References

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4.3 Talagrand

Here we use Talagrand's Inequality to show that 35 holds almost surely. Fix (p,k,δ)-superquasirandom G. Fix $l-1 \le t \le t-1$ and distinct $x_1,\ldots,x_t$. Set $N_G = N_G(x_1,\ldots,x_t)$, $N_H = N_H(x_1,\ldots,x_t)$, $X = |N_H|$ so that X is a random variable dependent on the choices of k-cliques in $\mathcal{F}$. Let A,B,C,q,ϵ be given by 38. Set

$$D = \binom{t}{l-1} B \tag{42}$$

For each $x \in N_G$ there are $(1 \pm \delta^*)D$ k-cliques that contain x and at least $l-1$ of the points $x_1,\ldots,x_t$. Let I_x be the indicator random variable for $x \in N(H)$ so that $X = \sum I_x$. Then

$$E[I_x] = (1-q)^{(1\pm\delta^*)D} = (1 \pm \delta^*)e^{-qD} \tag{43}$$

so that

$$E[X] = \sum_{x \in N_G} E[I_x] = (1 \pm \delta^*) n (pe^{-\epsilon})^{\binom{t}{l-1}} \tag{44}$$

We use Talagrand's Inequality to give a large deviation bound for X. Set $Y = |N_G| - X$, the number of vertices dropped because of cliques.

• Y is k-certifiable. Changing $K \notin \mathcal{K}$ to $K \in \mathcal{K}$ can only effect whether $y \in N_H$ for those $y \in K$.

• Y is f-certifiable where $f(s) = s$. If $Y \ge s$ then some $y_1,\ldots,y_s \in N_G - N_H$ so there are cliques $K_1,\ldots K_s \in \mathcal{K}$ with K_i containing y_i and $l-1$ of the $x_1,\ldots,x_t$. These $K_1,\ldots,K_s$ certify that $Y \ge s$.

From 16, letting $\mu_Y = E[Y]$

$$\Pr[|Y - \mu_Y| > \epsilon' \mu_Y] = e^{-\Omega(\mu_Y)} \tag{45}$$

for any $\epsilon' > 0$. Fortunately, $\mu_X = E[X]$ and μ_Y are both $\Theta(n)$, within constant multiples of each other. Thus for any $\epsilon^* > 0$

$$\begin{aligned} \Pr[|X - \mu_X| > \epsilon^* \mu_X] &< \Pr[|Y - \mu_Y| > \epsilon' \mu_Y] \\ &= e^{-\Omega(\mu_Y)} \\ &= e^{-\Omega(\mu_X)} \end{aligned} \tag{46}$$

As $\mu_X = \Omega(n)$ this is exponentially small and therefore certainly $o(n^{-t})$. Hence almost surely for *all* $x_1,\ldots,x_t$ $X \sim \mu_X$ and so N_H is $(pe^{-\epsilon},k,\delta)$-superquasirandom.

we may, for any positive ϵ', find ϵ, L, δ so that $g(\epsilon, L, \delta) > 1-\epsilon'$ and so for n sufficiently large there is a family of at least $(1-\epsilon')\binom{n}{l}/\binom{k}{l}$ disjoint k-cliques.

Now we aim for Theorem xxx. We set

$$\begin{aligned} A &= \binom{n}{k} p^{\binom{k}{l}} \\ B &= \binom{n-l}{k-l} p^{\binom{k}{l}-1} \\ C &= \binom{k}{l} B \\ q &= \epsilon / B \end{aligned} \tag{38}$$

We consider a random family $\mathcal{F}$ of k-cliques of G where for each such k-clique K

$$\Pr[K \in \mathcal{F}] = q \tag{39}$$

and the choices are mutually independent. It suffices to show that 34,35 both hold almost surely.

Remark The random hypergraph has $\sim A$ k-cliques, each edge is in $\sim B$ k-cliques, each k-clique overlaps $\sim C$ other k-cliques and for each edge $e \in E(G)$ the expected number of $K \in \mathcal{K}$ containing e is $\sim \epsilon$.

