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GAMES

MATHEMATICIANS PLAY

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A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks.
– Gian -Carlo Rota

THE TENURE GAME

$$\begin{array}{rrrr}
YAN & & & \\
ALON & CHEN & & \\
KARP & KNUTH & & LOVASZ \\
------ & ---- & --- & ------ \\
PostD & AP1 & AP2 & Assoc
\end{array}$$

Each year, Chair Paul gives promotion list L to Dean Carole. Carole Either

- Promotes L, Fires $\overline{L}$ *or*
- Promotes $\overline{L}$, Fires L

Carole wins if *nobody* gets tenure.

a_k people k rungs from Tenure

Theorem. If $\sum a_k 2^{-k} < 1$ then Carole wins.

Proof1. Carole plays randomly.

$T =$ number getting Tenure.

Pr[Paul wins] $\leq E[T] = \sum a_k 2^{-k} < 1$

Therefore Carole can *always* win.

Proof2. (Derandomization)

Carole plays to minimize $E[T]$.

Theorem. If $\sum a_k 2^{-k} \geq 1$ then Paul wins.

Lemma. If $E[T] \geq 1$ there is a move for Paul so that $E[T^{yes}] \geq 1$ and $E[T^{no}] \geq 1$.

Proof of Theorem:

Paul makes that splitting move.

BALANCING VECTOR GAME

n rounds. Initial $P \leftarrow 0 \in R^n$

Paul picks $v_i \in \{-1,+1\}^n$

Carole picks $\epsilon_i \in \{-1,+1\}$

$P \leftarrow P + \epsilon_i v_i$

Payoff to Paul: $|P^{final}|_\infty$

$VAL(n)$: value of Game.

Similar to:

- On Line Coloring of $A_1,\ldots,A_n \subseteq \{1,\ldots,n\}$
- On Line Roundoff of $x_1,\ldots,x_n \in [0,1]$ to minimize max error in linear $L_1,\ldots,L_n$

Carole $\sim$ Worst Case Analysis

Theorem. If

$$\Pr[|S_n| > \alpha] < n^{-1}$$

then Carole can keep $|P^{final}|_\infty < \alpha$

Proof1 . Carole plays randomly

$T =$ number of coordinates L_i with $|L_i| > \alpha$

$$E[T] = n \Pr[|S_n| > \alpha] < 1$$

$$\Pr[\text{Paul wins}] \leq E[T] < 1$$

Therefore Carole can *always* win

Proof2 (Derandomization)

$P = (L_1, \ldots, L_n)$ with t rounds remaining.

$$E[T] = w_t(P) = \sum \Pr[|L_i + S_t| > \alpha]$$

Carole plays to minimize $E[T]$

Theorem. If

$$\Pr[|S_n| > \alpha] > cn^{-1/2}$$

then Paul can force $|P^{final}|_\infty > \alpha$

Proof. With $t+1$ rounds remaining Paul picks $v = (\delta_1, \ldots, \delta_n)$ with

$$|w_t(P+v) - w_t(P-v)| \leq$$

$$\leq \max|\Pr[|L_i+1+S_t| > \alpha] - \Pr[|L_i-1+S_t| > \alpha]|$$

$$= O(t^{-1/2})$$

Then $w(P^{new}) > w(P^{old}) - O(t^{-1/2})$

$w(P^{final}) > w(P^{init}) - \sum O(t^{-1/2}) >$

$> w(P^{init}) - O(n^{1/2}) > 0$

Corollary. $VAL(n) = \Theta(\sqrt{n \ln n})$

PAUL AND CAROLE GAMES

- RANDOMIZATION

Carole plays randomly. If she wins with positive probability she can always win.

- DERANDOMIZATION

Conditional Expectation gives weight function for Carole to minimize deterministicly.

- ANTIRANDOMIZATION

Paul uses *this* weight function for effective counterplay.

Paul = Paul Erdős

Carole is anagram for ??

Paul versus Carole

N Possibilities

Q Yes/No Paul Queries

K (or fewer) Carole Lies

Try it with $N = 100$, $Q = 10$, $K = 1$

Carole plays Adversary Strategy

$\Rightarrow$ Perfect Information

$\Rightarrow$ Winning Strategy for Paul or Carole

$B_K(Q) =$ maximal N so that Paul Wins

Theorem:

$$B_K(Q) \sim \frac{2^Q}{\binom{Q}{K}}$$

Carole Strategy

Notation

$$\binom{Q}{\leq K} = \sum_{I=0}^{K} \binom{Q}{I}$$

Theorem: $N\binom{Q}{\leq K} > 2^Q \Rightarrow$ Carole Wins

Proof 1: Preserve Ministrategies

Proof 2: Random Play

Proof 1 $\Rightarrow$ Proof 2: Derandomization

Vector Format

Position $(3,14)$ $((x_0,\ldots,x_K))$

Paul Move $(1,9)$ $((a_0,\ldots,a_K))$

Yes: $(1,11)$; No: $(2,6)$

Perfect Split: Yes=No

Position $(8,20)$, Move $(4,10)$, Yes/No $(4,14)$

$L:(x,y)\rightarrow(\frac{x}{2},\frac{x}{2}+\frac{y}{2})$ $(L:R^{K+1}\rightarrow R^{K+1})$

Position after perfect split.

