Some Recent Tools and a BDDC Algorithm for 3D Problems in H(curl)

Clark R. Dohrmann[1] and Olof B. Widlund[2]

Abstract We present some recent domain decomposition tools and a BDDC algorithm for 3D problems in the space $H(\text{curl};\Omega)$. Of primary interest is a face decomposition lemma which allows us to obtain improved estimates for a BDDC algorithm under less restrictive assumptions than have appeared previously in the literature. Numerical results are also presented to confirm the theory and to provide additional insights.

1 Introduction

We investigate a BDDC algorithm for three-dimensional (3D) problems in the space $H_0(\text{curl};\Omega)$. The subject problem is to obtain edge finite element approximations of the variational problem: Find $\boldsymbol{u} \in H_0(\text{curl};\Omega)$ such that

$$a_\Omega(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_\Omega \quad \forall \boldsymbol{v} \in H_0(\text{curl};\Omega),$$

where

$$a_\Omega(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega [(\alpha \nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}) + (\beta \boldsymbol{u} \cdot \boldsymbol{v})]\, dx, \quad (\boldsymbol{f},\boldsymbol{v})_\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

The norm of $\boldsymbol{u} \in H(\text{curl};\Omega)$, for a domain with diameter 1, is given by $a_\Omega(\boldsymbol{u},\boldsymbol{u})^{1/2}$ with $\alpha = 1$ and $\beta = 1$; the elements of $H_0(\text{curl})$ have vanishing tangential compo-

Sandia National Laboratories, Albuquerque, New Mexico, 87185-0346, USA. Sandia is a multi-program laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000, crdohrm@sandia.gov · Courant Institute, 251 Mercer Street, New York, NY 10012, USA. This work supported in part by the U.S. Department of Energy under contracts DE-FG02-06ER25718 and in part by National Science Foundation Grant DMS-0914954, widlund@cims.nyu.edu, http://www.cs.nyu.edu/cs/faculty/widlund

nents on $\partial\Omega$. We could equally well consider cases where this boundary condition is imposed only on one or several subdomain faces which form part of $\partial\Omega$. We will assume that $\alpha \geq 0$ and $\beta > 0$ are constant in each of the subdomains $\Omega_1, \ldots, \Omega_N$. Our results could be presented in a form which accommodates properties which are not constant or isotropic in each subdomain, but we avoid this generalization for purposes of clarity.

In the pioneering work of [11], two different cases were analyzed for FETI-DP algorithms:
Case 1:

$$\alpha_i = \alpha \quad \text{for} \quad i = 1, \ldots, N$$

The condition number bound reported for the preconditioned operator is

$$\kappa \leq C \max_i (1 + H_i^2 \beta_i / \alpha)(1 + \log(H/h))^4, \tag{1}$$

where $H/h := \max_i H_i/h_i$.
Case 2:

$$\beta_i = \beta \quad \text{for} \quad i = 1, \ldots, N$$

for which the reported condition number bound is

$$\kappa \leq C \max_i (1 + H_i^2 \beta / \alpha_i)(1 + \log(H/h))^4. \tag{2}$$

We address the following basic questions regarding [11] in this study.

1. Is is possible to remove the assumption of $\alpha_i = \alpha$ or $\beta_i = \beta$ for all i?
2. Is it possible to remove the factor of $H_i^2 \beta_i / \alpha_i$ from the estimates?
3. Is is possible to reduce the logarithmic factor from four powers to two powers as is typical of other iterative substructuring algorithms?
4. Do FETI-DP or BDDC algorithms for 3D H(curl) problems have certain complications not present for problems with just a single parameter?

We find in the following sections that the answers are yes to all four questions. However, due to page limitations, we only consider here the relatively rich coarse space of Algorithm C of [11]. We remark that the analysis of 3D H(curl) problems with material property jumps between subdomains is quite limited in the literature. A comprehensive treatment of problems in 2D can be found in [3]. A different iterative substructuring algorithm for 3D problems is given in [6], but the authors were unable to conclude whether their condition number bound was independent of material property jumps.