Our superquasirandom G behaves almost like the random hypergraph. Here and in §4.3 we let δ^* denote a function of δ which can be made arbitrarily small by making δ appropriately small. Then G has $(1 \pm \delta^*)A$ k-cliques, each edges is in $(1 \pm \delta^*)B$ k-cliques, each k-clique overlaps $(1 \pm \delta^*)C$ other k-cliques. For each k-clique K of G let I_K be the indicator random variable for K being an isolated k-clique of H and let $X = \sum I_K$ be the number of such K. Then

$$E[I_K] = q(1-q)^{(1\pm\delta^*)C} \tag{40}$$

and

$$\begin{aligned} E[X] &= (1 \pm \delta^*)Aq(1-q)^{(1\pm\delta^*)C} \\ &= (1 \pm \delta^*)Aqe^{-qC} \\ &= (1 \pm \delta^*)\frac{\binom{n}{l}}{\binom{k}{l}}\epsilon e^{-\binom{k}{l}\epsilon} \end{aligned} \tag{41}$$

Further calculation gives $Var[X] = o(E[X]^2)$ so that 34 holds almost surely.

where $N_G(x_1, \ldots, x_t)$ denotes those y so that all l-sets consisting of y and $l-1$ of the the x's are hyperedges of G.

Remark The random hypergraph with edge probability p is, for any fixed $k, \delta > 0$, almost surely (p, k, δ)-superquasirandom.

Theorem For every $2 \leq l < k$, $0 < p \leq 1$, $\epsilon > 0$ and $\delta' > 0$ there exists $\delta > 0$ with the following property: Let G be any (p, k, δ)-superquasirandom l-graph on n vertices with n sufficiently large. Then there is a family $\mathcal{F}$ of k-cliques of G so that

$$\mathcal{F} \text{ has at least } p\frac{\binom{n}{l}}{\binom{k}{l}}\left[\epsilon e^{-\epsilon\binom{k}{l}} - \delta'\right] \text{ isolated } k\text{-cliques} \tag{34}$$

and, setting $H = G - \bigcup \mathcal{F}$,

$$H \text{ is } (pe^{-\epsilon}, k, \delta') - \text{superquasirandom} \tag{35}$$

We derive the proof of Theorem xxx from Theorem xxx, following the ideas of [4]. Indeed, the idea is sometimes called the Rödl nibble. Fix $0 < \epsilon < 1$,L a positive integer and $\delta > 0$. (Think of ϵ small and L big.) Let G_0 be the complete l-graph so that G_0 is $(1, k, 0)$-superquasirandom. Find $0 = \delta_0 < \delta_1 < \ldots < \delta_L = \delta$ so that (for n sufficiently large) we may apply Theorem xxx L times (keeping ϵ constant) giving $G_0 \supset G_1 \supset \ldots \supset G_L$ with G_i $(e^{-\epsilon i}, k, \delta_i)$-superquasirandom and $G_i - G_{i+1}$ containing at least

$$e^{-\epsilon i}\frac{\binom{n}{l}}{\binom{k}{l}}\left[\epsilon e^{-\epsilon\binom{k}{l}} - \delta_i\right]$$

disjoint k-cycles. In total this gives at least $g(\epsilon, L, \delta)\binom{n}{l}/\binom{k}{l}$ disjoint k-cliques where

$$g(\epsilon, L, \delta) = \sum_{i=0}^{L-1} e^{-\epsilon i}[\epsilon e^{-\epsilon\binom{k}{l}} - \delta] \tag{36}$$

As

$$\begin{aligned}
\lim_{\epsilon\to 0}\lim_{L\to\infty}\lim_{\delta\to 0} g(\epsilon, L, \delta) &= \lim_{\epsilon\to 0}\lim_{L\to\infty}\sum_{i=0}^{L-1} e^{-\epsilon i}[\epsilon e^{-\epsilon\binom{k}{l}}] \\
&= \lim_{\epsilon\to 0}\sum_{i=0}^{\infty} e^{-\epsilon i}\epsilon e^{-\epsilon\binom{k}{l}} \\
&= \lim_{\epsilon\to 0} e^{-\epsilon\binom{k}{l}}\frac{\epsilon}{1 - e^{-\epsilon}} \\
&= 1
\end{aligned} \tag{37}$$

