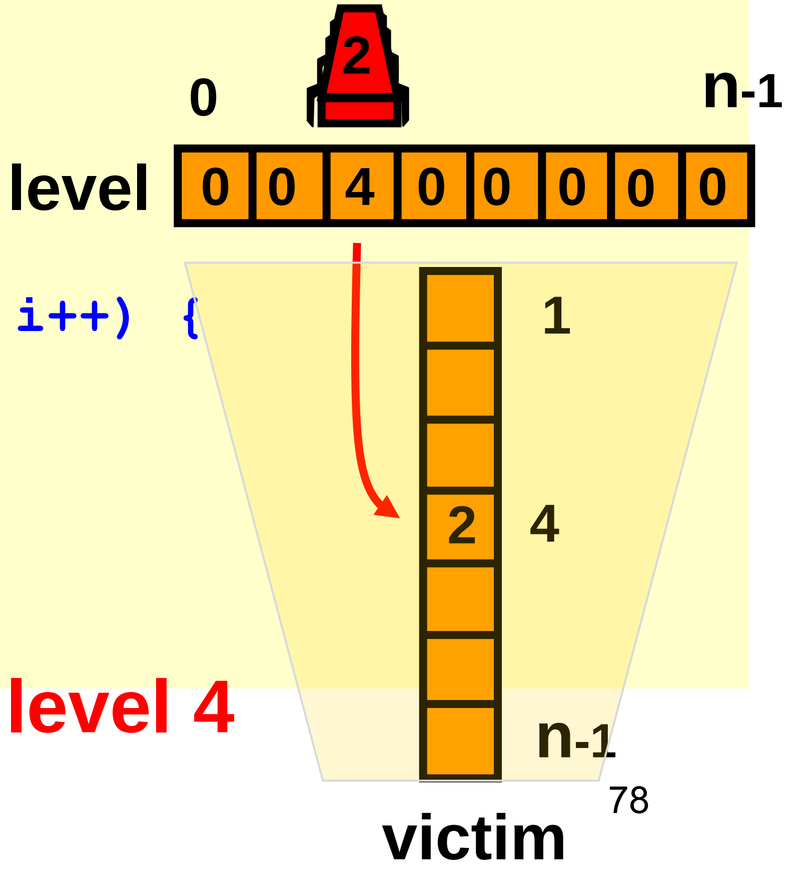
- At each level
  - At least one enters level
  - At least one blocked if many try
- Only one thread makes it through



Filter

```
class Filter implements Lock {
   int[] level;  // level[i] for thread i
   int[] victim; // victim[L] for level L

  public Filter(int n) {
      level  = new int[n];
      victim = new int[n];
      for (int i = 1; i < n; i++) {
          level[i] = 0;
      }}
   …
}
```



Thread 2 at level 4

Filter

```
class Filter implements Lock {
  …

  public void lock(){
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i level[k] >= L) &&
                 victim[L] == i ) {};
    }}
  public void unlock() {
    level[i] = 0;
  }}
```

Filter

```
class Filter implements Lock {
  …

  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i) {};
    }}
  public void release(int i) {
    level[i] = 0;
  }}
```

One level at a time

Filter

```
class Filter implements Lock {
  …

  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i)
    }}
  public void release(int i) {
    level[i] = 0;
  }}
```

Announce intention to enter level L

Filter

```
class Filter implements Lock {
  int level[n];
  int victim[n];
  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i) {};
    }}
  public void release(int i)
    level[i] = 0;
  }}
```

Give priority to anyone but me

Filter

Wait as long as someone else is at same or higher level, and I'm designated victim

```
public void lock() {
  for (int L = 1; L < n; L++) {
    level[i]  = L;
    victim[L] = i;
    while ((∃ k != i) level[k] >= L) &&
           victim[L] == i) {};
  }}
public void release(int i) {
  level[i] = 0;
}}
```

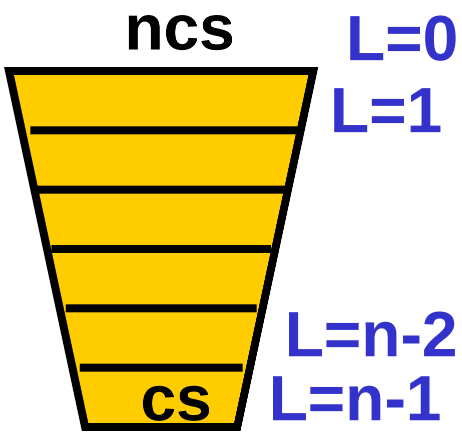
Filter

```
class Filter implements Lock {
  int level[n];
  int victim[n];
  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i) {};
    }}
```

Thread *enters* level L when it completes the loop

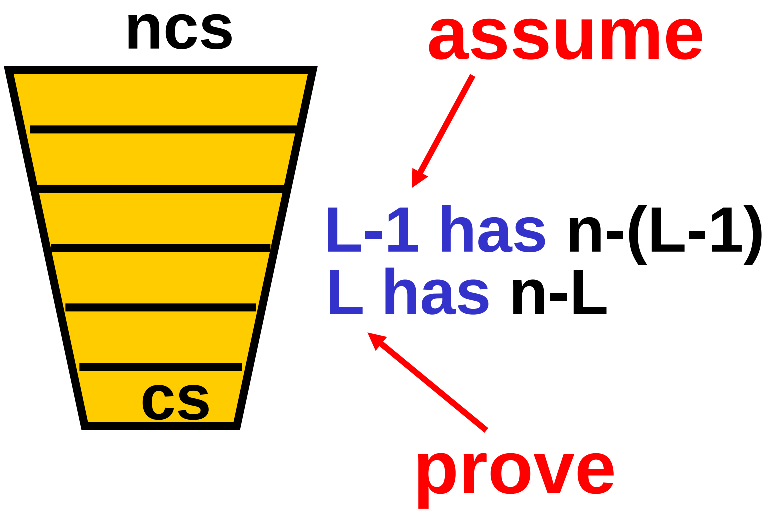
Claim

- Start at level L=0
- At most n-L threads enter level L
- Mutual exclusion at level L=n-1

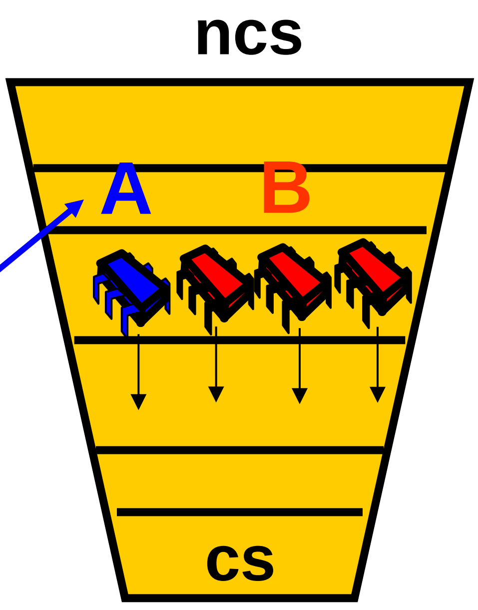


Induction Hypothesis

- No more than n-(L-1) at level L-1
- Induction step: by contradiction
- Assume all at level L-1 enter level L
- A last to write victim[L]
- B is any other thread at level L



Proof Structure



Assumed to enter L-1

n-L+1 = 4

n-L+1 = 4

Last to write victim[L]

By way of contradiction all enter L

Show that A must have seen L in level[B] and since victim[L] == A could not have entered

Just Like Peterson

(1) $\text{write}_B(\text{level}[B]=L) \rightarrow \text{write}_B(\text{victim}[L]=B)$

```
public void lock() {
 for (int L = 1; L < n; L++) {
   level[i] = L;
   victim[L]  = i;
   while ((∃ k != i) level[k] >= L)
          && victim[L] == i) {};
   }}
```

From the Code

From the Code

(2) $\text{write}_A(\text{victim}[L]=A) \rightarrow \text{read}_A(\text{level}[B])$
$\rightarrow \text{read}_A(\text{victim}[L])$

```
public void lock() {
 for (int L = 1; L < n; L++) {
   level[i] = L;
   victim[L]  = i;
   while ((∃ k != i) level[k] >= L)
          && victim[L] == i) {};
   }}
```

By Assumption

(3) $\text{write}_B(\text{victim}[L]=B) \rightarrow \text{write}_A(\text{victim}[L]=A)$

By assumption, A is the last thread to write **victim[L]**

Combining Observations

(1) $\text{write}_B(\text{level[B]=L}) \rightarrow \text{write}_B(\text{victim[L]=B})$

(3) $\text{write}_B(\text{victim[L]=B}) \rightarrow \text{write}_A(\text{victim[L]=A})$

(2) $\text{write}_A(\text{victim[L]=A}) \rightarrow \text{read}_A(\text{level[B]})$
$\rightarrow \text{read}_A(\text{victim[L]})$

Combining Observations

(1) $\text{write}_B(\text{level}[B]=L) \rightarrow$

(3) $\text{write}_B(\text{victim}[L]=B) \rightarrow \text{write}_A(\text{victim}[L]=A)$

(2) $\rightarrow \text{read}_A(\text{level}[B])$

$\rightarrow \text{read}_A(\text{victim}[L])$

Combining Observations

(1) $\text{write}_B(\text{level}[B]=L) \rightarrow$

(3) $\text{write}_B(\text{victim}[L]=B) \rightarrow \text{write}_A(\text{victim}[L]=A)$

(2) $\rightarrow \text{read}_A(\text{level}[B])$

$\rightarrow \text{read}_A(\text{victim}[L])$

A read level[B] ≥ L, and victim[L] = A, so it could not have entered level L!

No Starvation

- Filter Lock satisfies properties:
  - Just like Peterson Alg at any level
  - So no one starves
- But what about fairness?
  - Threads can be overtaken by others

Bounded Waiting

- Want stronger fairness guarantees
- Thread not “overtaken” too much
- If A starts before B, then A enters before B?
- But what does “start” mean?
- Need to adjust definitions ….