Problem: Integrality

Weight Function $W_Q(\vec{x})=L^Q(\vec{x})\cdot\vec{1}$

$W_Q(x,y)=2^{-Q}((Q+1)x+y)$

$(2^{-Q}(\binom{Q}{\leq K}x_0+\ldots+(Q+1)x_{K-1}+x_K))$

Paul Strategy

Theorem (JS): (K fixed, Q large)

$W \leq 1$ *and* $> cQ^K$ "pennies"

$\Rightarrow$ Paul Win

Keep Weight Equal (Perfect Split if Possible)

$Q = 10$. Position $(17, 837)$. $W = 1$

Paul $(8, 418 + x) \Rightarrow (8, 427 + x); (9, 427 - x)$

$W_9(1, -2x) = 0 \Rightarrow x = 5$

Problem: Nonnegativity

Proof Outline

First K Moves: Initial Penny Supply

Middle: Pennies Replenished from Nonpennies

End: Endgame Analysis

Halflie: No False Negatives

N Possibilities

Q Queries

K Halflies

$A_K(Q) =$ maximal N, Paul Wins

Theorem (Cicalese/Mundici): $A_1(Q) \sim 2^{Q+1}/Q$

Dumitriu/JS:

$$A_K(Q) \sim 2^K B_K(Q) \sim 2^K \frac{2^Q}{\binom{Q}{K}}$$

Position $\vec{x} = (x, y)$ $((x_0, \ldots, x_K))$

Paul Query: (a, b) $((a_0, \ldots, a_K))$

Yes $(a, b + x - a)$; No $(x - a, y - b)$

Perfect Split $(\frac{x}{2}, \frac{y}{2} - \frac{x}{4})$

Yes/No $L\vec{x} := (\frac{x}{2}, \frac{y}{2} + \frac{x}{4})$

Problems: Integrality, Nonnegativity

Weight $W_Q(\vec{x}) = L^Q(\vec{x}) \cdot \vec{1}$

$W_Q(x, y) = 2^{-Q}(x(1 + \frac{Q}{2}) + y)$

$2^{-Q}(x_0 p_K(Q) + \ldots + x_{K-1}(1 + \frac{Q}{2}) + x_K)$

Paul Strategy

Start $(N, 0)$, $N < (1-\epsilon)2^{Q+1}/Q$

- Roundup so $N = 2^T A$, A small.
- Give Ground to (N, N)
- T perfect splits to $L^T(N\vec{1})$
- Endgame, A fixed, R large:

$$\text{Win in } R \text{ from } (A, 2^R - 2A + 1)$$

A Combinatorial Approach

1-Set: Subset of $\{Y, N\}^Q$ with

$$
\begin{array}{rl}
\text{stem} & YNNYNY \\
\text{child} & Y\underline{Y}YNNY \\
\text{child} & YN\underline{Y}YYN \\
\text{child} & YNNY\underline{Y}N
\end{array}
$$

0-Set: Any Singleton

K-Set: Depth K tree with marked "lies."

$$
\begin{array}{rl}
\text{parent} & Y\underline{Y}YNNYN \\
\text{child} & Y\underline{Y}YN\underline{Y}NN \\
\text{grandchild} & Y\underline{Y}YN\underline{YY}Y
\end{array}
$$

Theorem: Paul Wins from $(x_0, \ldots, x_K)$ in Q

$\Leftrightarrow$ Can Pack x_i $K-i$-Sets in $\{Y, N\}^Q$

Bound Packing of K-Sets

- When all have $\geq L$ N, Size $> \binom{L}{\leq K}$

$L \sim \frac{Q}{2}$ Volume Bound $2^Q/\binom{Q/2}{K}$

$o(2^Q Q^{-K})$ have any $L < (1-o(1))\frac{Q}{2}$

$A_K(Q) < (1+o(1))2^Q/\binom{Q/2}{K}$

Careful Cutoff

Set $L = \frac{Q}{2} + c\sqrt{Q}\sqrt{\ln Q}\ Y$

$$A_K(Q) \leq \frac{2^Q}{\binom{Q/2}{K}}(1 + cQ^{-1/2}\sqrt{\ln Q})$$

Yan/JS: Remove $\sqrt{\ln Q}$

Two Batch Strategy

$\{Y,N\}^{r*}$: Number Y within $r^{0.6}$ of $\frac{r}{2}$

$|\{Y,N\}^{r*}| \sim 2^r$

"Assume" $N = |\{Y,N\}^{r*}| \sim 2^Q/(2Q)$

Associate $\sigma \in \{Y,N\}^{r*}$ with possibility

Batch 1: $1 \le i \le r$: Is $\sigma_i = Y$?

Carole *must* say No about half the time!

Endgame from $(1, \sim \frac{r}{2})$ in One Batch

Arbitrary Channel

T-ary queries

E lie patterns

Example with $T = 3$, $E = 4$

Ternary Answers A/B/C.

Carole may lie B for A, A for B, A or B for C.

Theorem (Dumitriu, JS):

$$A_K^*(Q) \sim \frac{T^K}{E^K} \frac{T^Q}{\binom{Q}{K}}$$

Working with Paul Erdős was like taking a walk in the hills. Every time when I thought that we had achieved our goal and deserved a rest, Paul pointed to the top of another hill and off we would go.

– Fan Chung