4 Asymptotic Packing

4.1 Notations

For $2 \le l < k < n$ let $m(n,k,l)$ denote the maximal size of a family F of k-element subsets of $\{1,\ldots,n\}$ with the property that no l points lie in more than one $A \in F$. Similarly, let $M(n,k,l)$ denote the minimal size of a family F of k-element subsets of $\{1,\ldots,n\}$ with the property that every l points lie in at least one $A \in F$. Elementary counting arguments give

$$m(n,k,l) \le \frac{\binom{n}{l}}{\binom{k}{l}} \le M(n,k,l)$$

Equality is achieved exactly when there is a family F so that every l points are in precisely one $A \in F$, what is called a tactical configuration. (For example, the case $k=3, l=2$ yields the well known Steiner Triple Systems.) In 1963 Paul Erdős and Haim Hanani [2]conjectured that these bounds were asymptotically achievable, more precisely that for any *fixed* k, l

$$\lim_{n\to\infty} m(n,k,l)\frac{\binom{k}{l}}{\binom{n}{l}} = 1 = \lim_{n\to\infty} M(n,k,l)\frac{\binom{k}{l}}{\binom{n}{l}} \tag{32}$$

This conjecture was proven in 1985 by Vojtech Rödl[4]. In this section we outline a proof that retains Rödl's key ideas but where Talagrand's Inequality allows some simplification.

It will be convenient for us to formulate the Erdős-Hanani conjecture in hypergraph terms. Let G be the complete l-graph on n points. Then $m(n,k,l)$ is the maximal number of disjoint k-cliques that may be packed into G, and $M(n,k,l)$ is the minimal number of k-cliques that cover G. Already in [2] it was known that the left hand limit of 32 is one if and only if the right hand limit is one. We'll restrict attention to the "packing problem", showing $m(n,k,l) \sim \binom{n}{l}/\binom{k}{l}$

4.2 Reductions.

We fix $2 \le l < k$ throughout this section, asymptotics are as $n \to \infty$. Let $0 < p \le 1$ and $\delta \ge 0$.

Definition. We call an l-graph G (p,k,δ)-superquasirandom if for every $l-1 \le t \le k-1$ and every set $x_1,\ldots,x_t$ of distinct vertices of G

$$1-\delta < \frac{|N_G(x_1,\ldots,x_t)|}{(n-t)p^{\binom{t}{l-1}}} < 1+\delta \tag{33}$$

As in §2 we have a problem with overlap. Set $r_k^*(n)$ equal to the maximal size of a family of *disjoint* representatives and set $\mu^* = \mu_k^*(n) = E[r_k^*(n)]$. In [3] is is shown that

$$r_k(n) - r_k^*(n) = O(1) \tag{27}$$

almost surely. We also claim

$$\mu_k^*(n) \sim \mu_k(n) \tag{28}$$

As in §2.2 we can define $r_k^{(i)}(n)$ as the number of isolated representations of n, sandwich $r_k^{(i)}(n) \leq r_k^*(n) \leq r_k(n)$ and calculate $E[r_k^{(i)}(n)] \sim E[r_k(n)]$.

We shall show

$$\Pr[|r_k^*(n) - \mu_k^*(n)| > \frac{1}{2}\mu_k^*(n)] = O(n^{-1.1}) \tag{29}$$

for c, and hence c_0, appropriately large. Given 29 the Borel-Cantelli Lemma yields that almost surely $|r_k^*(n) - \mu_k^*(n)| \leq \frac{1}{2}\mu_k^*(n)$ for all but finitely many n and then 27,28 yield the Theorem.

3.3 Talagrand

Here we use Talagrand's Inequality to show 29. We consider the probability space of choices of $S \cap \{1, \ldots, n\}$ as the product space of the n individual choices and $r^* = r_k^*(n)$ as a random variable on that product space.

• r^* is Lipschitz.

Consider two sets S, S^* on $[n]$ (i.e., elements of the probability space) identical except that S^* has an additional element i. A family of disjoint representatives of n with respect to S^* can have only one representative using i and deleting it gives such a family with respect to S.