Bounded Waiting

- Divide `lock()` method into 2 parts:
  - Doorway interval:
    - Written $\mathbf{D_A}$
    - always finishes in finite steps
  - Waiting interval:
    - Written $\mathbf{W_A}$
    - may take unbounded steps

First-Come-First-Served

- For threads A and B:
  - If $\mathbf{D_A^k \Rightarrow D_B^j}$
    - A's k-th doorway precedes B's j-th doorway
  - Then $\mathbf{CS_A^k \Rightarrow CS_B^j}$
    - A's k-th critical section precedes B's j-th critical section
    - B cannot overtake A

Fairness Again

- Filter Lock satisfies properties:
  - No one starves
  - But very weak fairness
    - Can be overtaken **arbitrary** # of times
  - That's pretty lame…

Bakery Algorithm

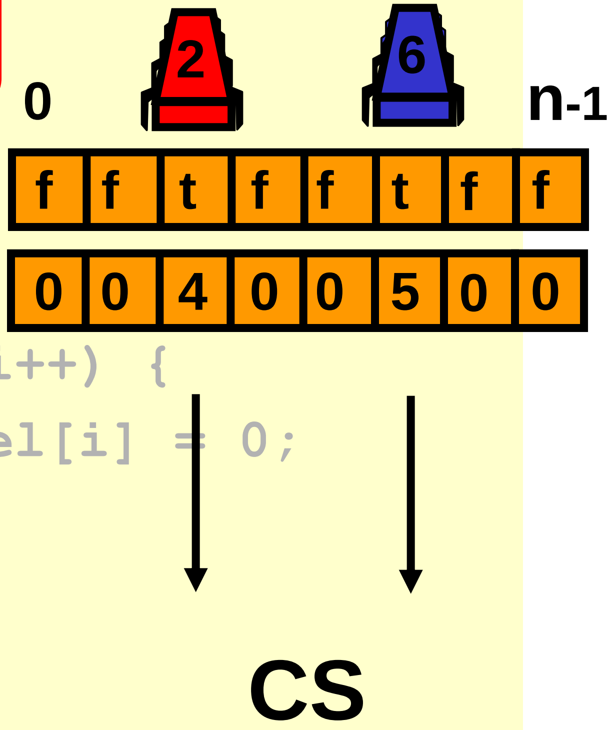
- Provides First-Come-First-Served
- How?
  - Take a "number"
  - Wait until lower numbers have been served
- Lexicographic order
  - $(a,i) > (b,j)$
    - If $a > b$, or $a = b$ and $i > j$

Bakery Algorithm

```
class Bakery implements Lock {
   boolean[] flag;
   Label[] label;
  public Bakery (int n) {
    flag  = new boolean[n];
    label = new Label[n];
    for (int i = 0; i < n; i++) {
       flag[i] = false; label[i] = 0;
    }
  }
 …
```

Bakery Algorithm

```
class Bakery implements Lock {
  boolean[] flag;
  Label[] label;
  public Bakery (int n) {
    flag  = new boolean[n];
    label = new Label[n];
    for (int i = 0; i < n; i++) {
       flag[i] = false; label[i] = 0;
    }
  }
 …
```



Bakery Algorithm

```
class Bakery implements Lock {
   …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) > (label[k],k));
 }
```

Bakery Algorithm

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
            && (label[i],i) > (label[k],k));
 }
```

Doorway

Bakery Algorithm

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) > (label[k],k));
 }
```

I'm interested

Bakery Algorithm

Take increasing label (read labels in some arbitrary order)

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
          && (label[i],i) > (label[k],k));
 }
```

Bakery Algorithm

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
                 && (label[i],i) > (label[k],k));
 }
```

Someone is interested

Bakery Algorithm

```
class Bakery implements Lock {
  boolean flag[n];
  int label[n];

 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
            && (label[i],i) > (label[k],k));
 }
```

Someone is interested …

… whose (label[k],k) in lexicographic order is lower

Bakery Algorithm

```
class Bakery implements Lock {

    …

 public void unlock() {
   flag[i] = false;
 }
}
```

Bakery Algorithm

```
class Bakery implements Lock {

    …

 public void unlock() {
   flag[i] = false;
 }
}
```

No longer interested

labels are always increasing

No Deadlock

- There is always one thread with earliest label
- Ties are impossible (why?)

First-Come-First-Served

- If $D_A \rightarrow D_B$ then
  - A's label is smaller
- And:
  - $\text{write}_A(\text{label}[A]) \rightarrow$
  - $\text{read}_B(\text{label}[A]) \rightarrow$
  - $\text{write}_B(\text{label}[B]) \rightarrow \text{read}_B(\text{flag}[A])$
- So B sees
  - smaller label for A
  - locked out while flag[A] is true

```
class Bakery implements Lock {

public void lock() {
  flag[i]  = true;
  label[i] = max(label[0],
                 …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) >
  (label[k],k));
 }
```

Mutual Exclusion

- Suppose A and B in CS together
- Suppose A has earlier label
- When B entered, it must have seen
  - flag[A] is *false*, or
  - label[A] > label[B]

```
class Bakery implements Lock {

public void lock() {
   flag[i]  = true;
   label[i] = max(label[0],
                      …,label[n-1])+1;
   while (∃k flag[k]
               && (label[i],i) >
   (label[k],k));
  }
```

Mutual Exclusion

- Labels are strictly increasing so
- B must have seen flag[A] == false

Mutual Exclusion

- Labels are strictly increasing so
- B must have seen flag[A] == false
- $\text{Labeling}_B \rightarrow \text{read}_B(\text{flag[A]}) \rightarrow \text{write}_A(\text{flag[A]}) \rightarrow \text{Labeling}_A$

Mutual Exclusion

- Labels are strictly increasing so
- B must have seen flag[A] == false
- $\text{Labeling}_B \rightarrow \text{read}_B(\text{flag[A]}) \rightarrow \text{write}_A(\text{flag[A]}) \rightarrow \text{Labeling}_A$
- Which contradicts the assumption that A has an earlier label

Bakery Y2^{32}K Bug

```
class Bakery implements Lock {
   …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
             && (label[i],i) > (label[k],k));
 }
```

