Basic Algorithms, Assignment 11

Solutions

1. Consider Prim’s Algorithm for MST on the complete graph with vertex set \{1, \ldots, n\}. Assume that edge \{i, j\} has weight \|j - i\|^3. Let the root vertex \(r = 1\). Show the pattern as Prim’s Algorithm is applied.

Solution: The set \(S\), initially \{1\}, will grow to \{1, 2\}, \ldots, \{1, 2, \ldots, i\}, \ldots, \{1, \ldots, n\}. When \(S = \{1, \ldots, i\}\) the closest point to \(S\) will be \(i + 1\) with \(\pi[i + 1] = i\) and \(key[i + 1] = 1\). In particular, Let \(n = 500\) and consider the situation when the tree created has 211 elements and \(\pi\) and \(key\) have been updated.

(a) What are these 211 elements.
Solution: 1, \ldots, 211

(b) What are \(\pi[309]\) and \(key[309]\).
Solution: \(\pi[309] = 211\) (all other of 1, \ldots, 202 are further) and \(key[309] = (309 - 211)^3\).

2. Find \(d = \gcd(144, 89)\) and \(x, y\) with \(144x + 89y = 1\). [Remark: This is part of a pattern with two consecutive numbers from the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots]

Solution:

\[
\text{EUCLID}(144, 89) = \\
\text{EUCLID}(89, 55) = \text{EUCLID}(55, 34) = \text{EUCLID}(34, 21) = \\
= \text{EUCLID}(21, 13) = \text{EUCLID}(13, 8) = \text{EUCLID}(8, 5) = \\
= \text{EUCLID}(5, 3) = \text{EUCLID}(3, 2) = \text{EUCLID}(2, 1) = \\
= \text{EUCLID}(1, 0) = 1
\]

with all quotients 1 except the last. For \(\text{EXTENDED - EUCLID}\) we get a chart like Figure 31.1:
so $x = 34$ and $y = -55$. (Note that the $x$’s and $y$’s form a Fibonacci like pattern as well!)

3. Find $\frac{311}{507}$ in $\mathbb{Z}_{1000}$.

Solution: Here we first find $\text{EUCLID}(1000, 507)$:

$$
\text{EUCLID}(1000, 507) = \text{EUCLID}(507, 493) = \text{EUCLID}(493, 14) =
$$

$$
= \text{EUCLID}(14, 3) = \text{EUCLID}(3, 2) = \text{EUCLID}(2, 1) =
$$

$$
= \text{EUCLID}(1, 0) = 1
$$

For EXTENDED - EUCLID we get a chart like Figure 31.1:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$\lfloor a/b \rfloor$</th>
<th>$d$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>507</td>
<td>1</td>
<td>1</td>
<td>181</td>
<td>-357</td>
</tr>
<tr>
<td>507</td>
<td>493</td>
<td>1</td>
<td>1</td>
<td>-176</td>
<td>181</td>
</tr>
<tr>
<td>493</td>
<td>14</td>
<td>1</td>
<td>35</td>
<td>5</td>
<td>-176</td>
</tr>
<tr>
<td>14</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>-1</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

so that $1000(181) - 357(507) = 1$ so in $\mathbb{Z}_{1000}$ we have $(-357)(507) = 1$ so $\frac{1}{507} = -357 = 643$. Finally $\frac{311}{507} = 311 \cdot 643 = 199973 = 973$. So the answer is 973. (You might prefer to write it as $-27$ which is the same.) To check: $973 \cdot 507 = 493311 = 311$. 

\[\text{EUCLID}(1000, 507) = \text{EUCLID}(507, 181) = \text{EUCLID}(181, 7) = \text{EUCLID}(7, 4) = \text{EUCLID}(4, 3) = \text{EUCLID}(3, 1) = \text{EUCLID}(1, 0) = 1\]
4. Solve the system
\[ x \equiv 34 \mod 101 \]
\[ x \equiv 59 \mod 103. \]
Solution: We write \( x = 103y + 59 \) (we could start with either and this one is a bit easier) so that in \( \mathbb{Z}_{101} \) we want \( 103y + 59 = 34 \) or \( 2y = -25 = 76 \) and \( y = 38 \). (Usually division is complicated but here it worked out like normal division.) Then \( x = 103(38) + 59 = 3973 \). The general answer is given as \( x \equiv 3973 \mod 10403 \) as \( 10403 = 103 \cdot 101 \).

5. Using the Island-Hopping Method to find \( 2^{1072} \) modulo 1073 using a Calculator but NOT using multiple precision arithmetic.
Solution:
\[
\begin{align*}
2^1 &= 2 \\
2^2 &= 2 \cdot 2 = 4 \\
2^4 &= 4 \cdot 4 = 16 \\
2^8 &= 16 \cdot 16 = 256 \\
2^{16} &= 256 \cdot 256 = 65536 = 83 \\
2^{32} &= 83 \cdot 83 = 6889 = 451 \\
2^{64} &= 451 \cdot 451 = 203401 = 604 \\
2^{128} &= 604 \cdot 604 = 364816 = 1069 = -4 \\
2^{256} &= (-4) \cdot (-4) = 16 \\
2^{512} &= 16 \cdot 16 = 256 \\
2^{1024} &= 256 \cdot 256 = 65536 = 83
\end{align*}
\]
As 1072 is 10000110000 base two, 1072 = 1024 + 32 + 16 and we have
\[ 2^{1072} = 83 \cdot 451 \cdot 83 \]
We calculate in stages: \( 83 \cdot 451 = 37433 = 951, 951 \cdot 83 = 78933 = 604 \) So the answer is 604. This shows that 1073 is definitely not a prime. (Of course, for numbers this small there are easier ways!)

6. Suppose that during Kruskal’s Algorithm (for MST) and some point we have \( \text{SIZE}[v] = 35 \). What is the interpretation of that in the case when \( \pi[v] = v \)?
Solution: At that moment \( v \) is in a component of size 35 and it is the root of the associated tree.
What is the interpretation of that in the case when $\pi[v] = u \neq v$?

**Solution:** $v$ had had size 35 at the moment when $\pi[v]$ was changed, and the component with $v$ was joined to the (larger) component with $u$.

How many different values can $\pi[w]$ have during the course of Kruskal’s algorithm?

**Solution:** Two. Initially $\pi[w] = w$ but once it changes to $\pi[w] = v$ it doesn’t change any more. Precisely one vertex does not ever change, it becomes the root of the final rooted tree.

How many different values (as a function of $V$, the number of vertices) can $\text{SIZE}[w]$ have during the course of Kruskal’s algorithm?

**Solution:** $V$. It is possible that $w$ is joined to isolated vertices $V - 1$ times and so $\text{SIZE}[w]$ goes from 1 to $V$ by ones.