• r^* is f-certifiable with $f(s) = ks$.

If $r^* \geq s$ there are s disjoint representatives which together involve ks elements and these elements certify that $r^* \geq s$.

Let $m^* = m_k^*(n)$ denote the median of $r_k^*(n)$. From 14

$$m^* \sim \mu^* \sim c_0(\ln n) \tag{30}$$

and from 16, for any fixed $\epsilon > 0$,

$$\Pr[|r^* - m^*| > \epsilon m^*] < e^{-\Omega(m^*)} = O(n^{-1.1}) \tag{31}$$

for sufficiently large c_0. This yields 29 with room to spare.

3 Representations as $x_1 + \ldots + x_k$

3.1 Notations

Fix a positive integer $k \geq 3$. For a set S of positive integers let $f_k(n) = f_k(n, S)$ denote the number of representations of n as the sum of k distinct elements of S. Our object will be the following result of Paul Erdős and Prasad Tetali [3].

Theorem There is an S and positive constants c_1, c_2 (dependent on k) so that

$$c_1 \ln n < f_k(n) < c_2 \ln n \tag{22}$$

for all sufficiently large n.

When $k = 2$ this result is one of the classic applications of the probabilistic method by Paul Erdős [1].

3.2 Reductions

The results in this section are basically from [3]. Consider a random set S of positive integers for which the events $x \in S$ are mutually independent and

$$\Pr[x \in S] = p_x = c \left(\frac{\ln x}{x^{k-1}} \right)^{1/k} \tag{23}$$

where c is a large constant. (When this $p_x > 1$ set $p_x = 1$.) Now $f_k(n)$ is a random variable with

$$E[f_k(n)] = \sum_{x_1+\ldots+x_k=n} p_{x_1} \cdots p_{x_k} \tag{24}$$

There are $\Theta(n^{k-1})$ terms. Most (though not all!) lie within a constant of $p_n^k = c^k (\ln n) n^{-(k-1)}$. Computation gives

$$E[f_k(n)] \sim K c^k (\ln n) \tag{25}$$

where K depends only on k. Let $\mu = \mu_k(n) = E[f_k(n)]$ for notational convenience. Pick c so that

$$\mu \sim c_0 (\ln n) \tag{26}$$

with c_0 large.

Call an extension g of f *isolated* if there is no other extension g' of f with $g(a) = g'(b)$ for some $a, b \notin R$. Conditioning on g being an extension of f for g to not be isolated there must be $R \subset R' \subset V(H)$ and a g' which is an (R', H)-extension of $g|_{R'}$. But (R', H) is dense ([5], page 106,F) so the expected number of such g' is $o(1)$. Letting $N^{(i)}$ be the number of isolated extensions g, $E[N^{(i)}] \sim E[N]$. As $N^{(i)} \leq N^* \leq N$, 19 follows.

Thus it suffices to show that almost surely $N^* \sim E[N^*]$ for all f. As there are only $O(n^r)$ possible f it suffices to show

$$\Pr[|N^* - E[N^*]| > \epsilon\mu] = o(n^{-r}) \tag{20}$$

for a fixed f.

2.3 Talagrand

Now we employ Talagrand's Inequality. Fix f and (R, H). The probability space $G(n, p)$ is considered as the product space of the choices on the individual edges. N^* is now a random variable on the product space.

• N^* is K'-Lipschitz for $K' = (v - r)(K - 1)$.

Let G, G^* be two graphs on $[n]$ (i.e., elements of the probability space $G(n, p)$.) identical except that G^* has an additional edge w_1, w_2. If $w_1, w_2 \in f(R)$ then $N^*(G) = N^*(G^*)$. Suppose $w_1 \notin f(R)$. Consider a family of extensions of f with respect to G^* that have no $g_1, \ldots, g_K$ with a common value $g_i(a) = v$. At most $(v - r)(K - 1)$ of them can have $w_1 \in g(V(H))$ as otherwise some K would have the same $a \in V(H) - R$ with $g(a) = w_1$. The remaining g form such a family with respect to G.

