FUNDAMENTAL ALGORITHMS MIDTERM

What is slight can still be great, if it is written in a natural, 
flowing and easy style - and at the same time bears the mark of sound composition.
– letter to Wolfgang Mozart from his father, 1778

Total Points 110. Do all problems.

1. (20) These questions concern a max-heap $A$ with auxilliary structures $\text{length}[A]$, $\text{Left}[i]$, $\text{Right}[i]$ and $\text{Parent}[i]$.

   (a) (5) Give the max-heap property. (That is, what is the relationship between the values $A[i]$ that one needs for a heap.)

   (b) (5) Suppose $\text{length}[A]=100$ and the values $A[i]$ are distinct. What are the possible $i$ for which $A[i]$ is the second largest value? (You must give all possible $i$.)

   (c) (10) Describe the algorithm $\text{MAX-HEAP-INSERT}(A,\text{key})$ which adds to heap $A$ a new entry with value $\text{key}$. (Either a description in words or a pseudocode program would be fine.)

2. (15) These questions concern $\text{QUICKSORT}$. We will take $\text{PARTITION}(A,p,r)$ as given. (This procedure rearranges $A[p\ldots r]$, placing all entries less than $A[r]$ to the left and all entries greater than $A[r]$ to the right. You are not being asked to write it!!)

   (a) (5) Give the recursive procedure for $\text{QUICKSORT}[A,p,r]$.

   (b) (10) Let $T(n)$ denote the average time for procedure $\text{QUICKSORT}$ on an array of length $n$. Give a recursive formula for $T(n)$ and (important!) give a description in words of where that recursive formula comes from.
3. (15) Consider the following procedure (* is multiplication):

FOR I=1 TO N
    J=I
    WHILE J ≤ N
        do J=2*J
    END WHILE
END FOR

(a) (5) How many times is the WHILE step reached for a given I and N

(b) (5) Write the total number of times the WHILE step is reached as a summation.

(c) (5) Evaluate that sum in Θ-land.

4. (20) These questions concern a binary search tree $T$ in which each node $x$ has a key value denoted $key[x]$. We assume $\text{Root}[T], \text{left}[x], \text{right}[x], \text{parent}[x]$ are part of the structure.

(a) (5) State the binary-search-tree property. (That is, the condition that the keys are required to fulfill.)

(b) (5) Suppose the tree has a million records. What is the smallest the height of the tree (recall this is defined as the maximal distance from the root to a leaf) can be? (Give a precise integer!)

(c) (5) Describe a procedure MIN that returns that node $x$ with minimal value of $key[x]$.

(d) (5) Suppose $T$ has height $h$. In the (warning: a bit tricky!) best-case, how long would your MIN procedure above take. Give a nice picture of $T$ in that best case.
5. (25) Prof. Squander decides to do Bucket Sort on \( n \) items with \( n^2 \) buckets while his student Ima Hogg decides to do Bucket Sort on \( n \) items with \( n^{1/2} \) buckets. Assume that the items are indeed uniformly distributed. Assume that Ima’s algorithm for sorting inside a bucket takes time \( O(m^2) \) when the bucket has \( m \) items.

(a) (10) Argue that Prof. Squander has made a poor choice of the number of buckets by analyzing the time of Bucket Sort in his case.

(b) (10) Argue that Ima has made a poor choice of the number of buckets by analyzing the time of Bucket Sort in her case.

(c) (5) Argue that Ima uses roughly the same amount of space as someone using \( n \) buckets.

6. (15) Let \( x_1x_2 \cdots x_n, y_1y_2 \cdots y_n \) be two sequences of zeroes and ones. For \( 1 \leq a \leq n \) and \( 1 \leq b \leq n \) let \( LCS[a, b] \) denote the length of the longest common subsequence of \( x_1 \cdots x_a \) and \( y_1 \cdots y_b \).

(a) (5) Give an efficient algorithm to find \( LCS[a, 1] \) for \( 1 \leq a \leq n \). How long (in \( \Theta \)-land) does your algorithm take.

(b) (10) Assume \( LCS[a, 1], 1 \leq a \leq n \) and \( LCS[1, b], 1 \leq b \leq n \) are already known. For \( 2 \leq a, b \leq n \) give a recursion for \( LCS[a, b] \) in terms of \( LCS[a, b-1], LCS[a-1, b] \) and \( LCS[a-1, b-1] \). Give a clear argument for why this recursion is correct.

The computer is useless. It can only answer questions.
– Pablo Picasso