2 Tools

We assume that Ω is decomposed into N non-overlapping subdomains, $\Omega_1, \ldots, \Omega_N$, each the union of elements of the triangulation of Ω. We denote by H_i the diameter of Ω_i. The interface of the domain decomposition is given by

$$\Gamma := \left(\bigcup_{i=1}^{N} \partial\Omega_i\right) \backslash \partial\Omega,$$

and the contribution to Γ from $\partial\Omega_i$ by $\Gamma_i := \partial\Omega_i \backslash \partial\Omega$. These sets are unions of subdomain faces, edges, and vertices. For simplicity, we assume that each subdomain is a shape-regular and convex tetrahedron or hexahedron with planar faces.

We assume a shape-regular triangulation $\mathscr{T}_{h_i}$ of each Ω_i with nodes matching across the interfaces. The smallest element diameter of $\mathscr{T}_{h_i}$ is denoted by h_i. Associated with the triangulation $\mathscr{T}_{h_i}$ are the two finite element spaces $W_{\text{grad}}^{h_i} \subset H(\text{grad}, \Omega_i)$ and $W_{\text{curl}}^{h_i} \subset H(\text{curl}, \Omega_i)$ based on continuous, piecewise linear, tetrahedral nodal elements and linear, tetrahedral edge (Nédeléc) elements, respectively. We could equally well develop our algorithms and theory for low order hexahedral elements.

The energy of a vector function $\boldsymbol{u} \in W_{\text{curl}}^{h_i}$ for subdomain Ω_i is defined as

$$E_i(\boldsymbol{u}) := \alpha_i(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{u})_{\Omega_i} + \beta_i(\boldsymbol{u}, \boldsymbol{u})_{\Omega_i}, \tag{3}$$

where α_i and β_i are assumed constant in Ω_i.

Let $\boldsymbol{N}_e \in W_{\text{curl}}^{h_i}$ and $\boldsymbol{t}_e$ denote the finite element shape function and unit tangent vector, respectively, for an edge e of $\mathscr{T}_{h_i}$. We assume that $\boldsymbol{N}_e$ is scaled such that $\boldsymbol{N}_e \cdot \boldsymbol{t}_e = 1$ along e. The *edge* finite element interpolant of a sufficiently smooth vector function $\boldsymbol{u} \in H(\text{curl}, \Omega_i)$ is then defined as

$$\Pi^{h_i}(\boldsymbol{u}) := \sum_{e \in \mathscr{M}_{\bar{\Omega}_i}} u_e \boldsymbol{N}_e, \quad u_e := (1/|e|) \int_e \boldsymbol{u} \cdot \boldsymbol{t}_e \, ds, \tag{4}$$

where $\mathscr{M}_{\bar{\Omega}_i}$ is the set of edges of $\mathscr{T}_{h_i}$, and $|e|$ is the length of e. We will also make use of other sets of edges of $\mathscr{T}_{h_i}$. Namely, $\mathscr{M}_{\partial\Omega_i}$, $\mathscr{M}_{\mathscr{E}}$, $\mathscr{M}_{\mathscr{F}}$, and $\mathscr{M}_{\partial\mathscr{F}}$ contain the edges of $\partial\Omega_i$, subdomain edge $\mathscr{E}$, subdomain face $\mathscr{F}$, and $\partial\mathscr{F}$, respectively. We denote by $\mathscr{G}_{i\mathscr{F}}$, $\mathscr{G}_{i\mathscr{E}}$, and $\mathscr{G}_{i\mathscr{V}}$ sets of subdomain faces, subdomain edges, and subdomain vertices for Ω_i. The wire basket $\mathscr{W}_i$ is the union of all subdomain edges and vertices for Ω_i. We will also make use of the symbol $\omega_i := 1 + \log(H_i/h_i)$, and bold faced symbols refer to vector functions. We denote by $\bar{p}_i$ the mean of p_i over Ω_i.

The estimate in the next lemma can be found in several references, see e.g., Lemma 4.16 of [12].

