Microsoft Research New England

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Colloquium

Finding Needles

in

Exponential Haystacks

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Working with Paul Erdős was like taking a walk in the hills. Every time when I thought that we had achieved our goal and deserved a rest, Paul pointed to the top of another hill and off we would go.

– Fan Chung

Six Standard Deviations Suffice

$S_1, \ldots, S_n \subseteq \{1, \ldots, n\}$

$\chi : \{1, \ldots, n\} \rightarrow \{-1 + 1\}$

$\chi(S) := \sum_{j \in S} \chi(j)$, $\mathtt{disc(S)} = |\chi(\mathtt{S})|$

Theorem (JS/1985): There *exists* χ

$$\mathtt{disc(S_i)} \leq 6\sqrt{n}$$

for *all* $1 \leq i \leq n$.

Conjecture (JS/1986-2009) You can't find χ in polynomial time.

Theorem (Bansal/2010): Yes I can!

Erdős Magic

Theorem (Erdős): There *exists* χ

$$\mathtt{disc(S_i)} \leq \sqrt{2n \ln 2n}$$

for *all* $1 \leq i \leq n$.

Proof: Pick χ randomly!

Linear Formulation

$|a_{ij}| \leq 1$, $1 \leq i, j \leq n$.

$$L_i(x_1, \ldots, x_n) := \sum_{j=1}^{n} a_{ij} x_j$$

Theorem (JS/1985): There exists $x_1, \ldots, x_n \in \{-1, +1\}$

$$|L_i(x_1, \ldots, x_n)| \leq 6\sqrt{n}$$

for *all* $1 \leq i \leq n$.

Simultaneous Roundoff

Old $x_j^{\mathtt{old}}$, New $x_j^{\mathtt{new}}$

$$\Delta_i := L_i^{\mathtt{new}} - L_i^{\mathtt{old}}$$

Theorem (JS/1985): Given $x_j^{\mathtt{old}} \in [-1, +1]$ there exists a simultaneous roundoff $x_j^{\mathtt{new}} \in \{-1, +1\}$ with

$$|\Delta_i| \le 6\sqrt{n}$$

for *all* $1 \le i \le n$.

Entropy

With $\Pr[Z = \alpha] = p_\alpha$, $\alpha \in I$:

$$H[Z] := \sum_{\alpha \in I} p_\alpha (-\lg p_\alpha)$$

For $p \in (0, 1)$:

$$H(p) := -p \lg p - (1-p) \lg(1-p)$$

- Subadditivity:

$$H((Z_1, \ldots, Z_n)) \leq \sum_{j=1}^{n} H(Z_i)$$

- Pigeonhole:

$$H(Z) \leq s \Rightarrow \text{ some } \Pr[Z = \alpha] \geq 2^{-s}$$

The Cost Function

Definition: $\mathtt{COST}[\beta]$ is the entropy of the round-off of the standard normal N to the nearest multiple of β.

Asymptotics:

β large:

$$\mathtt{COST}[\beta] = \Theta(\beta e^{-\beta^2/8})$$

β small:

$$\mathtt{COST}[\beta] = \Theta(\lg \beta^{-1})$$

The Cost Equation

$|a_{ij}| \leq 1$, $1 \leq i \leq n$, $1 \leq j \leq m$

$$L_i(x_1, \ldots, x_m) := \sum_{j=1}^{m} a_{ij} x_j$$

Theorem: If

$$\sum_{i=1}^{n} \mathtt{COST}[\beta_{\mathtt{i}}] \leq \mathtt{m}(1 - \mathtt{H}(\mathtt{c}))$$

then there exists $x_1, \ldots, x_m \in \{-1, 0, +1\}$:

- Substantial: $|\{j : x_j \neq 0\}| \geq 2cm$.
- Good: For $1 \leq i \leq n$

$$\left| \sum_{j=1}^{m} a_{ij} x_j \right| \leq \frac{\beta_i}{2} \sqrt{m}$$

Proof of The Cost Equation

$x_j \in \{-1, +1\}$, uniform, independent.

$$\Lambda : (x_1, \ldots, x_m) \to (Z_1, \ldots, Z_n)$$

with Z_i roundoff of $L_i(x_1, \ldots, x_m)$ to nearest multiple of $\beta_i\sqrt{m}$.

$H(Z_i) \le \mathtt{COST}[\beta_\mathtt{i}]$

$H(\Lambda) \le \sum \mathtt{COST}[\beta_\mathtt{i}] \le \mathtt{m(1 - H(c))}$

Some $\Lambda(\vec{x})$ hit $\ge 2^{mH(c)}$ times.

Kleitman: $\Lambda(\vec{x'}) = \Lambda(\vec{x''})$, $\rho(\vec{x'}, \vec{x''}) \ge 2cm$.

Beck Idea: Set

$$x_j = \frac{x'_j - x''_j}{2} \text{ for } 1 \le j \le m$$

Coloring by Phases

Phase Zero

$c = \frac{1}{4}$. $\mathtt{COST}[\beta] = \mathtt{1} - \mathtt{H(c)}$.

Color Half. $|\Delta_i| \leq \beta\sqrt{n}$.

Phase t: When $\sim 2^{-t}n$ uncolored.

$c = \frac{1}{4}$. $\mathtt{COST}[\beta] = \mathtt{2}^{-\mathtt{t}}(\mathtt{1} - \mathtt{H(c)})$.