• N^* is f-certifiable with $f(s) = es$, where (R, H) has type (v, e).

If $N^* \geq s$ then there are extensions $g_1, \ldots, g_s$. The edges of these g_i-copies certify that $N^* \geq s$. Each copy has s edges. While they may overlap, together they have at most es edges.

Applying 16 to N^*

$$\Pr[|N^* - \mu^*| > \epsilon\mu^*] < e^{-\Omega(\mu^*)} \tag{21}$$

As μ^* is at least a positive power of n this quantity is certainly $o(n^{-r})$ completing the proof.

and let $f : R \to V(G)$ be an injection. We say an injection $g : V(H) \to V(G)$ is a set-extension if $g|_R = f$ and is an extension if, in addition,

$$\{x, y\} \in E(H), y \notin R \Rightarrow \{g(x), g(y)\} \in E(G) \tag{17}$$

When this occurs we call the set of such edges $\{g(x), g(y)\}$ the g-copy of H. We let $N = N(f, (R, H), G)$ denote the number of such extensions.

Examples Let H be a triangle on $\{a, b, c\}$ with $R = \{a\}$. Suppose $f(a) = v$. Then N is twice the number of triangles of G containing v. Note that for triangle v, w_1, w_2 we can extend by $g(b) = w_1, g(c) = w_2$ or by $g(b) = w_2, g(c) = w_1$. With the same H let now $R = \{a, b\}$ and suppose $f(a) = v_1, f(b) = v_2$. Then N is the number of common neighbors of v_1, v_2 in G. Note that the edge $\{a, b\}$ is immaterial.

Our next definitions are with respect to a fixed α, $0 < \alpha < 1$.

(R, H) of type (v, e) is *sparse* if $v - e\alpha > 0$

(R, H) of type (v, e) is *dense* if $v - e\alpha < 0$

(R, H) is *safe* if (R', H) is sparse for all $R \subseteq R' \subset V(H)$.

(R, H) is *hinged* if it is safe but for all $R \subset R' \subset V(H)$ (R', H) is not safe.

Now let G be the random graph $G(n, p)$ with $p = n^{-\alpha}$. Our object will be the following result.

Theorem Let (R, H) be safe. Then almost surely

$$N \sim (n - r)_v p^e \sim n^v p^e \sim n^{v - \alpha e} \tag{18}$$

for *all* injections $f : R \to V(G)$.

2.2 Reductions

A weaker form of this result was given in [5] and this result was shown by different means in [6]. Observe that, for fixed $f, (R, H)$, N is a random variable with expectation $(n-r)_v p^e$. The arguments of [5] (Theorem 3, Page 107) allow us to assume (R, H) hinged. (Otherwise, roughly, extensions from R to H split into two safe parts and we use induction.) From [5] (Page 106,condition F; Page 109, Lemma 4) there is a constant K so that almost surely no $f : R \to V(G)$ has extensions $g_1, \ldots, g_K$ with a common value $g_i(a) = v$ where $a \notin R$. Set $N^* = N^*(f, (R, H), G)$ equal the maximal size of a family of extensions g_i of f without such $g_1, \ldots, g_K$. Then almost surely $N^* = N$ for all f. We claim

$$E[N] \sim E[N^*] \tag{19}$$

while for $y \geq m$, $\lambda = y(y+m)^{-1/2} \geq y(2y)^{-1/2}$ and so

$$\Pr[X - m \geq im] \leq 2e^{y/8K^2c} = 2e^{-im/8K^2c} \text{ for } i = 1, 2, \ldots \tag{12}$$

Combining

$$\begin{aligned} E[|X-m|] &\leq 2\sqrt{m}\int_{\lambda=0}^{\infty} \lambda(\lambda/4cK^2)e^{-\lambda^2/4cK^2}d\lambda \\ &+ 2\sqrt{m}\int_{\tau=0}^{\infty} \tau(\tau/8cK^2)e^{-\tau^2/8cK^2}d\tau \\ &+ \sum_{i=0}^{\infty} 2(i+1)me^{-im/8cK^2} \\ &= O(\sqrt{m}) \end{aligned} \tag{13}$$

so that

$$|\mu - m| \leq E[|X-m|] \leq c_1\sqrt{m} \tag{14}$$

for m sufficiently large where c_1 depends only on c, K.