Bakery $Y2^{32}K$ Bug

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) > (label[k],k));
 }
```

Mutex breaks if `label[i]` overflows

Does Overflow Actually Matter?

- Yes
  - Y2K
  - 18 January 2038 (Unix `time_t` rollover)
  - 16-bit counters
- No
  - 64-bit counters
- Maybe
  - 32-bit counters

Timestamps

- Label variable is really a timestamp
- Need ability to
  - Read others' timestamps
  - Compare them
  - Generate a **later** timestamp
- Can we do this without overflow?

The Good News

- One can construct a
  - Wait-free (no mutual exclusion)
  - Concurrent
  - Timestamping system
  - That never overflows

The Good News

Bad

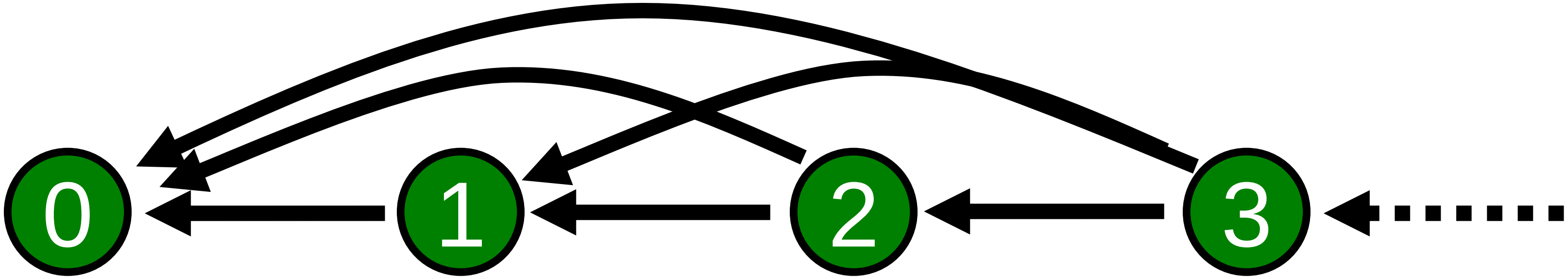
- One can construct a
  - Wait-free (no mutual exclusion)
  - Concurrent
  - Timestamping system
  - That never overflows

This part is hard

Instead …

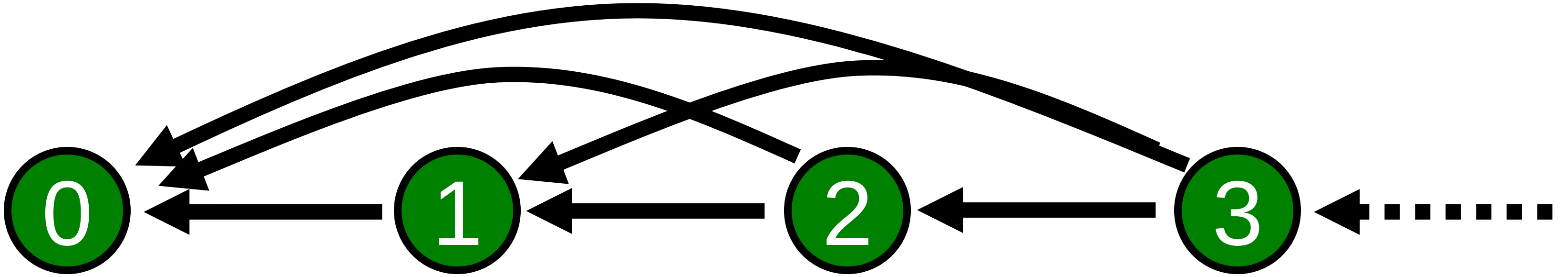
- We construct a Sequential timestamping system
  – Same basic idea
  – But simpler
- As if we use mutex to read & write atomically
- No good for building locks
  – But useful anyway

Precedence Graphs



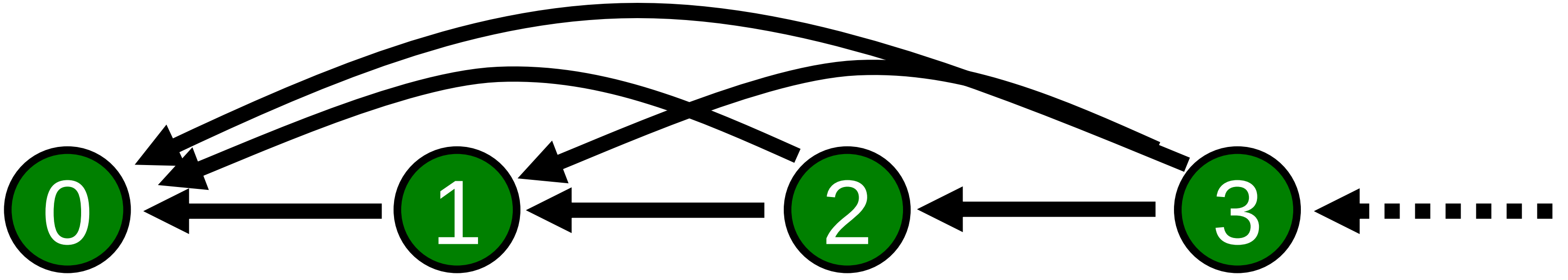
- Timestamps form directed graph