Lemma 1. *For any $p_i \in W_{grad}^{h_i}$ and subdomain edge $\mathscr{E}$ of Ω_i,*

$$\|p_i\|_{L^2(\mathscr{E})}^2 \leq C\omega_i \|p_i\|_{H^1(\Omega_i)}^2. \tag{5}$$

Lemma 2. *For any $p_i \in W_{grad}^{h_i}$, there exist $p_{i\mathscr{V}}, p_{i\mathscr{E}}, p_{i\mathscr{F}} \in W_{grad}^{h_i}$ such that*

$$p_i|_{\partial\Omega_i} = \sum_{\mathscr{V}\in\mathscr{G}_{i\mathscr{V}}} p_{i\mathscr{V}}|_{\partial\Omega_i} + \sum_{\mathscr{E}\in\mathscr{G}_{i\mathscr{E}}} p_{i\mathscr{E}}|_{\partial\Omega_i} + \sum_{\mathscr{F}\in\mathscr{G}_{i\mathscr{F}}} p_{i\mathscr{F}}|_{\partial\Omega_i}, \tag{6}$$

where the nodal values of $p_{i\mathscr{V}}$, $p_{i\mathscr{E}}$, and $p_{i\mathscr{F}}$ on $\partial\Omega_i$ may be nonzero only at the nodes of $\mathscr{V}$, $\mathscr{E}$, and $\mathscr{F}$, respectively. Further,

$$|p_{i\mathscr{V}}|^2_{H^1(\Omega_i)} \le C\|p_i\|^2_{H^1(\Omega_i)}, \tag{7}$$

$$|p_{i\mathscr{E}}|^2_{H^1(\Omega_i)} \le C\omega_i\|p_i\|^2_{H^1(\Omega_i)}, \tag{8}$$

$$|p_{i\mathscr{F}}|^2_{H^1(\Omega_i)} \le C\omega_i^2\|p_i\|^2_{H^1(\Omega_i)}. \tag{9}$$

Proof. The estimates in (7-9) are standard, and follow from Corollary 4.20 and Lemma 4.24 of [12] and elementary estimates.

We note that a Poincaré inequality allows us to replace the H^1-norm of p_i by its H^1-seminorm in Lemmas 1 and 2 if $\bar{p}_i = 0$.

The next lemma is stated without proof due to page restrictions.

Lemma 3. *Let $f_i \in W^{h_i}_{grad}$ have vanishing nodal values everywhere on $\partial\Omega_i$ except on the wire basket $\mathscr{W}_i$ of Ω_i. For each subdomain face $\mathscr{F}$ of Ω_i and $Ch_i \le d \le H_i/C$, $C > 1$, there exists a $\boldsymbol{v}_i \in W^{h_i}_{curl}$ such that $v_{ie} = \nabla f_{ie}$ for all $e \in \mathscr{M}_{\mathscr{F}}$, $v_{ie} = 0$ for all other edges of $\partial\Omega_i$, and*

$$\|\boldsymbol{v}_i\|^2_{L^2(\Omega_i)} \le C(\omega_i\|f_i\|^2_{L^2(\partial\mathscr{F})} + d^2\|\nabla f_i\cdot\boldsymbol{t}_{\partial\mathscr{F}}\|^2_{L^2(\partial\mathscr{F})}), \tag{10}$$

$$\|\nabla\times\boldsymbol{v}_i\|^2_{L^2(\Omega_i)} \le C(\tau(d)\|f_i\|^2_{L^2(\partial\mathscr{F})} + \|\nabla f_i\cdot\boldsymbol{t}_{\partial\mathscr{F}}\|^2_{L^2(\partial\mathscr{F})}), \tag{11}$$

where $\boldsymbol{t}_{\partial\mathscr{F}}$ is a unit tangent along $\partial\mathscr{F}$, and

$$\tau(d) = \begin{cases} 0 & \text{if } d > H_i/C \\ d^{-2} & \text{otherwise.} \end{cases}$$

The Helmholtz-type decomposition and estimates in the next lemma will allow us to make use of and build on existing tools for scalar functions in $H^1(\Omega_i)$. We refer the reader to Lemma 5.2 of [4] for the case of convex polyhedral subdomains; this important paper was preceded by [5], which concerns other applications of the same decomposition.