Color Half.

$$|\Delta_i| \leq \beta\sqrt{n2^{-t}} = \sqrt{n}O(2^{-t/2}\sqrt{t})$$

At end

$$|\Delta| \leq \sum_{t=0}^{\infty} |\Delta_i^{(T)}| = \sqrt{n} \cdot O(1)$$

What is Semidefinite Programming

Linear Programming on a_{ij}, $1 \leq i, j \leq m$

$A = (a_{ij})$ Semidefinite

- Unknowns $\vec{v_1}, \ldots, \vec{v_m} \in R^m$

Linear Programming on $a_{ij} = \vec{v_i} \cdot \vec{v_j}$.

Feasibility: If system feasible, Semidefinite Programming will *find* $\vec{v_1}, \ldots, \vec{v_m} \in R^m$.

Maybe not the ones you were thinking of!

The Semidefinite Program

Assume β_i, c, m, n satisfy Cost Equation.

$|\vec{v_j}|^2 \leq 1,\ 1 \leq j \leq m$

$\sum_{j=1}^{m} |\vec{v_j}|^2 \geq cm$

$\left|\sum_{j=1}^{m} a_{ij}\vec{v_j}\right|^2 \leq [\frac{\beta_i}{2}\sqrt{m}]^2$

Solution $\vec{v_j} = x_j \in \{-1, 0, +1\} \in R^1$.

Find solution in R^m!

Random Projection

$\vec{G} = (g_1, \ldots, g_m)$, $g_i \sim N(0,1)$, i.i.d.

$$x_j \leftarrow x_j + \epsilon \vec{v_j} \cdot \vec{G}$$

$$L_i \leftarrow L_i + \epsilon [\sum_{j=1}^{m} a_{ij} \vec{v_j}] \cdot \vec{G}$$

$\epsilon \vec{z} \cdot \vec{G} \sim N(0, \epsilon^2 |\vec{z}|^2)$

x_j, L_i martingales. Not independent.

Brownian motion as $\epsilon \to 0^+$.

Roughly $x_j \leftarrow x_j \pm \epsilon$, $L_j \leftarrow L_j \pm \epsilon \beta_i \sqrt{m}/2$

Problem: A few L_i get big.

Moving by Phases

Time $T := \frac{1}{n}\sum x_i^2$

Phase 0: Start Arbitrary. End $1 - T \leq \frac{1}{2}$.

Phase t: Start $1 - T < 2^{-t}$. End $1 - T \leq 2^{-t-1}$

x_j frozen if "near" ± 1.

$m \geq \frac{n}{2}$ floating in Phase 0.

Claim: Phase 0 (others similar) with all

$$|\Delta_i| \leq K\sqrt{n}$$

$T \leftarrow T + \epsilon^2 \sum |\vec{v_i} \cdot \vec{G}|^2 \geq T + \epsilon^2 (cm/n)$

Number of steps $\Theta(\epsilon^{-2})$

Danger Levels

i *safe* if $|\Delta_i| n^{-1/2} \le K_1$

Danger Level u: $|\Delta_i| n^{-1/2} \in (K_u, K_{u+1}]$

$K_u \to K$

Speeds $\gamma_0 > \gamma_1 > \gamma_2 > \ldots$

When i at level u

$$\left| \sum_{j=1}^{m} a_{ij} \vec{v_j} \right|^2 \le [\frac{\gamma_u}{2}\sqrt{m}]^2$$

More Dangerous $\Rightarrow$ Slow Down!

Does it Work?

Phase 0: $m = \frac{n}{2}$, $c = \frac{1}{4}$

$$\sum_{i=1}^{n} \mathtt{COST}[\beta_\mathtt{i}] \leq (\mathtt{1} - \mathtt{H(c)})\mathtt{m} = \mathtt{c_1 n}$$

Danger Levels: $\frac{K}{2}, \frac{2K}{3}, \frac{3K}{4}, \ldots$

- $\mathtt{COST}[\gamma_0] \leq \frac{c_1}{10}$

While safe $L_i \leftarrow L_i \pm \epsilon\gamma_0\sqrt{m}/2$

- In $\Theta(\epsilon^{-1})$ steps 1% of $|\Delta_i|$ reach $\frac{K}{2}\sqrt{n}$

Pick γ_0 large and small enough.

$\gamma_1 = \frac{\gamma_0}{10}$. Expensive but only 1%.

Tenth of speed, Third of distance.

$10^{-6}n$ reach $\frac{2K}{3}\sqrt{n}$.

$\gamma_u = \gamma_0 10^{-u}$. Expensive but very few. OK!

Forcing Perpendicularity

At $(x_1, \ldots, x_m)$ add condition

$$\left| \sum_{j=1}^{m} x_j \vec{v_j} \right|^2 \leq n^{-10}$$

Expensive but only one.

$x_j \leftarrow x_j + \delta_j$ with

$$(x_1, \ldots, x_m) \cdot (\delta_1, \ldots, \delta_m) \sim 0$$

Each step $T^{\texttt{new}} \geq T^{\texttt{old}} + \epsilon^2 (cm/n)$ definitely.

Number of steps $\Theta(\epsilon^{-2})$ definitely.

Two Exponential Needles

- Bansal!
- Moser/Tardos on Local Lovász Lemma

Moser/Tardos: Indpendent Proof

Bansal: Uses existence to find algorithm

It is six in the morning.

The house is asleep.

Nice music is playing.

I prove and conjecture.

– Paul Erdős, in letter to Vera Sós