- Edge x to y
  - Means x is later timestamp
  - We say x dominates y

Unbounded Counter Precedence Graph

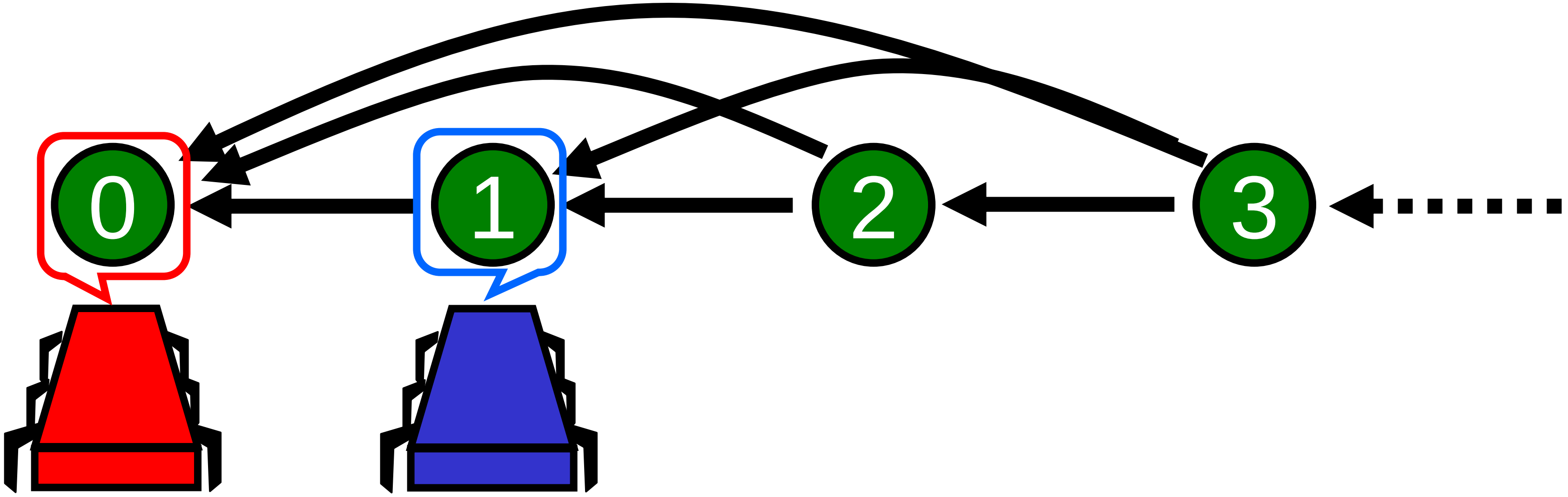


- Timestamping = move tokens on graph
- Atomically
  - read others' tokens
  - move mine
- Ignore tie-breaking for now

Unbounded Counter Precedence Graph

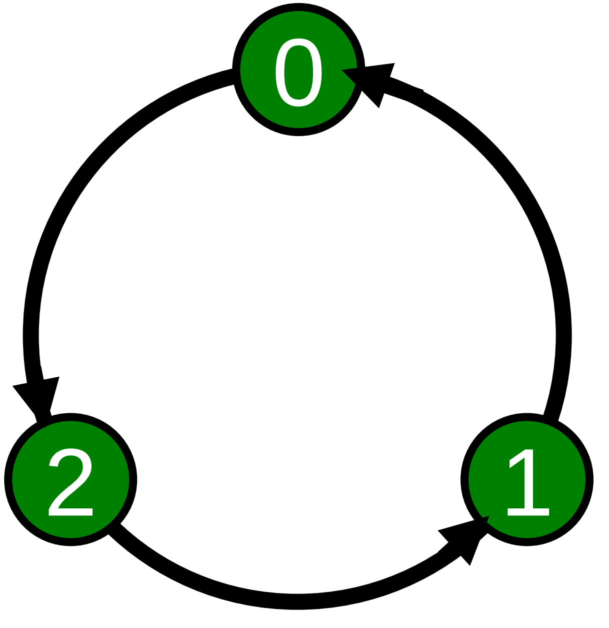


Unbounded Counter Precedence Graph

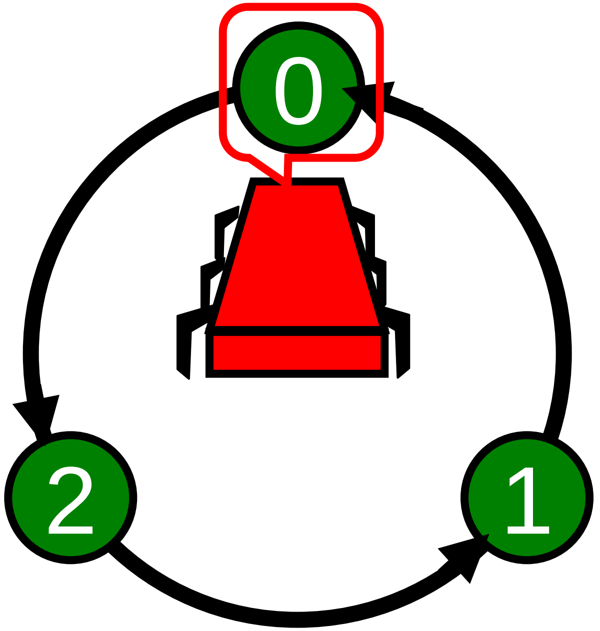


takes 0 takes 1 takes 2

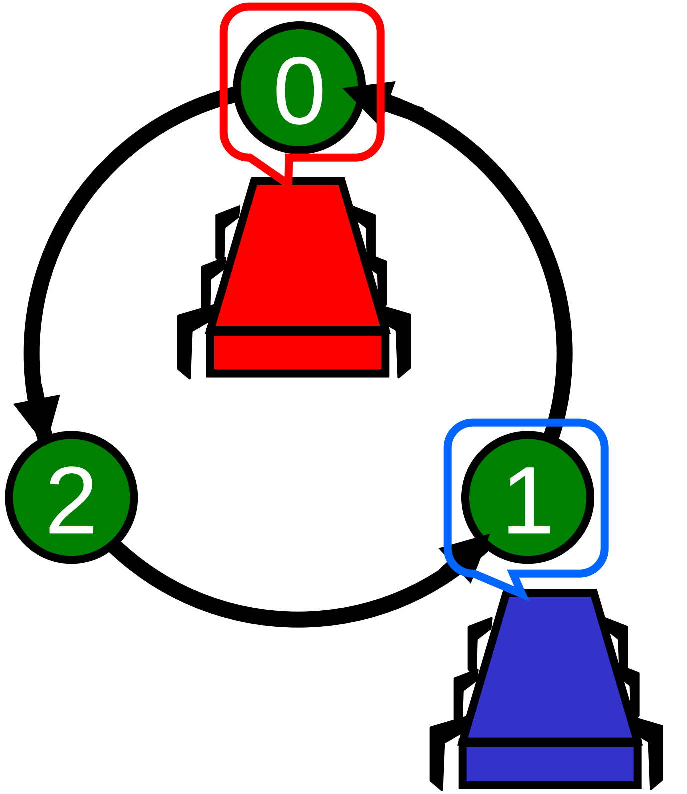
Two-Thread Bounded Precedence Graph



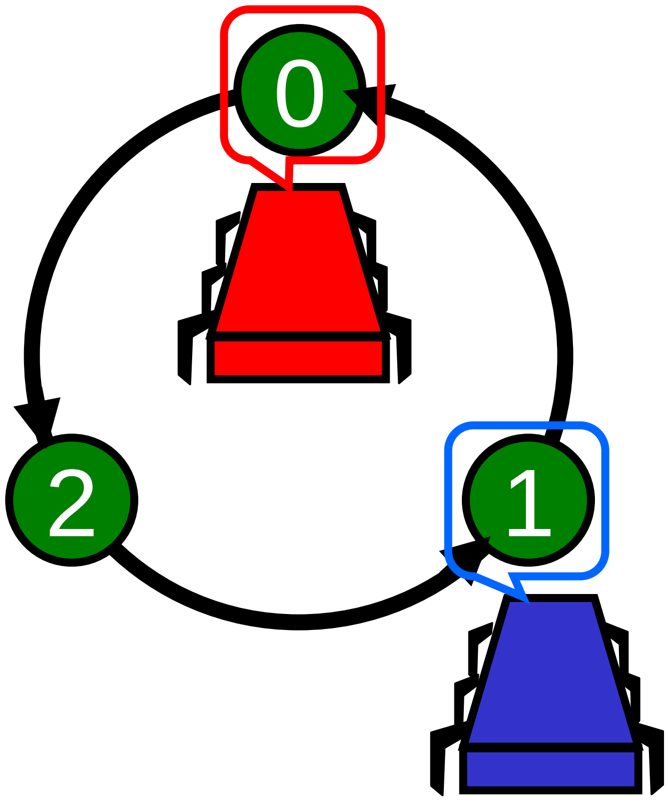
Two-Thread Bounded Precedence Graph



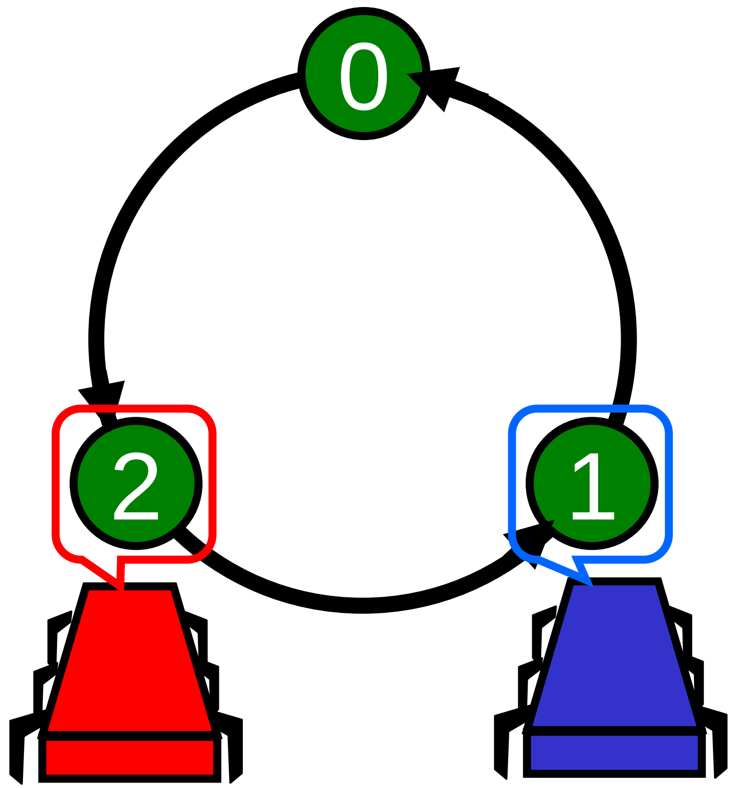
Two-Thread Bounded Precedence Graph



Two-Thread Bounded Precedence Graph

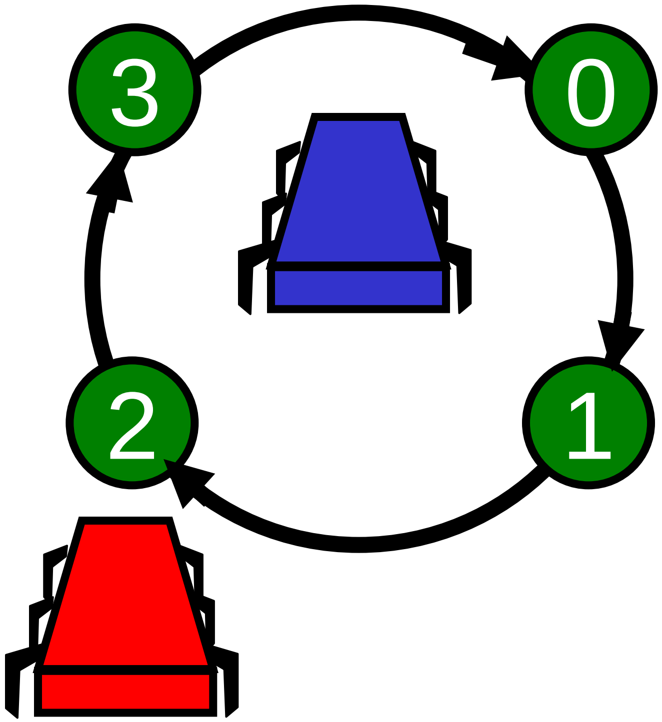


Two-Thread Bounded Precedence Graph T^2

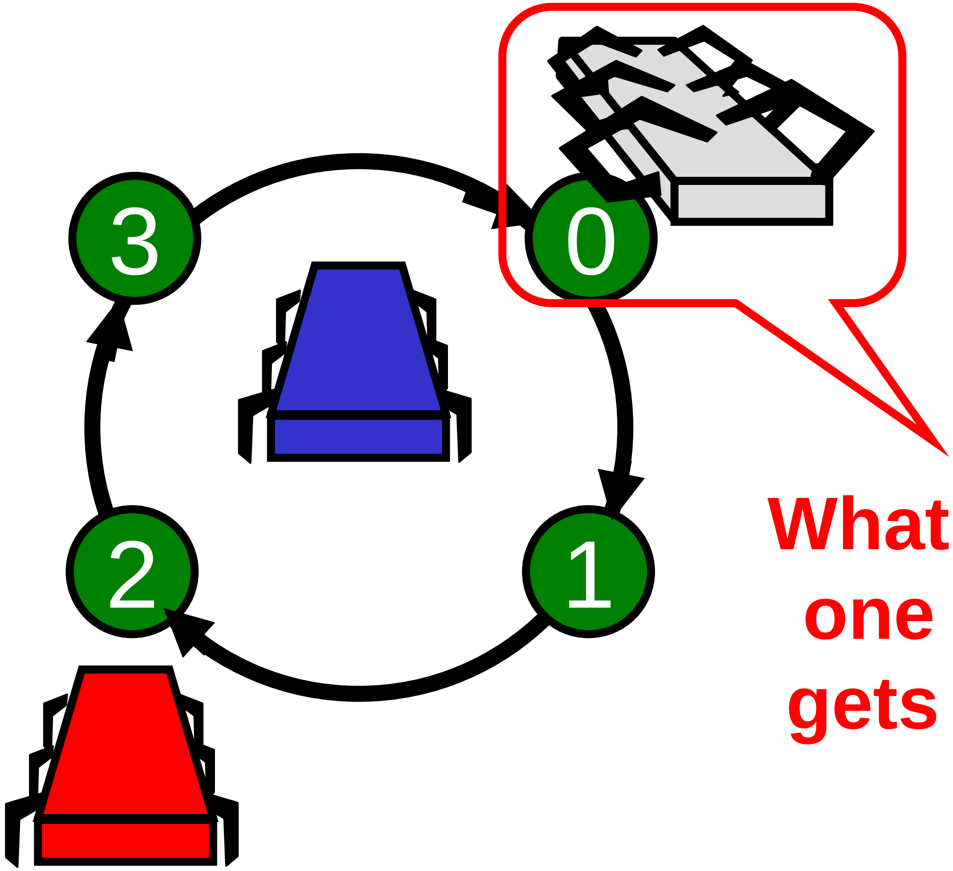


and so on …

Three-Thread Bounded Precedence Graph?

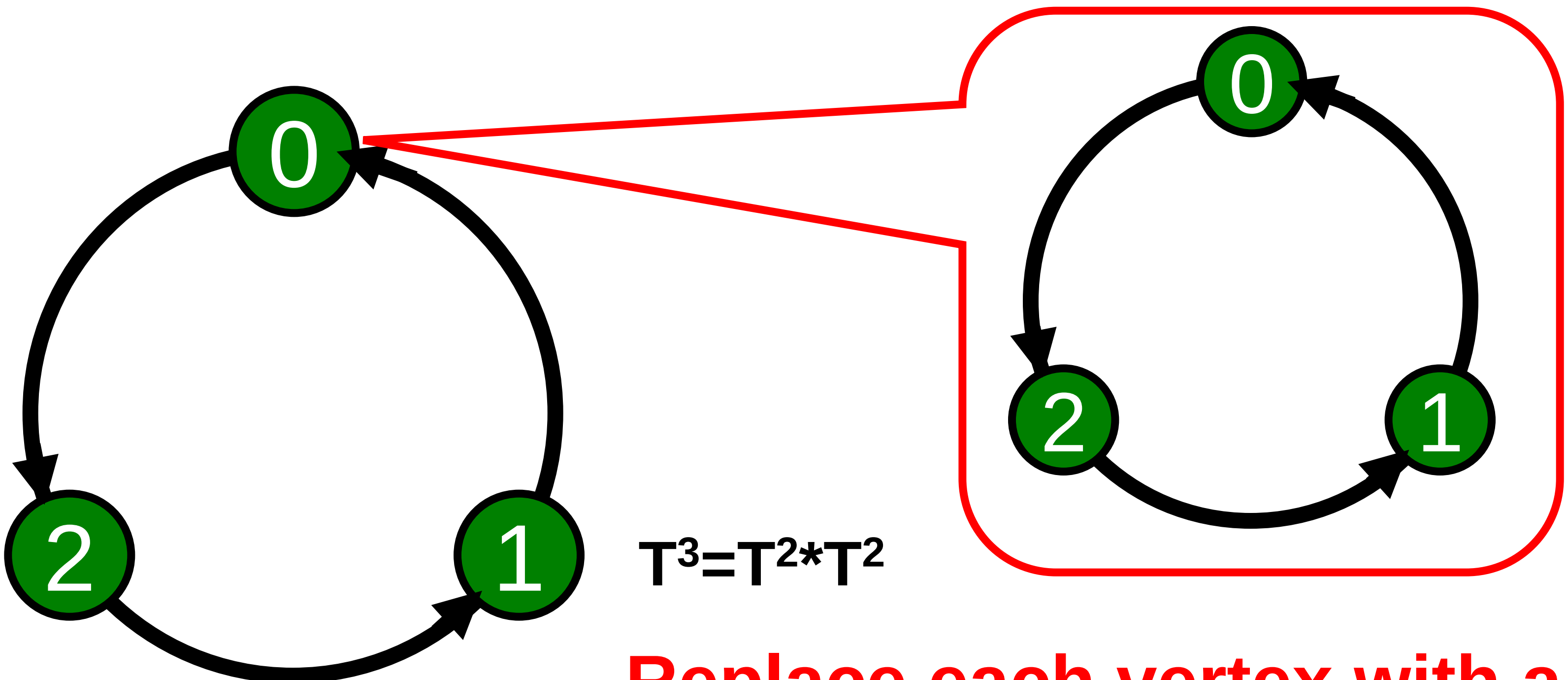


Three-Thread Bounded Precedence Graph?



What to do if one thread gets stuck?

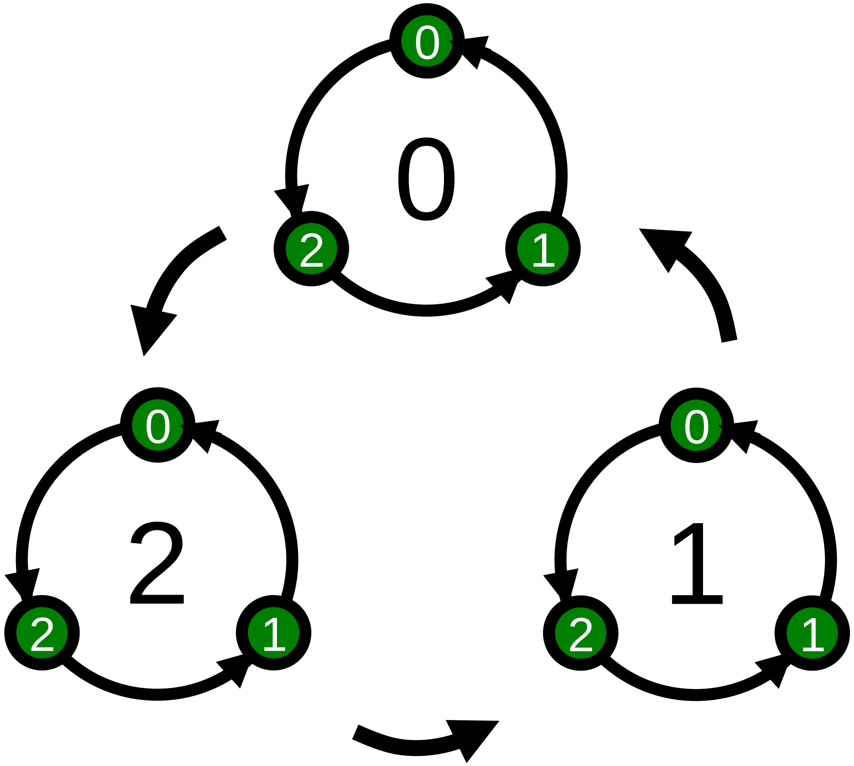
Graph Composition



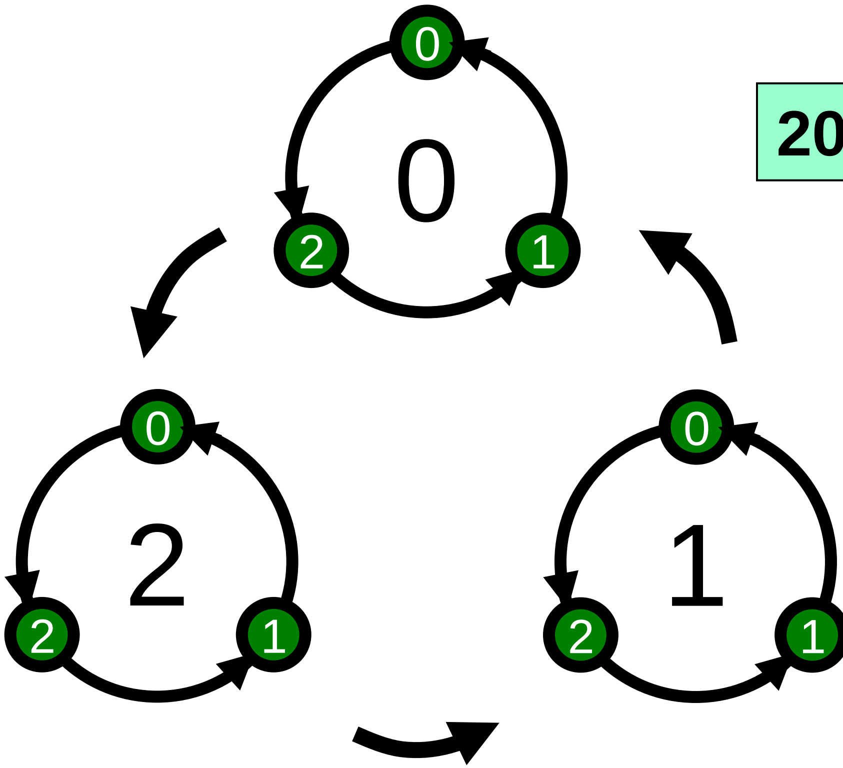