Lemma 4. *For a convex and polyhedral subdomain Ω_i and any $\boldsymbol{u}_i \in W^{h_i}_{curl}$, there is a $\boldsymbol{q}_i \in W^{h_i}_{curl}$, $\boldsymbol{\Psi}_i \in (W^{h_i}_{grad})^3$, and $p_i \in W^{h_i}_{grad}$ such that*

$$\boldsymbol{u}_i = \boldsymbol{q}_i + \Pi^{h_i}(\boldsymbol{\Psi}_i) + \nabla p_i, \tag{12}$$

$$\|\nabla p_i\|_{L^2(\Omega_i)} \le C\|\boldsymbol{u}_i\|_{L^2(\Omega_i)}, \tag{13}$$

$$\|\boldsymbol{\Psi}_i\|_{L^2(\Omega_i)} \le C\|\boldsymbol{u}_i\|_{L^2(\Omega_i)}, \tag{14}$$

$$\|h_i^{-1}\boldsymbol{q}_i\|^2_{L^2(\Omega_i)} + \|\boldsymbol{\Psi}_i\|^2_{H^1(\Omega_i)} \le C\|\nabla\times\boldsymbol{u}_i\|^2_{L^2(\Omega_i)}. \tag{15}$$

Lemma 5. *For any $\boldsymbol{u}_i \in W_{curl}^{h_i}$ with $u_{ie} = 0$ for all $e \in \mathscr{M}_{\partial\mathscr{F}}$, there exists a $\boldsymbol{v}_{i\mathscr{F}} \in W_{curl}^{h_i}$ such that $v_{i\mathscr{F}e} = u_{ie}$ for all $e \in \mathscr{M}_{\mathscr{F}}$, $v_{i\mathscr{F}e} = 0$ for all $e \in \mathscr{M}_{\partial\Omega_i} \setminus \mathscr{M}_{\mathscr{F}}$, and*

$$E_i(\boldsymbol{v}_{i\mathscr{F}}) \leq C\omega_i^2 E_i(\boldsymbol{u}_i), \tag{16}$$

where the energy E_i is defined in (3).

Proof. Let p_i in (12) be chosen so $\bar{p}_i = 0$. This is possible since a constant can be added to p_i without changing its gradient. Because $u_{ie} = 0$ for all $e \in \mathscr{M}_{\partial\mathscr{F}}$, it follows from Lemmas 1 and 4 and elementary estimates that

$$\begin{aligned} \|\nabla p_i \cdot \boldsymbol{t}_{\mathscr{E}}\|_{L^2(\partial\mathscr{F})}^2 &= \|(\Pi^{h_i}(\boldsymbol{\Psi}_{\boldsymbol{i}}) + \boldsymbol{q}_i) \cdot \boldsymbol{t}_{\mathscr{E}}\|_{L^2(\partial\mathscr{F})}^2 \\ &\leq C\omega_i \|\nabla \times \boldsymbol{u}_i\|_{L^2(\Omega_i)}^2. \end{aligned} \tag{17}$$

We then find from Lemmas 2 and 4 that

$$\|\nabla p_{i\mathscr{F}}\|_{L^2(\Omega_i)}^2 \leq C\omega_i^2 \|\boldsymbol{u}_i\|_{L^2(\Omega_i)}^2. \tag{18}$$

Define

$$p_{i\mathscr{W}} := \sum_{\mathscr{V} \in \mathscr{G}_{i\mathscr{V}}} p_{i\mathscr{V}} + \sum_{\mathscr{E} \in \mathscr{G}_{i\mathscr{E}}} p_{i\mathscr{E}}, \quad d := \begin{cases} H_i & \text{if } d_i \geq H_i \\ \max(d_i, Ch_i) & \text{otherwise,} \end{cases}$$

where $d_i := \sqrt{\alpha_i/\beta_i}$. Further, let $p_{i\mathscr{W}}$ and $\boldsymbol{p}_{i\mathscr{F}}$ denote the functions f_i and $\boldsymbol{v}_i$, respectively, of Lemma 3. We then find from Lemmas 1 and 3 and (17) that

$$E_i(\boldsymbol{p}_{i\mathscr{F}}) \leq C\omega_i^2 E_i(\boldsymbol{u}_i), \tag{19}$$

where $p_{i\mathscr{F}e} = \nabla p_{i\mathscr{W}e}\ \forall e \in \mathscr{M}_{\mathscr{F}}$ and $p_{i\mathscr{F}e} = 0\ \forall e \in \mathscr{M}_{\partial\Omega_i} \setminus \mathscr{M}_{\mathscr{F}}$. With reference to (12) and (4), we define

$$\boldsymbol{q}_{i\mathscr{F}} := \sum_{e \in \mathscr{M}_{\mathscr{F}}} q_{ie} \boldsymbol{N}_e, \tag{20}$$

