Horizons of Combinatorics

Balatonalmádi 2006

The

GIANT COMPONENT Revisited

Joel Spencer

Erdős came in the Fall of 1940 and immediately began questioning the graduate students. This was my final year at Penn, the year in which I completed my thesis. I told him I was doing a problem in the theory of partitions. He looked thoughtful and said, “I have often thought about partitions but I have never written anything down. Tell me about your problem.”
– Letter of Prof. Joe Lehner, 1998

Paul Erdős and Alfred Rényi

On the Evolution of Random Graphs

Magyar Tud. Akad. Mat. Kutató Int. Közl

volume 8, 17-61, 1960

$\Gamma_{n,N(n)}$: n vertices, random $N(n)$ edges

[...] the largest component of $\Gamma_{n,N(n)}$ is of order $\log n$ for $\frac{N(n)}{n} \sim c < \frac{1}{2}$, of order $n^{2/3}$ for $\frac{N(n)}{n} \sim \frac{1}{2}$ and of order n for $\frac{N(n)}{n} \sim c > \frac{1}{2}$. This double "jump" when c passes the value $\frac{1}{2}$ is one of the most striking facts concerning random graphs.

The Giant Component

$G(n,p)$, $p = \frac{c}{n}$ (or $\sim \frac{c}{2}n$ edges)

$c > 1$ constant

$|C_1| \sim yn$, $y = y(c) > 0$

$$1 - y = e^{-cy}$$

The Dominant Component

(Bollobás) $p = \frac{1+\epsilon}{n}$, $\epsilon = o(1)$

`Critical Scaling:` $n^{-1/3} \ll \epsilon$

$|C_1| \sim 2\epsilon n$

Tilted Balls in Bins

$k-1$ balls, k bins, $p \in (0,1]$

Truncated Geometric

Ball j in Bin T_j (i.i.d.)

$$\Pr[T_j = i] = \frac{p(1-p)^{i-1}}{1-(1-p)^k}$$

Z_i balls in bin i

$Y_0 = 1$, $Y_i = Y_{i-1} + Z_i - 1$ (so $Y_k = 0$)

$\mathsf{TREE} = \mathsf{TREE}_{k,p}$: $Y_t > 0$, $0 \le t < k$

$A_2 = A_2(k,p) = \Pr[\mathsf{TREE}_{k,p}]$

$$M := \sum_{i=0}^{k} (Y_i - 1) = \binom{k}{2} - \sum_{j=1}^{k} T_j$$

$M^* := M|_{\mathsf{TREE}}$

Vertices $\{0, 1, \ldots, n-1\}$

$C(0)$ has X vertices, $X - 1 + Y$ edges

$\Pr[X = k] = A_1 A_2$ with

$A_1 = \Pr[\text{BIN}[n-1, 1-(1-p)^k] = k-1]$

$A_2 = A_2(k, p)$

In particular

$$\begin{aligned} \Pr[G(n,p) \text{ connected}] &= \Pr[X = n] \\ &= (1-(1-p)^n)^{n-1} A_2(n,p) \end{aligned}$$

Further $\Pr[X = k \text{ and } Y = l] = A_1 A_2 A_3$ with

$A_3 := \Pr[Y = l | X = k] = \Pr[\text{BIN}[M^*, p] = l]$

BFS on $G(n,p)$

$T_j^* = i$: Vertex j joins BFS tree at i-th opportunity (*fictitious continuation!*) T_j^* geometric

$X = k \Rightarrow$ precisely $k-1$ of $T_j \le k$

Condition on that. (A_1 factor)

WLOG $T_j^* \le k$ for $1 \le j \le k-1$

$T_j^* \to T_j$, truncated geometric

$X = k$ iff queuesize Y_t never zero iff TREE

$\{j, j'\}_<$ unexposed iff pop j when j' in queue *or* $\{j', j\}_<$ exposed for T_j vertices j'

Precisely M^* unexposed

Expose unexposed, additional BIN$[M^*, p]$ edges

1	2	3	4	5
N	N	Y	Y	N
N	N	-	-	N
Y	N	-	-	N
-	Y	-	-	Y
-	-	-	-	-
-	-	-	-	-

$T_3 = T_4 = 1$, $T_1 = 3$, $T_2 = T_5 = 4$

A_1: All T_j defined

$\vec{Z} = (2, 0, 1, 2, 0, 0)$

Walk $\vec{Y} = (1, 2, 1, 1, 2, 1, 0)$

TREE: BFS doesn't terminate early

Tree Edges $03, 04, 41, 12, 15$

$M = 2$ Unexposed $34, 25$

Asymptotics of Pr[TREE]