$T^3 = T^2 * T^2$

Replace each vertex with a copy of the graph

Three-Thread Bounded Precedence Graph T^3

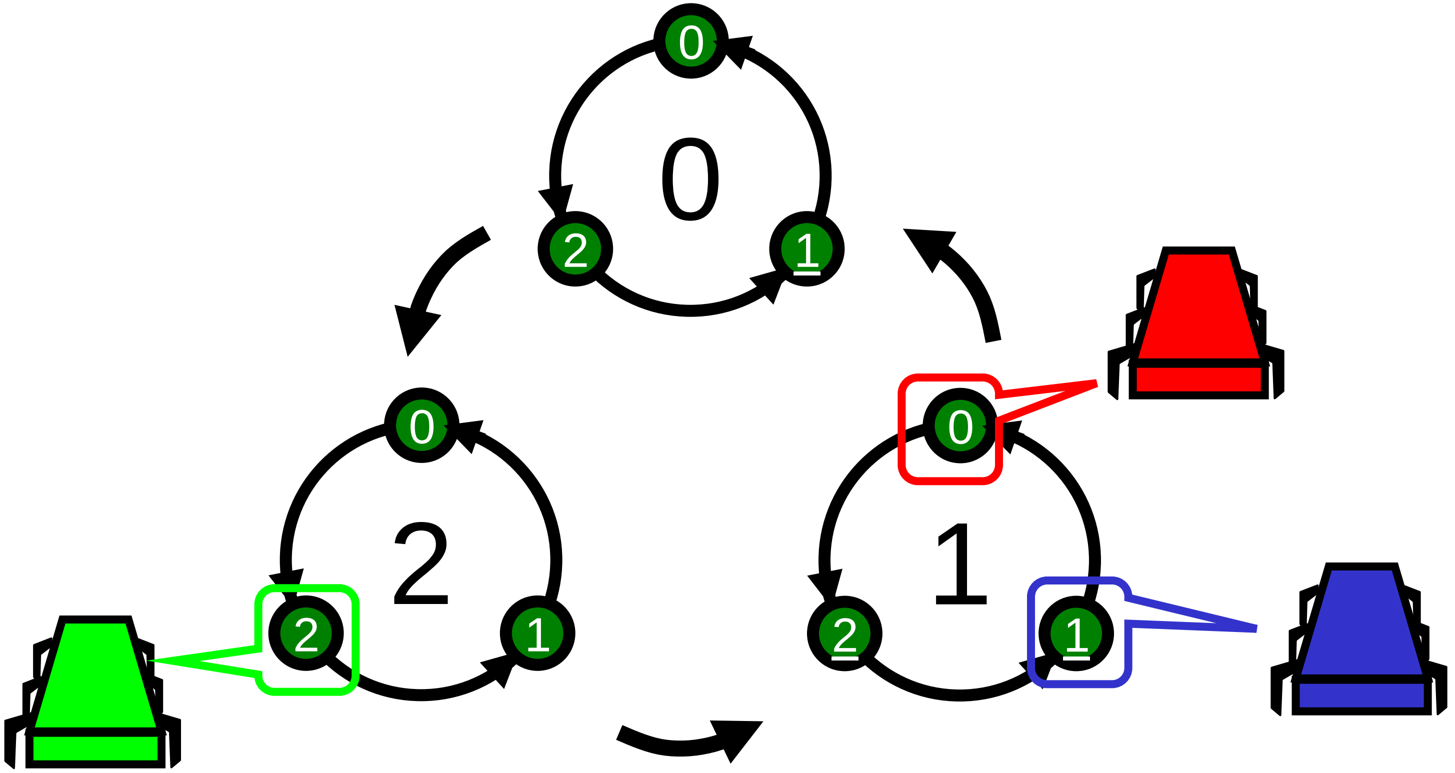


Three-Thread Bounded Precedence Graph T^3



$\boxed{20} < \boxed{02} < \boxed{12}$

Three-Thread Bounded Precedence Graph T^3



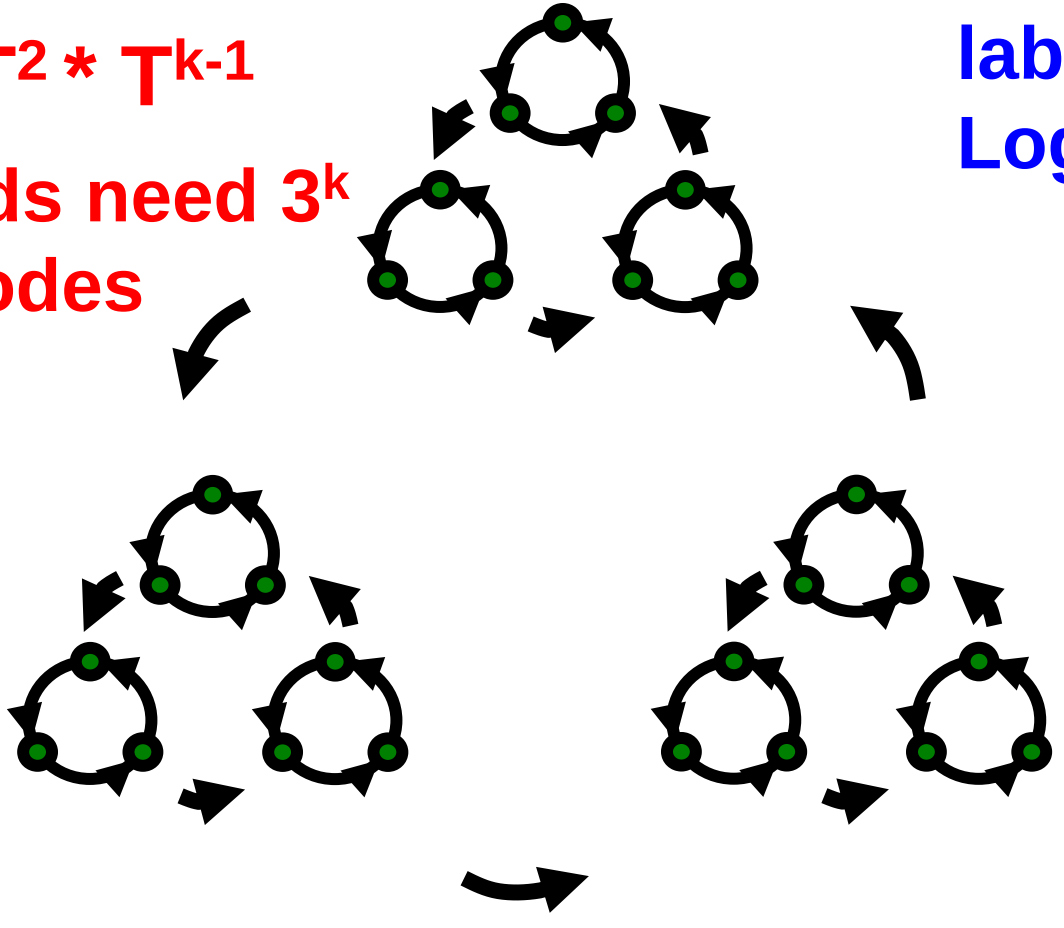
and so on…

In General

$T^k = T^2 * T^{k-1}$

K threads need 3^k nodes

label size = $Log_2(3^k) = 2k$



Deep Philosophical Question

- The Bakery Algorithm is
  - Succinct,
  - Elegant, and
  - Fair.
- Q: So why isn't it practical?
- A: Well, you have to read N distinct variables

Shared Memory

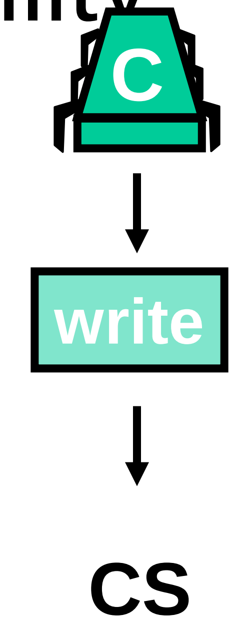
- Shared read/write memory locations called Registers (historical reasons)
- Come in different flavors
  - Multi-Reader-Single-Writer (**`Flag[]`**)
  - Multi-Reader-Multi-Writer (**Victim`[]`**)
  - Not that interesting: SRMW and SRSW

Theorem

At least N MRSW (multi-reader/single-writer) registers are needed to solve deadlock-free mutual exclusion.

N registers like Flag[]…

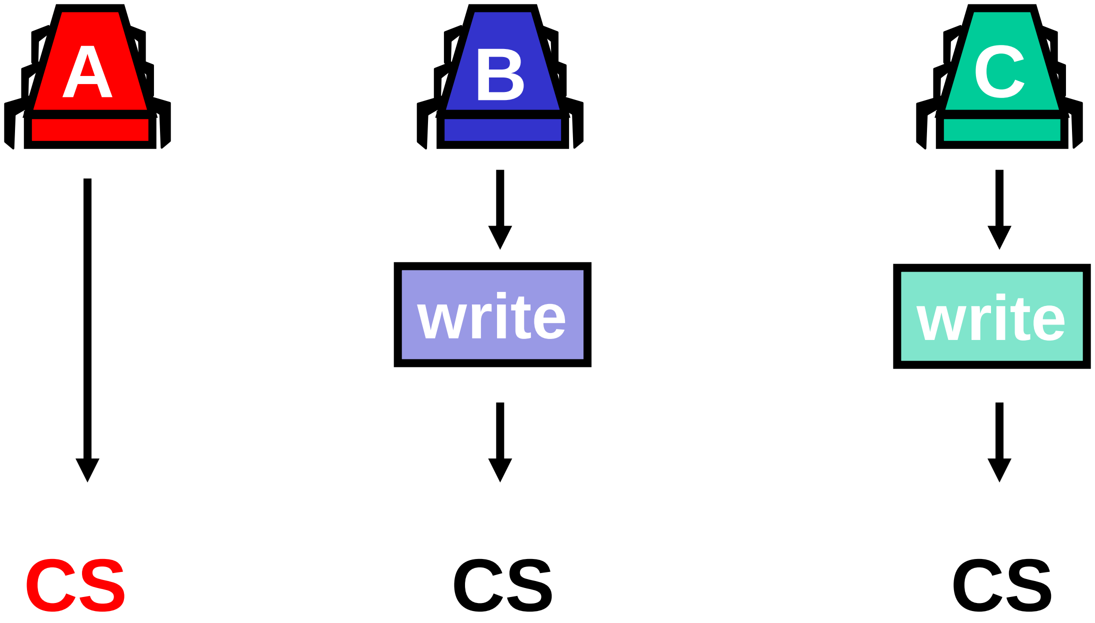
Proving Algorithmic Impossibility



- To show no algorithm exists:
  - assume by way of contradiction one does,
  - show a **bad execution** that violates properties:
  - in our case assume an alg for deadlock free mutual exclusion using < N registers

Proof: Need N-MRSW Registers

Each thread must write to some register



…can't tell whether A is in critical section

Upper Bound

- Bakery algorithm
  - Uses **2N** MRSW registers
- So the bound is (pretty) tight
- But what if we use MRMW registers?
  - Like **victim[]** ?

Bad News Theorem

At least N MRMW multi-reader/multi-writer registers are needed to solve deadlock-free mutual exclusion.

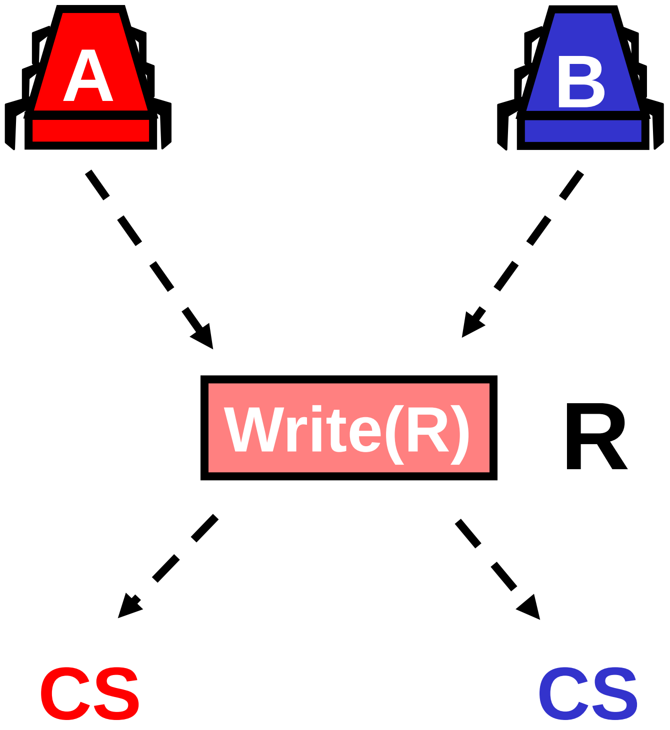
(So multiple writers don't help)

Theorem (For 2 Threads)

Theorem: Deadlock-free mutual exclusion for 2 threads requires at least 2 multi-reader multi-writer registers

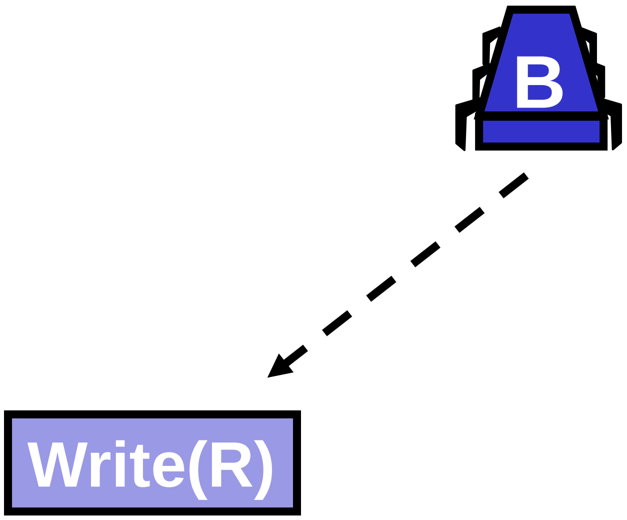
Proof: assume one register suffices and derive a contradiction

Two Thread Execution



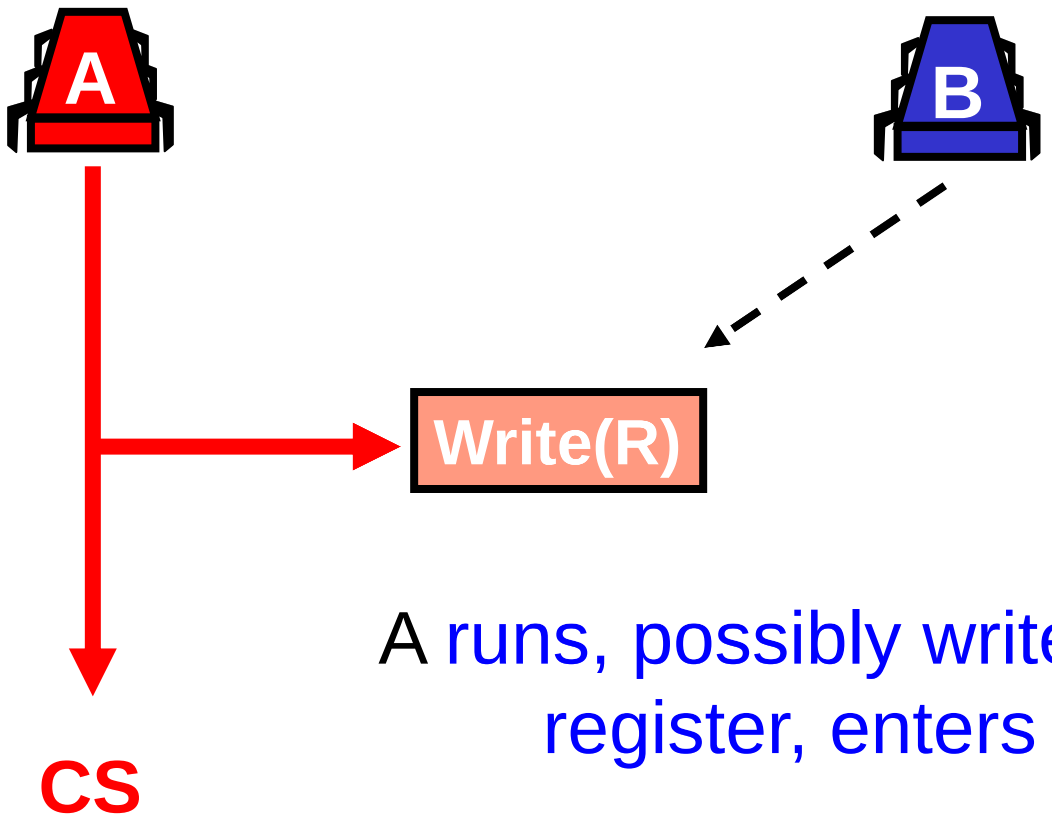
- Threads run, reading and writing R
- Deadlock free so at least one gets in

Covering State for One Register Always Exists



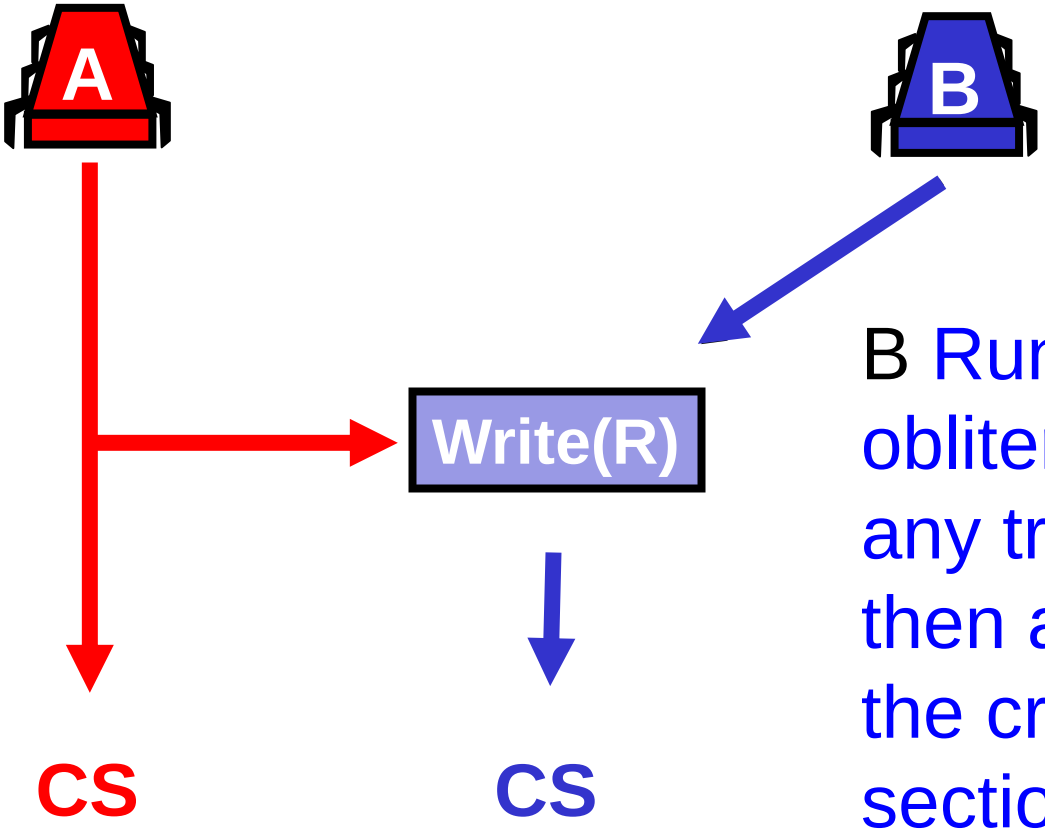