and from elementary finite element estimates and Lemma 4 find

$$\|\boldsymbol{q}_{i\mathscr{F}}\|_{L^2(\Omega_i)}^2 \leq Ch_i^3 \sum_{e \in \mathscr{M}_{\mathscr{F}}} q_{ie}^2 \leq C\|\boldsymbol{q}_i\|_{L^2(\Omega_i)}^2 \leq C\|\boldsymbol{u}_i\|_{L^2(\Omega_i)}^2, \tag{21}$$

$$\|\nabla \times \boldsymbol{q}_{i\mathscr{F}}\|_{L^2(\Omega_i)}^2 \leq Ch_i \sum_{e \in \mathscr{M}_{\mathscr{F}}} q_{ie}^2 \leq C\|\nabla \times \boldsymbol{u}_i\|_{L^2(\Omega_i)}^2. \tag{22}$$

It follows from Lemmas 2 and 4 that there exists a $\boldsymbol{\Psi}_{i\mathscr{F}} \in (W_{\text{grad}}^{h_i})^3$ such that $\boldsymbol{\Psi}_{i\mathscr{F}} = \boldsymbol{\Psi}_i$ at all nodes of $\mathscr{F}$, that vanishes at all other nodes of $\partial\Omega_i$, and

$$\|\boldsymbol{\Psi}_{i\mathscr{F}}\|_{L^2(\Omega_i)}^2 \leq C\|\boldsymbol{\Psi}_i\|_{L^2(\Omega_i)}^2 \leq C\|\boldsymbol{u}_i\|_{L^2(\Omega_i)}^2, \tag{23}$$

$$\|\nabla \times \boldsymbol{\Psi}_{i\mathscr{F}}\|_{L^2(\Omega_i)}^2 \leq C\omega_i^2 \|\boldsymbol{\Psi}_i\|_{H^1(\Omega_i)}^2 \leq C\omega_i^2 \|\nabla \times \boldsymbol{u}_i\|_{L^2(\Omega_i)}^2. \tag{24}$$

From Lemmas 1 and 4, we obtain

$$\|\boldsymbol{\Psi}_i\|^2_{L^2(\partial\mathscr{F})} \leq C\omega_i \|\boldsymbol{\Psi}_i\|^2_{H^1(\Omega_i)} \leq C\omega_i \|\nabla \times \boldsymbol{u}_i\|^2_{L^2(\Omega_i)}. \quad (25)$$

Let $\boldsymbol{\Psi}_{i\partial\mathscr{F}} \in (W^{h_i}_{\text{grad}})^3$ be identical to $\boldsymbol{\Psi}_i$ at all nodes of $\partial\mathscr{F}$ and vanish at all other nodes of Ω_i. For $\boldsymbol{g} := \Pi^{h_i}(\boldsymbol{\Psi}_{i\partial\mathscr{F}})$, we define

$$\boldsymbol{g}_{i\mathscr{F}} := \sum_{e \in \mathscr{M}_{\mathscr{F}}} g^{h_i}_e \boldsymbol{N}_e. \quad (26)$$

From elementary estimates and (25,) we then obtain

$$\|\boldsymbol{g}_{i\mathscr{F}}\|^2_{L^2(\Omega_i)} \leq C h_i^2 |\boldsymbol{\Psi}_i\|^2_{L^2(\partial\mathscr{F})} \leq C\omega_i h_i^2 \|\nabla \times \boldsymbol{u}_i\|^2_{L^2(\Omega_i)}, \quad (27)$$

$$\|\nabla \times \boldsymbol{g}_{i\mathscr{F}}\|^2_{L^2(\Omega_i)} \leq C\omega_i \|\nabla \times \boldsymbol{u}_i\|^2_{L^2(\Omega_i)}. \quad (28)$$

