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# Firing Neurons


Joel Spencer

Problem from Charles Peskin (Courant)

Robert DeVille (Courant $\longrightarrow$ Urbana)

Angelika Steger (ETH) & Students

Juliana Friere (Courant graduate student)

Fabian Kuhn (ETH $\longrightarrow$ MIT $\longrightarrow$ ??)

$n$ nodes (neurons) ($n \to \infty$)

$k$ regular levels ($k$ fixed, $k = 10$)

Levels $0, \ldots, k-1$ and special $F$ (fire)

Firing Parameter $p$. Parametrize $p = \frac{ck}{n}$

State: Each neuron has level $0 \leq i \leq k-1$

Initial State: You pick it!

Transition: Pick random node. Increment level.

# Firing

When neuron $v$ incremented to $F$

Queue set to $\{v\}$

PopQueue, neuron stays at $F$ but spent

Nonspent neurons:

$$\text{increment with probability } p = \frac{ck}{n}$$

Continue until Queue empty

Reset all fired neurons to level zero

F*: Firing

F-: Already Fired, remains at F

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| initial | 3 | 9* | 9 | 8 | 1 | 9 | 7 |
| fires | 3 | F* | 9 | 8 | 1 | 9 | 7 |
| fires | 3 | F- | F* | 8 | 2 | F | 7 |
| fires | 3 | F- | F- | 9 | 3 | F* | 7 |
| fires | 3 | F- | F- | F* | 4 | F- | 8 |
| fires | 3 | F- | F- | F- | 4 | F- | 8 |
| reset | 3 | 0 | 0 | 0 | 4 | 0 | 8 |

`burst size` $= 4$

`Big Burst`, burst size $\Omega(n)$.

`Small Burst`, burst size $O(\ln n)$.

## Asymptotic Description

State $(\alpha_0, \ldots, \alpha_{k-3}, \beta, \alpha)$

$(n/k)\alpha_i$ at level $i$

$(n/k)\beta$ at level $k-2$

$(n/k)\alpha$ at level $k-1$

Level $k-1$ to $F$

Basic Erdős-Rényi

If $p(n/k)\alpha = c\alpha < 1$ small burst

If $p(n/k)\alpha = c\alpha > 1$ big burst

But, what happens *dynamically*??

## The Dyanamic Picture

$p = cn/k$

$(n/k)\alpha$ at level $k-1$

$(n/k)\beta$ at level $k-2$

$\beta > 1$ relatively fixed

$\alpha$ increasing to $c\alpha = 1$.

When $\alpha$ "reaches" $c\alpha = 1$

`Big Burst` fed by level $k-2$

**Critical Window** $c\alpha = 1 + \Theta(n^{-1/3})$

## Underlying Walk

Parameter $p = n^{-1}$

$X_0 = 1$

$X_t = X_{t-1} + \text{BIN}[t, p] - 1$

If $X_t = 0$ reset $X_t \leftarrow 1$

For $t < (1 - \epsilon)n$ keep crashing

For $t > (1 + \epsilon)n$ can go to infinity

Last Crash at $n + \Theta(n^{-1/3})$

Longest Crash $\Theta(n^{2/3})$

# Erdős-Rényi Double Jump

$G(n,p)$. Traditional: $p = \frac{c}{n}$

$c < 1$ Subcritical. Components $O(\ln n)$

$c > 1$ Supercritical. Giant Component $\Theta(n)$

$c = 1$ Delicate. Largest $|C| = \Theta(n^{2/3})$

Erdős-Rényi Critical Window

$$p = \frac{1}{n} + \lambda n^{-4/3}$$

$\lambda \to -\infty$ Barely Subrcritical.

- $|C_1| \sim |C_2| \sim \ldots \sim |C_{100}|$
- Big components trees

$\lambda \to +\infty$ Barely Supercritical.

**Dominant Component** $|C_1| \gg |C_2|$

$C_1$ complex

$\lambda$ constant. Critical Window.

## Erdős-Rényi and Neuron Firing

ER with $p = n^{-1} + \lambda n^{-4/3}$

Generate $C(v)$ by BFS. $X_t =$ queue size

Add Poisson $1 + \lambda n^{-1/3}$, Subtract one.

Ecological Limitation.

When $cn^{2/3}$ found, Poisson $1 + (\lambda - c)n^{-1/3}$.

Neuron Firing $X_t =$ queue size

Add Poisson $1 + \lambda n^{-1/3}$, Subtract one.

BUT: Suppose double at penultimate level.

Ecological Positive Feedback.

When $cn^{2/3}$ fire have $cn^{2/3}$ *more*!

When $cn^{2/3}$ found, Poisson $1 + (\lambda + c)n^{-1/3}$.

Limiting Brownian motion

# Two Regimes

$c$ is expected new to $F$ *if* uniform

- $c > 1$

  With *any* initial state

  Soon reach near periodic behavior

  Big Burst

  Buildup to Big Burst

- $c$ small

  With *any* initial state

  Soon reach quiet behavior

  Only Small Burst

  Approach Uniform Level Distribution

## A Third Regime! $c$ slightly less than one

Uniform is *SemiStable*

Takes Exponential Time to Leave

• • • Another Semistable Behavior

Big Burst

Buildup to Big Burst

Takes Exponential Time to Leave

# Limiting Behavior

State $\vec{x} = (x_0, \ldots, x_{k-1})$, $(n/k)x_i$ at level $n$.

Loading Phase

Merry Go Round (MGR)

$x'_i = x_{i-1} - x_i$ for $1 \leq i \leq k-1$

$x'_0 = x_{k-1} - x_0$

Valid if no big bursts

$\vec{x}(t) \rightarrow (1, \ldots, 1)$ (uniform)

# Burst Equation (BST)

Start when $cx_{k-1}(t_0) = 1$

Auxilliary $F(t)$ = proportion fired. $F(t_0) = 0$

$x_i' = x_{i-1} - x_i$ for $1 \le i \le k-1$

$x_0' = -x_0$

$F' = x_{k-1}$. So $F'(t_0) = c^{-1}$

END at first $t_1$ with $F(t_1) = c^{-1}(t_1 - t_0)$

`MGR-BST Toggle`

Do MGR until $cx_{k-1} = 1$

Do BST until END

Reset $x_0 \leftarrow x_0 + F$

Back to MGR

## The Third Regime: $k = 100$, $c = 0.99$

Position $(x_0, \ldots, x_k)$ *good* if

- (Dust) $x_i < k^{-10}$, $1 \leq i \leq k/2$
- (Zero) At least 99.9% at Zero
- (Right) Starting with only Right, MGR would have $cx_k < 0.1$ forever.

Claim: good $\longrightarrow$ good

Mass at Zero becomes moving Gaussian

Triggers BST

Huge Burst swallows Zero and Right

Dust either in Burst or at Right

Kuhn,JS,Steger,Panagiotou:

$c < 2^{5/4}(k \ln k)^{-1/4}(1-\epsilon)$: Quiet

$c > 2^{5/4}(k \ln k)^{-1/4}(1+\epsilon)$: Start all at 0. Stable state of big bursts.

`Open Questions:`

$c > 1$ Is there *unique* stable state of big bursts.

Middle $c$: Are there only two stable states.

I have no home, the world is my home.

– Paul Erdős