In any protocol B has to write to the register before entering CS, so stop it just before

Proof: Assume Cover of 1



A runs, possibly writes to the register, enters CS

Proof: Assume Cover of 1

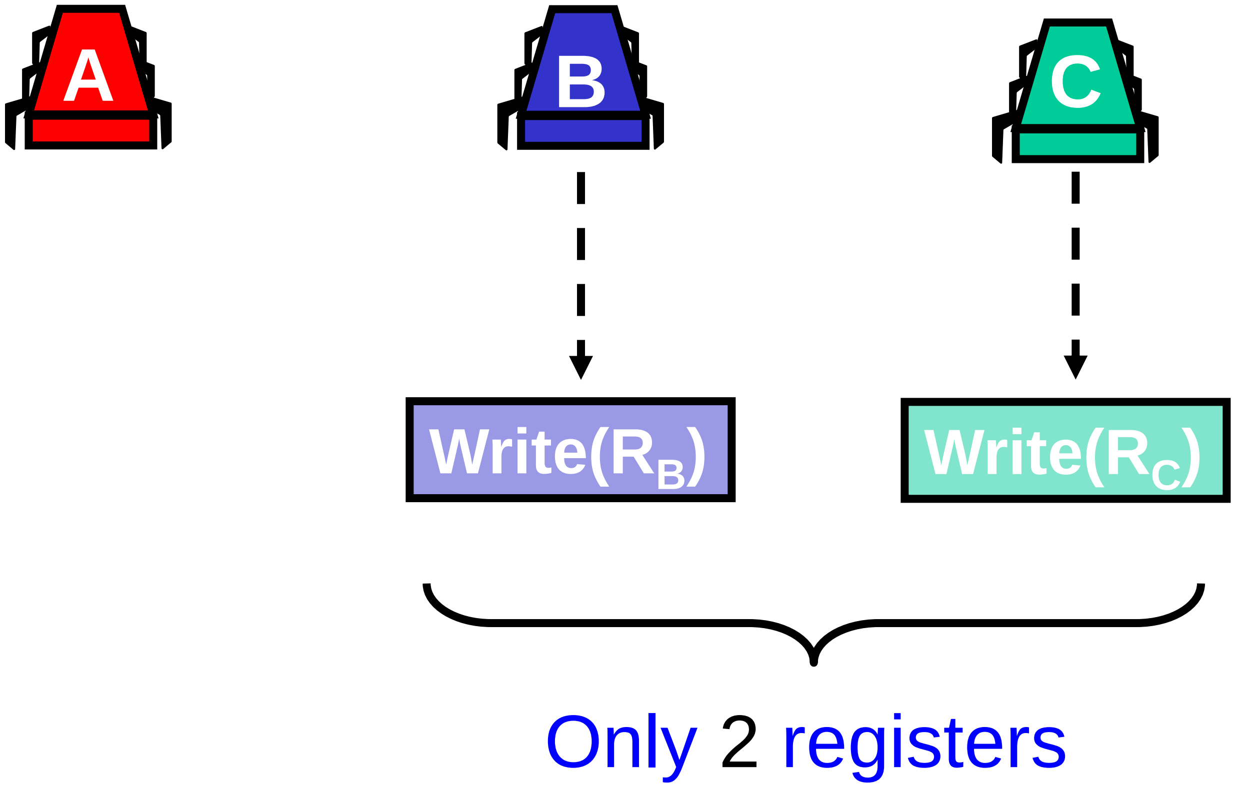


B Runs, first obliterating any trace of A, then also enters the critical section

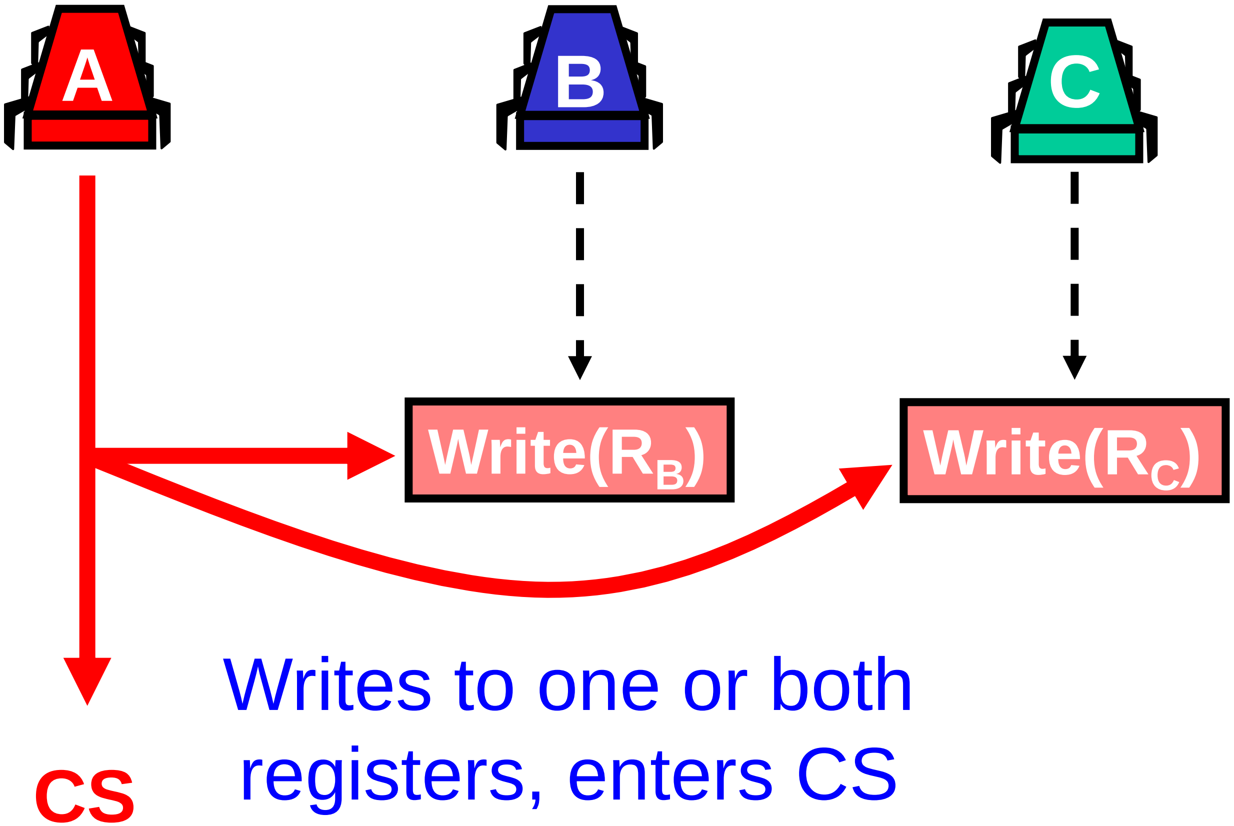
Theorem

Deadlock-free mutual exclusion for 3 threads requires at least 3 multi-reader multi-writer registers

Proof: Assume Cover of 2

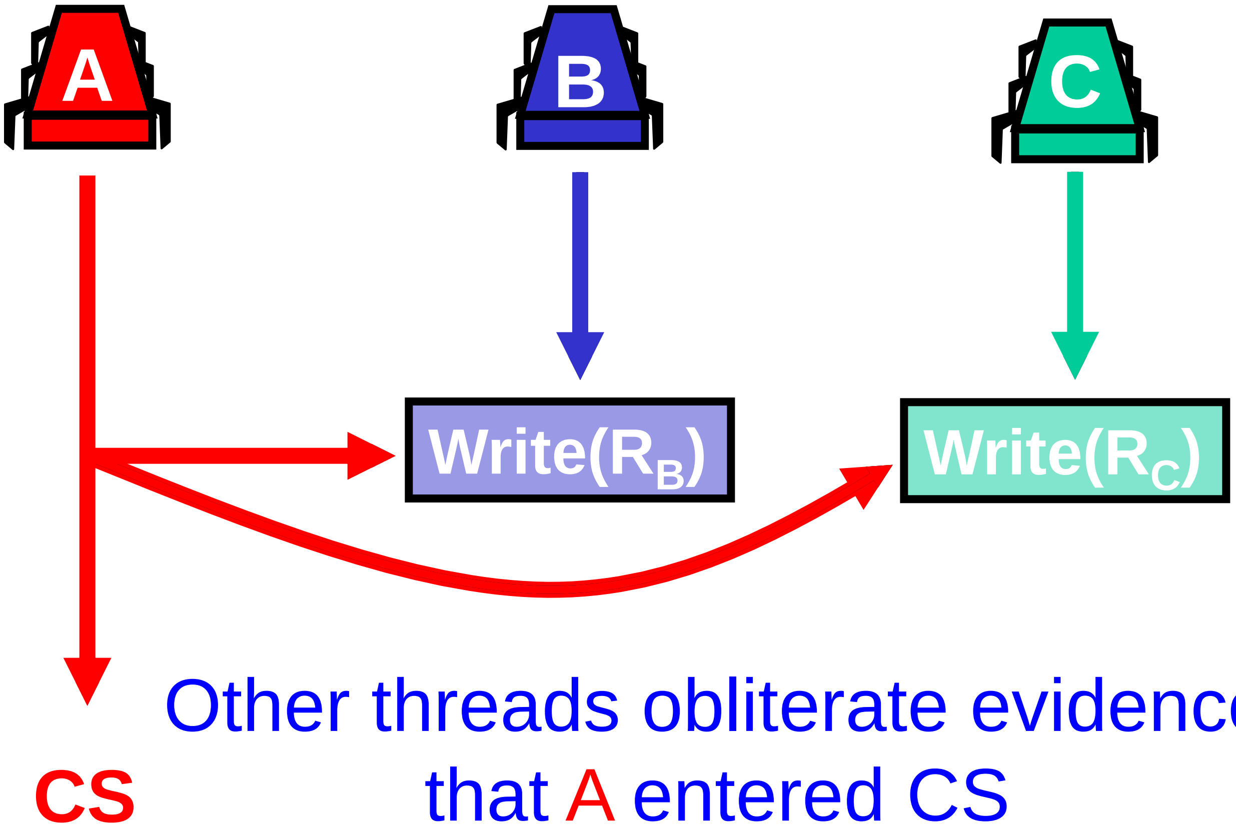


Run A Solo



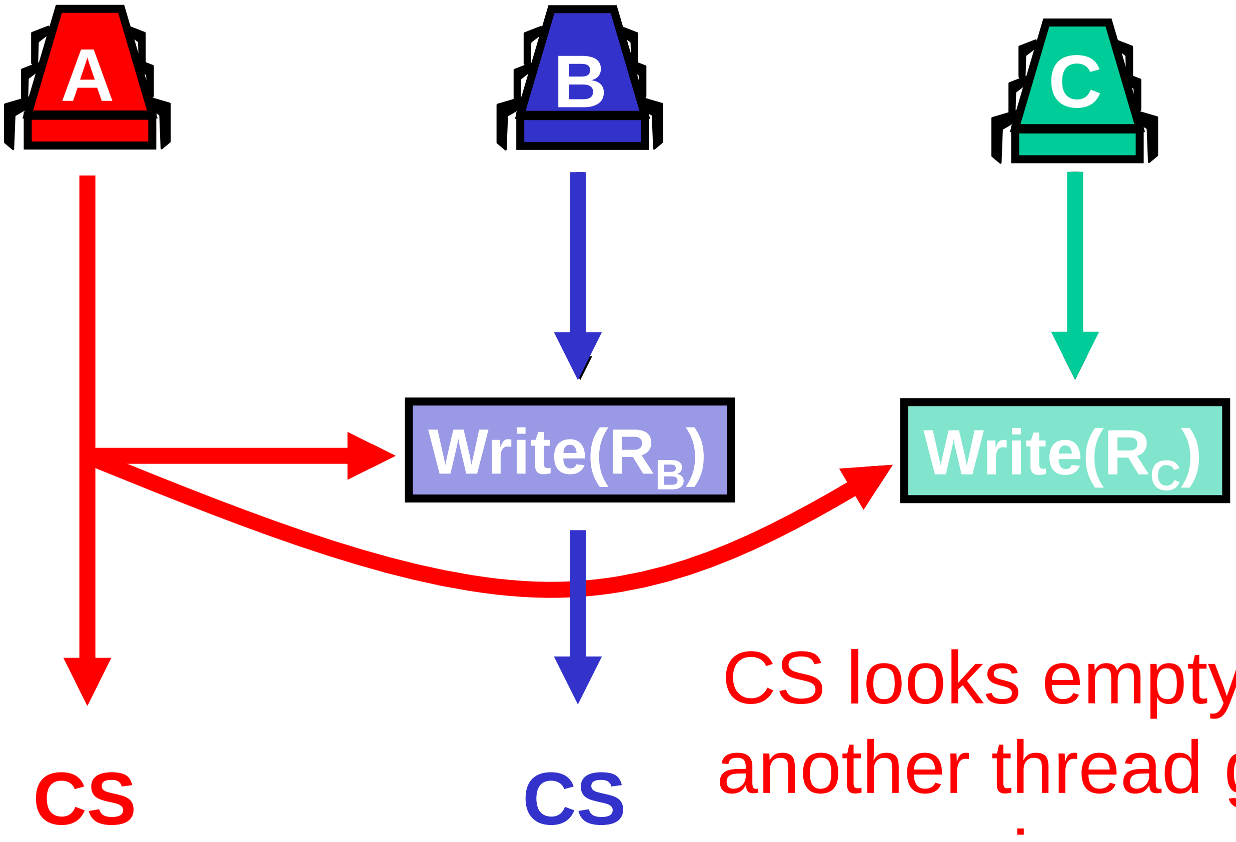
Writes to one or both registers, enters CS

Obliterate Traces of A



Other threads obliterate evidence that A entered CS

Mutual Exclusion Fails

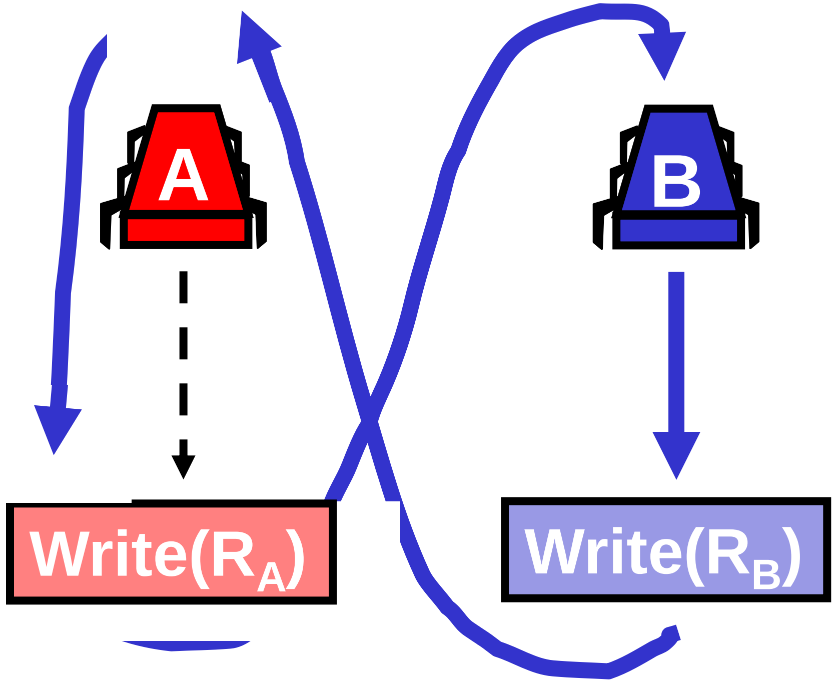


CS looks empty, so another thread gets in

Proof Strategy

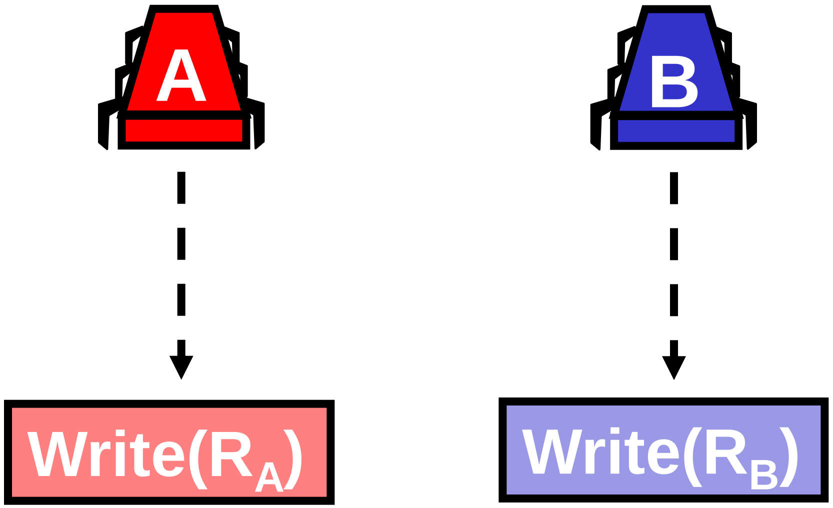
- Proved: a contradiction starting from a covering state for 2 registers
- Claim: a covering state for 2 registers is reachable from any state where CS is empty

Covering State for Two



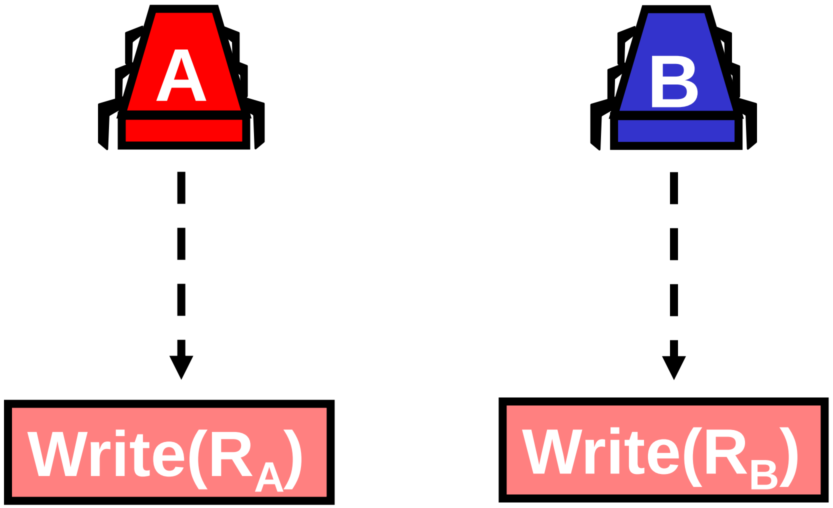
- If we run B through CS 3 times, B must return twice to cover some register, say R_B