Defining

$$\boldsymbol{v}_{i\mathscr{F}} := \nabla p_{i\mathscr{F}} + \boldsymbol{p}_{i\mathscr{F}} + \boldsymbol{q}_{i\mathscr{F}} + \Pi^{h_i}(\boldsymbol{\Psi}_{i\mathscr{F}}) + \boldsymbol{g}_{i\mathscr{F}}, \quad (29)$$

we find that $v_{i\mathscr{F}e} = u_{ie}\ \forall e \in \mathscr{M}_{\mathscr{F}}$ and $v_{i\mathscr{F}e} = 0\ \forall e \in \mathscr{M}_{\partial\Omega_i} \setminus \mathscr{M}_{\mathscr{F}}$. The estimate in (16) then follows from the bounds for each of the terms on the right-hand-side of (29) along with elementary estimates for $\Pi^{h_i}(\boldsymbol{\Psi}_{i\mathscr{F}})$. □

3 BDDC

Background information and related theory for BDDC can be found in several references including [2, 9, 10, 8, 1]. Let u_i and u denote vectors of finite element coefficients associated with Γ_i and Γ. In general, entries in u_i and u_j are allowed to differ for $j \neq i$ even though they refer to the same finite element edge. Entries in the vector $\tilde{u}_i$ are partially continuous in the sense that specific edge values or edge averages over certain subsets of Γ are required to match for adjacent subdomains. In order to obtain consistent entries, we define the weighted average

$$\hat{u}_i = R_i \sum_{j=1}^{N} R_j^T D_j \tilde{u}_j, \quad (30)$$

where R_j is a 0-1 (Boolean) matrix that selects the rows of u_j from u and D_j is a weight matrix. The weight matrices form a partition of unity in the sense that

$$\sum_{i=1}^{N} R_i^T D_i R_i = I, \quad (31)$$

where I is the identity matrix. To summarize, $\hat{u}_i$ is fully continuous while $\tilde{u}_i$ is only partially continuous. The number of continuity constraints that must be satisfied by all the $\tilde{u}_i$ determines the dimension of the coarse space.

The energy of $\boldsymbol{u}$ for Ω_i can be expressed as

$$E_i(\boldsymbol{u}) = E_i(u_i) = u_i^T S_i u_i, \tag{32}$$

where S_i is the Schur complement matrix associated with Ω_i and Γ_i. The system operator for BDDC is the assembled Schur complement

$$S = \sum_{i=1}^{N} R_i^T S_i R_i. \tag{33}$$

From Theorem 25 of [10], the condition number of the BDDC preconditioned operator is bounded above by

$$\kappa(M^{-1}S) \leq \sup_{\tilde{u}_i} \frac{\sum_{i=1}^{N} \hat{u}_i^T S_i \hat{u}_i}{\sum_{i=1}^{N} \tilde{u}_i^T S_i \tilde{u}_i}. \tag{34}$$

This remarkably simple expression shows that the continuity constraints for $\tilde{u}_i$ should be chosen so that large increases in energy do not result from the averaging operation in (30).

Let $R_{i\partial\mathscr{F}_{ij}}$ select the rows of u_i corresponding to the edge coefficients on the boundary of the face $\mathscr{F}_{ij}$, the closure of which is $\partial\Omega_i \cap \partial\Omega_j$. Similarly, let $R_{i\mathscr{F}_{ij}}$ select the rows of u_i corresponding to the interior of the face $\mathscr{F}_{ij}$. We define the vector of face edge coefficients by $u_{iF} := R_{i\mathscr{F}_{ij}} u_i$ and the face Schur complement matrix by $S_{iFF} := R_{i\mathscr{F}_{ij}} S_i R_{i\mathscr{F}_{ij}}^T$.

Because of page restrictions, we only consider a very rich coarse space which includes every edge variable of each subdomain edge. This coarse space corresponds to Algorithm C of [11]. For this case, we choose the weighted average of u_{iF} and u_{jF} as

$$\hat{u}_F = (S_{iFF} + S_{jFF})^{-1}(S_{iFF} u_{iF} + S_{jFF} u_{jF}). \tag{35}$$

Thus,

$$u_{iF} - \hat{u}_F = (S_{iFF} + S_{jFF})^{-1} S_{jFF}(u_{iF} - u_{jF}). \tag{36}$$

Using the eigenvectors of the generalized eigenvalue problem $S_{iFF}x = \lambda S_{jFF}x$ as a convenient basis, we find

$$u_{kF}^T \bar{S}_{iFF} u_{kF} \leq u_{kF}^T S_{kFF} u_{kF}, \quad \forall u_{kF} \quad k \in \{i, j\}, \tag{37}$$

where

$$\bar{S}_{iFF} := S_{jFF}(S_{iFF} + S_{jFF})^{-1} S_{iFF}(S_{iFF} + S_{jFF})^{-1} S_{jFF} \tag{38}$$