In our applications we're concerned with deviations commensurate with the median. Again suppose h is f-certifiable with $f(s) = cs$ and is K-Lipschitz. Applying 6 with $b = m, \lambda = \epsilon\sqrt{m}$ and then with $b = (1+\epsilon)m, \lambda = \epsilon(1+\epsilon)^{-1/2}\sqrt{m}$ gives

$$\Pr[|X - m| > \epsilon m] < e^{-\Omega(m)} \tag{15}$$

with the constant dependent on c, K, ϵ. In view of 14 we have further

$$\Pr[|X - \mu| > \epsilon\mu] < e^{-\Omega(\mu)} \tag{16}$$

with the constant again dependent only on c, K, ϵ.

2 Extension Statements

2.1 Notations

We follow the notation of [5] which we repeat for convenience. A *rooted graph* is a pair (R, H) consisting of a graph $H = (V(H), E(H))$ and a subset $R \subset V(H)$. The vertices of R are called roots. We say (R, H) has type (v, e) where v is the number of non-root vertices and e is the number of edges of H, excluding edges between two roots. Let $G = (V(G), E(G))$ be any graph

A small generalization will prove useful. Call $h : \Omega \to R$ *K-Lipschitz* if $|h(x) - h(y)| \leq K$ whenever x, y differ in only one coordinate. Applying 3 to h/K and renormalizing we find

$$\Pr[X < b - tK\sqrt{f(b)}] \Pr[X \geq b] \leq e^{-t^2/4} \tag{4}$$

In applications one often takes b to be the median so that for t large the probability of being $t\sqrt{f(b)}$ under the median goes sharply to zero. But it works both ways, by parametrizing so that $d = b - t\sqrt{f(b)}$ is the median one usually gets $b \sim d + t\sqrt{f(d)}$ and that the probability of being $t\sqrt{f(b)}$ above the median goes sharply to zero. Martingales, via Azuma's Inequality, generally produce a concentration result around the mean of X while Talagrand's Inequality yields a concentration result about the median. Means tend to be easy to compute, medians notoriously difficult, but our tight concentration result will allow us to show that the mean and median are not far away.

Let us suppose h is f-certifiable for $f(s) = cs$ and K-Lipschitz. Then the random variable $X = h(\cdot)$ satisfies

$$\Pr[X \leq b - \lambda K\sqrt{c}\sqrt{b}] \Pr[X \geq b] \leq e^{-\lambda^2/4} \tag{5}$$

or, changing parameters,

$$\Pr[X \leq b - \lambda\sqrt{b}] \Pr[X \geq b] \leq e^{-\lambda^2/4K^2c} \tag{6}$$

for all b, λ. Let $\mu = E[X]$, the mean of X, and let m denote the median of X. Then

$$|\mu - m| = |E[X - m]| \leq E(|X - m|) \tag{7}$$

Setting $b = m$

$$\Pr[X - m \leq -\lambda\sqrt{m}] \leq 2e^{-\lambda^2/4K^2c} \tag{8}$$

On the other side let $y = y(\lambda)$ denote the solution to

$$y + m - \lambda\sqrt{y + m} = m \tag{9}$$

so that

$$\Pr[X - m \geq y] \leq 2e^{-\lambda^2/4K^2c} \tag{10}$$

For $y \leq m$, $\lambda = y(y + m)^{-1/2} \geq y(2m)^{-1/2}$ so

$$\Pr[X - m \geq \tau\sqrt{m}] \leq 2e^{-\tau^2/8K^2c} \text{ for } 0 \leq \tau \leq \sqrt{m} \tag{11}$$

The notation takes some getting used to. Suppose $\Omega_i = \{0, 1\}$ with the uniform distribution so that Ω is the Hamming w-cube with uniform distribution. If $y \in A_t$ then, taking all $\alpha_i = 1$, y must be within Hamming distance $t\sqrt{w}$ of A. In this sense the inequality is remniscent of isoperimetric inequalities. However there are certainly further conditions on membership in A_t that make this inequality far stronger.