Covering State for Two



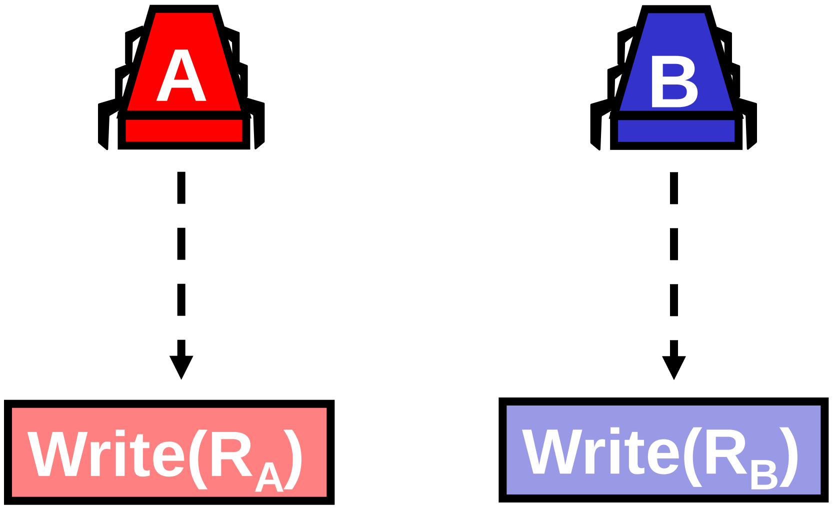
- Start with B covering register R_B for the 1st time
- Run A until it is about to write to uncovered R_A
- Are we done?

Covering State for Two



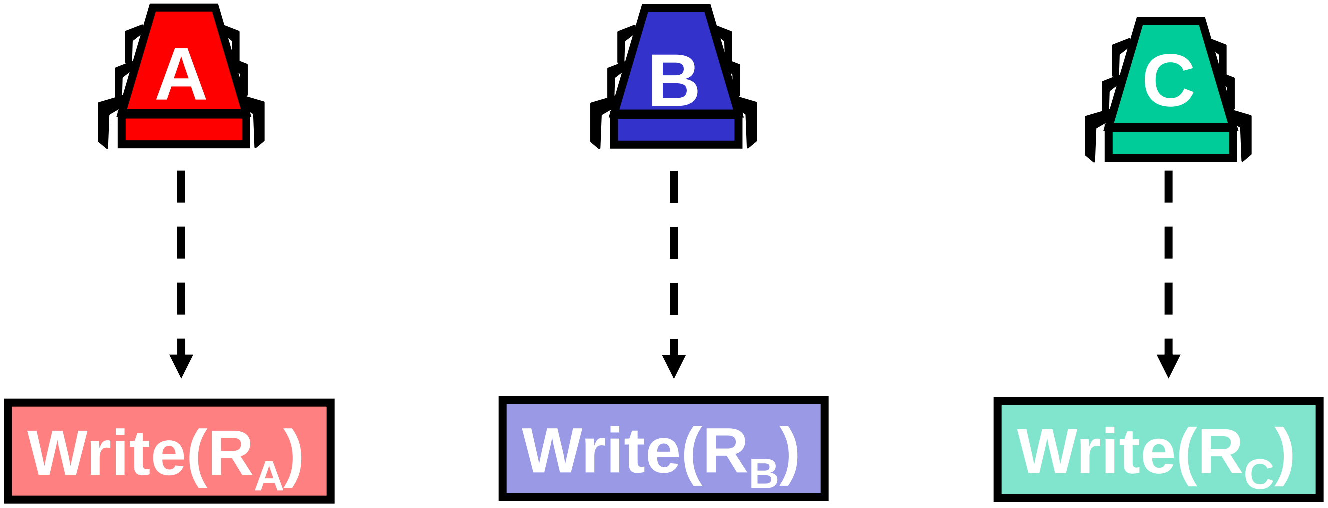
- NO! A could have written to R_B
- So CS no longer looks empty to thread C

Covering State for Two



- Run **B** obliterating traces of **A** in $\mathbf{R_B}$
- Run **B** again until it is about to write to $\mathbf{R_B}$
- Now we are done

Inductively We Can Show



- There is a covering state
  - Where **k** threads not in CS cover **k** distinct registers
  - Proof follows when **k = N-1**

Summary of Lecture

- In the 1960's several incorrect solutions to starvation-free mutual exclusion using RW-registers were published…
- Today we know how to solve FIFO N thread mutual exclusion using 2N RW-Registers

Summary of Lecture

- N RW-Registers inefficient
  - Because writes **“cover”** older writes
- Need stronger hardware operations
  - that do not have the **“covering problem”**
- In next lectures - understand what these operations are…