Let us assume for the moment that there are vectors u_{ij}, u_{ji}, and a scalar $\hat{C} > 0$ such that

$$R_{i\partial\mathscr{F}_{ij}} u_{ij} = R_{j\partial\mathscr{F}_{ij}} u_{ji} = u_{\partial F}, \tag{39}$$

$$R_{i\mathscr{F}_{ij}} u_{ij} = R_{j\mathscr{F}_{ij}} u_{ji}, \tag{40}$$

$$u_{ij}^T S_i u_{ij} + u_{ji}^T S_j u_{ji} \leq \hat{C}(u_i^T S_i u_i + u_j^T S_j u_j). \tag{41}$$

In other words, u_{ij}, u_{ji}, u_i and u_j are all identical along the boundary of $\mathscr{F}_{ij}$. Further, u_{ij} and u_{ji} are identical in the interior of $\mathscr{F}_{ij}$, and the sum of their energies is bounded uniformly by the sum of the energies of u_i and u_j.

In order to establish a condition number bound for Algorithm C, we need an estimate for $E_i(R_{i\mathscr{F}_{ij}}^T(u_{iF} - \hat{u}_F))$; see (34). By construction, we have $R_{i\partial\mathscr{F}_{ij}}(u_i - u_{ij}) = 0$ and $R_{j\partial\mathscr{F}_{ij}}(u_j - u_{ji}) = 0$. Since $u_{iF} - u_{jF} = (u_{iF} - u_{ijF}) - (u_{jF} - u_{jiF})$, it then follows from (36), (37), (41), and Lemma 5 that

$$\begin{aligned} E_i(R_{iF_{ij}}^T(u_{iF} - \hat{u}_F)) =& E_i(R_{iF_{ij}}^T(S_{iFF} + S_{jFF})^{-1} S_{jFF}(u_{iF} - u_{jF})) \\ \leq& 2(u_{iF} - u_{ijF})^T S_{iFF}(u_{iF} - u_{ijF}) + \\ & 2(u_{jF} - u_{jiF})^T S_{jFF}(u_{jF} - u_{jiF}) \\ \leq& \hat{C} C \omega_i^2 (E_i(u_i) + E_j(u_j)). \end{aligned} \tag{42}$$

We are able to show there exist u_{ij} and u_{ji} which satisfy the conditions in (39-41) with $\hat{C}$ independent of mesh parameters and the material properties α_i, β_i, α_j, and β_j under the assumption

$$\alpha_m \leq C\alpha_n \quad \text{and} \quad \beta_m \leq C\beta_n \quad \text{for } \{m,n\} = \{i,j\} \text{ or } \{m,n\} = \{j,i\}. \tag{43}$$

This can be done using Lemma 4 together with an extension theorem for H^1 functions on Lipschitz domains. We note that numerical experiments suggest that no assumptions on subdomain material properties are needed, other than them being constant in each subdomain, for $\hat{C}$ in (41) to be uniformly bounded.

Our main result follows from the estimate in (42).

Theorem 1 (Condition Number Estimate). *Under the assumption in (43), the condition number of the BDDC preconditioned operator for this study is bounded by*

$$\kappa \leq C\omega^2, \tag{44}$$

where

$$\omega = \max_i (1 + \log(H_i/h_i)). \tag{45}$$

In summary, we have obtained a favorable condition number estimate with less restrictive assumptions on the material properties of the subdomains than in previous studies. Comparing the condition number estimate of Theorem 1 with those in (1) and (2), we see that the factor of $H_i^2 \beta_i/\alpha_i$ can be removed provided the assumption in (43) holds. In addition, the logarithmic factor has been reduced from four powers to two. We note that the estimate in Theorem 1 also holds for FETI-DP due its spectral equivalence with BDDC.

We note that the algorithm involves a non-standard averaging given by (35). This averaging requires the solution of Dirichlet problems over the union of each pair of subdomains sharing a face. The importance of this method of averaging for some problems is shown in the next section.