$A_2(k,p) = \text{Pr}[\text{TREE}_{k,p}] \sim$

$$\begin{array}{rll} 1 & \text{for} & p \gg k^{-1} \\ 1-(c+1)e^{-c} & \text{for} & p \sim ck^{-1} \\ \frac{1}{2}\epsilon^2 & \text{for} & p \sim \epsilon k^{-1}, k^{-1/2} \ll \epsilon = o(1) \\ \texttt{complicated!} & \text{for} & p \sim ck^{-3/2} \\ k^{-1} & \text{for} & 0 \le p \ll k^{-3/2} \end{array}$$

First Two Cases: Giant Component

Third Case: Dominant Component

($p \sim \frac{1}{n}$, $k \sim 2\epsilon n$: $\epsilon \gg n^{-1/3} \leftrightarrow k^{-1/2} \ll \epsilon$)

Pr[TREE] with $p \sim \frac{c}{k}$

Left Z_i Poisson $\frac{c}{1-e^{-c}}$

Galton-Watson Pr[ESC] $\sim 1 - e^{-c}$

Right $Z_i^* = Z_{k-i}; Y_i^* = Y_{k-i}$

$Y_0^* = 0$, $Y_i^* = Y_{i-1}^* + 1 - Z_i^*$

Z_i^* Poisson $\frac{ce^{-c}}{1-e^{-c}}$

Pr[ESC*] $\sim 1 - \frac{ce^{-c}}{1-e^{-c}}$

Chernoff: $Y_i > 0$ in middle

Pr[TREE] $\sim$ Pr[ESC] Pr[ESC^*] $\rightarrow 1-(c+1)e^{-c}$

If $c \rightarrow \infty$ Pr[TREE] $\rightarrow 1$

If $c \rightarrow 0^+$ Pr[TREE] $\sim \frac{1}{2}c^2$ for a while!

Why $\epsilon \gg k^{-1/2}$?

$p = \epsilon k^{-1}$, $\epsilon = o(1)$

Left Z_i Poisson $1 + \frac{\epsilon}{2}$

Left Drift $\frac{\epsilon}{2}$

Right Z_{k-i} Poisson $1 - \frac{\epsilon}{2}$

Right Drift $-\frac{\epsilon}{2}$

Drift takes $\Theta(\epsilon^{-2})$ “time” to be “felt”

Left/Right/Middle separation iff $\epsilon^{-2} \ll k$

$0 \leq p \ll k^{-3/2}$

p effectively zero. Uniform Distribution

k^{k-2} placements with TREE

$\Pr[\text{TREE}] \sim k^{k-2}(1/k)^{k-1} = k^{-1}$

$\Pr[G \text{ connected}] \sim \Pr[G \text{ tree}] \sim$

$\sim k^{k-2}p^{k-1}(1-p)^{k^2/2}$

$p \sim ck^{-3/2}$, $k \sim c'\epsilon^{-2}$ `complicated`

Limiting behavior (???) Brownian bridge

$\Pr[G \text{ connected}] \sim \Pr[G \text{ tree}] \sum_{i=0}^{\infty} c_i c^{3i/2}$

c_i Wright Constants

$$A_3 = \Pr[\mathrm{BIN}[M^*, p] = l | X = k]$$

$M = \binom{k}{2} - \sum_{j=1}^{k} T_j \sim N(\mu, \sigma^2)$ (CLT$\cdots$)

Claim: $M^* \sim N(\mu, \sigma^2)$

$p \gg k^{-1}$ trivial

$p \sim ck^{-1}$ easy

$p = \epsilon k^{-1}$, ϵ not near $k^{-1/2}$: hard

$p = \epsilon k^{-1}$, $\epsilon \gg k^{-1/2}$ barely: very hard

When l near $p\mu$ find asymptotics of A_3 and Joint Distribution of X, Y

Counting Connected Graphs

$C(k,l)$ labelled connected graphs

with k vertices and $k-1+l$ edges

$C(k,0) = k^{k-2}$, $C(k,l) \sim c_l k^{k-2} k^{3l/2}$

$k, l \to \infty$: Bender, McKay, Canfield

JS, van der Hofstad approach:

$\Pr[(X,Y) = (k,l)] =$

$= \binom{n-1}{k-1} C(k,l) p^{k-1+l} (1-p)^{k(n-k)+\binom{k}{2}-(k-1+l)}$

Reverse Engineering: Select n, p so k, l usual

values of Giant/Dominant component

Asymptotics of $\Pr[(X,Y) = (k,l)] = A_1 A_2 A_3$

Imply $C(k,l)$ known asymptotically!

I have no home.

The world is my home.

– Paul Erdős