Fundamental Algorithms
OLD FINAL EXAM

1. (20) Describe the algorithm TOPDOLLAR(PRICE, N). The input is an array of nonnegative real numbers PRICE[1 · · · N] where PRICE[I] represents the price for a rod of length I. Your output should be VALUE, which should be the optimal total price among all cuttings of the rod. You may, and should, use an auxiliary array. You may write in pseudocode or actual code but in either case you must give clear comments describing what you are doing and why it works. Analyze (just the answer will not suffice!), in Θ-land, the time for your algorithm.

2. (5) For n large which is faster: Θ(lg^3 n) or Θ(√n) algorithm?

3. (15) Given A[1 · · · N] with 0 ≤ A[I] < N^N for all I.
   (a) How long will COUNTING-SORT take?
   (b) How long will RADIX-SORT take using base N?
   (c) Give an algorithm (no proof required!) that does better than both of the above, and does so in worst case.

4. (5) In a max Heap what is the property of the value at the root? (That is, how does it compare to the values of the other nodes.)

5. (20) Let G be a connected graph on vertex set V. Let a nonnegative weight w(e) be assigned to every edge e. Let S ⊂ V with S ≠ θ and S ≠ V.
   (a) Prove (yes, prove!) that the Minimal Spanning Tree necessarily contains that edge e = {x, y} with x ∈ S and y ∉ S which has minimal weight. A good picture won’t hurt. Assume no two edges have the same weight. (Warning: Saying that an algorithm tells you to pick this edge e is not a proof!)
   (b) In what algorithm is the above result used.

6. (10) Consider the recursion T(2n) = 3T(n) + n + 1 with initial value T(1) = 5.
   (a) What is T(4)?
   (b) What it T(n) in Θ-land.
7. (20) If multiplication mod $p$ takes 1 second then show, using Island-Hopping, how to calculate $x^{1023}$ quickly. How many seconds did it take? (Hint: What is special about 1023?)

8. (20) Find the Huffman code on letters $a, b, c, d, e, f$ with frequencies $0.13, 0.16, 0.09, 0.45, 0.05, 0.12$ respectively. Give pictures and words clearly indicating the intermediate steps in finding the code.

9. (5) State the Chinese Remainder Theorem.

10. (15) We call a positive integer $n$ SQUAREFREE if there is no integer $d > 1$ with $d^2$ dividing $n$. We call $n$ NOTSQUAREFREE if $n$ is not SQUAREFREE.

   (a) Argue NOTSQUAREFREE is in NP.
   (b) (This is harder!) Argue SQUAREFREE is in NP.

11. (5) What was the breakthrough of Agrawal, Kayal and Saxena?

12. (20) In Depth-First Search each vertex $v$ gets a discovery time $d[v]$ and a finishing time $f[v]$. Let $G$ be a graph and $v, w$ two distinct vertices for which $w \in \text{Adj}[v]$.

   (b) Now further assume that $G$ is a DAG (a Directed Acyclic Graph.) Assume $d[w] < d[v]$. Argue that $f[w] < f[v]$.

13. (5) What is the difference, if any, between a $\Theta(\lg n)$ algorithm and a $\Theta(\log_{10} n)$ algorithm? (Short reason please.)