4 Numerical Results

In this section, we present some numerical results to verify the theory and also to provide some additional insights. The domain is a unit cube discretized into smaller cubic elements. All the examples are solved to a relative residual tolerance of 10^{-8} for random right-hand-sides using the conjugate gradient algorithm with BDDC as the preconditioner. The number of iterations and condition number estimates from conjugate gradients are under the headings of *iter* and *cond* in the tables. We consider three different types of weights for the averaging operator. The first one, designated *SC*, is the one based on (35). Unless otherwise specified in the tables, this is the weighting used. The second type, *stiff*, is based on a conventional approach in which the weights are proportional to the entries on the diagonals of subdomain matrices. The third, *card*, uses the inverse of the cardinality of an edge, i.e. the reciprocal of the number of subdomains sharing the edge, for the weight.

The results in Table 1 are consistent with theory, suggesting condition numbers that are bounded independently of the number of subdomains, while the results in Table 2 are consistent with the $\log(H/h)^2$ estimate of Theorem 1.

We also consider a checkerboard distribution of material properties in which (α,β) for a subdomain is either (α_1,β_1) or (α_2,β_2), and note that subdomains with the same properties only share a subdomain vertex and no degrees of freedom. Results for 64 cubic subdomains each with $H/h = 4$ are shown in Table 3. Notice that for only one choice of material properties in the table do all three types of weighting lead to small condition numbers, and only the *SC* approach always gives condition numbers which are independent of the material properties. We have also investigated another type of weighting similar to *card*, but with weights γ, $0 < \gamma < 1$ for faces of subdomains with properties α_1,β_1 and $1-\gamma$ for faces of subdomains with properties α_2,β_2. Regardless of the choice of γ, large condition numbers were observed for the coefficients of the final row of Table 3. We note also that the choice of material properties in the final row is not covered by the theory of [11].

In the final example, we consider a cubic mesh of 20^3 elements that is partitioned into different numbers of subdomains using the graph partitioner Metis [7]. Although this example is not covered by our theory because the subdomains have irregular shapes, the results in Table 4 indicate that the algorithm of this study continues to perform well. The results in Tables 3 and 4 suggest that the *SC* weighting of this study may be necessary in order to effectively solve problems with material property jumps or with subdomains of irregular shape.

Table 1 Results for N cubic subdomains, each with $\beta = 1$ and $H/h = 4$.

N	$\alpha = 10^2$ iter (cond)	$\alpha = 1$ iter (cond)	$\alpha = 10^{-2}$ iter (cond)
4^3	15 (2.70)	14 (2.63)	10 (1.77)
6^3	16 (2.88)	15 (2.81)	11 (2.05)
8^3	16 (2.95)	15 (2.87)	12 (2.23)
10^3	17 (2.98)	16 (2.91)	13 (2.33)

Table 2 Results for 64 cubic subdomains, each with $\beta = 1$.

H/h	$\alpha = 10^2$ iter (cond)	$\alpha = 1$ iter (cond)	$\alpha = 10^{-2}$ iter (cond)
4	15 (2.70)	14 (2.63)	10 (1.77)
6	17 (3.30)	16 (3.21)	11 (2.14)
8	18 (3.77)	16 (3.66)	13 (2.46)
10	19 (4.16)	18 (4.03)	13 (2.72)

Table 3 Checkerboard material property results for 64 cubic subdomains with $H/h = 4$.

α_1	β_1	α_2	β_2	*SC* iter (cond)	*stiff* iter (cond)	*card* iter (cond)
1	1	10^3	1	10 (1.59)	19 (4.57)	196 (1.64e3)
1	1	1	10^3	11 (1.96)	84 (2.69e2)	109 (4.72e2)
1	1	1	1.01	14 (2.63)	14 (2.63)	14 (2.63)
10^2	10^{-2}	1	1	6 (1.07)	65 (3.17e2)	74 (1.65e2)

Table 4 Results for 20^3 elements partitioned into N subdomains using a graph partitioner. Material properties are constant with $\alpha = 1$ and $\beta = 1$.

N	*SC* iter (cond)	*stiff* iter (cond)	*card* iter (cond)
60	19 (4.30)	189 (6.31e2)	24 (9.06)
65	19 (4.40)	184 (6.34e2)	29 (1.55e3)
70	18 (3.89)	188 (6.47e2)	23 (7.48)
75	19 (4.16)	176 (6.12e2)	23 (6.49)

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