In [8] Talagrand developes numerous applications of this Inequality. There he chooses to make all derivations directly from the Inequality. Here we take the opposite tact, designing a convenient framework into which all our applications can be placed. We should emphasize that this format is implicit in [8] and that the specific notations were the joint effort of several mathematicians visiting IMA in Fall 1993, including Svante Janson, Eli Shamir and Michael Steele. Let $h : \Omega \to R$. We call h *Lipschitz* if $|h(x) - h(y)| \leq 1$ for all $x, y \in \Omega$ which differ in only one coordinate. For $f : N \to N$ (e.g., $f(b) = b$) we call h f-certifiable if whenver $h(x) \geq s$ there exists an index set $I \subseteq \{1, \ldots w\}$ with $|I| \leq f(s)$ so that all $y \in \Omega$ that agree with x on the coordinates I have $h(y) \geq s$. (I.e., there will be a set of coordinate values of size $f(s)$ that will certify that $h \geq s$.) Let h satisfy the above and consider the random variable $X = h(\cdot)$.

Corollary. Under the above assumptions and for all b, t

$$\Pr[X \leq b - t\sqrt{f(b)}]\Pr[X \geq b] \leq e^{-t^2/4} \tag{3}$$

Proof. Set $A = \{x : h(x) < b - t\sqrt{f(b)}\}$. Now suppose $h(y) \geq b$. We claim $y \notin A_t$. Let I be a set of indices of size at most $f(b)$ that certifies $h(y) \geq b$ as given above. Define $\alpha_i = 0$ when $i \notin I$, $\alpha_i = 1$ when $i \in I$. If $y \in A_t$ there exists a $z \in A$ that differs from y in at most $t\sqrt{f(b)}$ coordinates of I though at arbitrary coordinates outside of I. Let y' agree with y on I and agree with z outside of I. By the certification $h(y') \geq b$. Now y', z differ in at most $t\sqrt{f(b)}$ coordinates and so, by Lipschitz,

$$h(z) > h(y') - t\sqrt{f(b)} \geq b - t\sqrt{f(b)}$$

but then $z \notin A$, a contradiction. So $\Pr[X > b] \leq 1 - \Pr[A_t]$ so from 2.

$$\Pr[X < b - t\sqrt{f(b)}]\Pr[X \geq b] \leq e^{-t^2/4}$$

As the right hand side is continuous in t we may replace $<$ by $\leq$ giving the Corollary. $\Box$

Applications of Talagrand's Inequality

Joel Spencer

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Probability Theorems with application to probabilistic methods are rare gems. One thinks of the Lovász Local Lemma, the Janson Inequalities, Azuma's martingale inequality. In all those cases the results themselves can be stated purely in the language of (mostly) elementary probability theory. There strength is in their applicability. They are convenient tools – one needn't go back to the original proofs of these results. They fit many naturally occuring problems and their application requires only relatively straightforward calculations. A new inequality due to Michel Talagrand ([7], esp. equation (1.3)) seems also in that mode. In [8] Talagrand himself gives numerous applications of his inequality, mostly to random structures per se. Here we discuss applications involving probabilistic methods.

In §1 we state, without proof, Talagrand's Inequality. We then give a consequence of it in a form that seems particularly useful. We also give some technical results that will allow for easier applicability. In §2,3,4 we give three separate applications. In all cases these are known results with new and, we believe, simpler proofs.

1 Talagrand.

Let $\Omega = \prod_{i=1}^{w} \Omega_i$ be a product probability space, we write x_i for the i^{th} coordinate of an $x \in \Omega$. For $A \subset \Omega$ and t an arbitrary positive real we define A_t by saying $y \in A_t$ if and only if for all $\alpha_1, \ldots, \alpha_w$ there exists an $x \in A$ with

$$\sum_{x_i \neq y_i} \alpha_i < t \left(\sum_{1 \leq i \leq w} \alpha_i^2 \right)^{1/2} \tag{1}$$

Talagrand's Inequality:

$$\Pr[A](1 - \Pr[A_t]) \leq e^{-t^2/4} \tag{2}$$