OLD FINAL EXAM
SOLUTIONS

1. (20) Describe the algorithm `TOPDOLLAR(`$PRICE$`,`N`)`. The input is an array of nonnegative real numbers $PRICE[1 \cdots N]$ where $PRICE[I]$ represents the price for a rod of length I. Your output should be $VALUE$, which should be the optimal total price among all cuttings of the rod. You may, and should, use an auxilliary array. You may write in pseudocode or actual code but in either case you **must** give clear comments describing what you are doing and *why* it works. Analyze (just the answer will not suffice!), in Θ-land, the time for your algorithm.
   `Solution:`This is the `RODCUTTER` algorithm, as discussed in the text and in class.

2. (5) For n large which is faster: $\Theta(\lg^3 n)$ or $\Theta(\sqrt{n})$ algorithm?
   `Solution:`$\Theta(\lg^3 n)$. Any power of the log grows slower than any power of n so that the time is less so it is faster.

3. (15) Given $A[1 \cdots N]$ with $0 \leq A[I] < N^N$ for all I.

   (a) How long will `COUNTING-SORT` take?
   `Solution:`$O(N^N)$. The main part is creating the array of lenght N^N.

   (b) How long will `RADIX-SORT` take using base N?
   `Solution:`N applications of `COUNTING-SORT` on N, each one takes $O(N)$, so total $O(N^2)$.

   (c) Give an algorithm (no proof required!) that does better than both of the above, and does so in worst case.
   `Solution:`Two possibles are `MERGESORT` and `HEAPSORT` but not `QUICKSORT` which does badly in worst case.

4. (5) In a max Heap what is the property of the value at the root? (That is, how does it compare to the values of the other nodes.)
   `Solution:`It is the biggest value.

5. (20) Let G be a connected graph on vertex set V. Let a nonnegative weight $w(e)$ be assigned to every edge e. Let $S \subset V$ with $S \neq \emptyset$ and $S \neq V$.

   (a) Prove (yes, prove!) that the Minimal Spanning Tree necessarily contains that edge $e = \{x, y\}$ with $x \in S$ and $y \notin S$ which has

minimal weight. A good picture won't hurt. Assume no two edges have the same weight. (Warning: Saying that an algorithm tells you to pick this edge e is **not** a proof!)
`Solution:` Take the MST T not containing e. There is a path from x to y in the tree and at some point an edges must cross from some $v \in S$ to some $w \notin S$. Replace $\{v, w\}$ by $\{x, y\}$ to get a tree with smaller weight.

(b) In what algorithm is the above result used.
`Solution:` Prim's Algorithm.

6. (10) Consider the recursion $T(2n) = 3T(n) + n + 1$ with initial value $T(1) = 5$.

   (a) What is $T(4)$?
   `Solution:` With $n = 1$, $T(2) = 3 \cdot 5 + 1 + 1 = 17$ so with $n = 2$, $T(4) = 3 \cdot 17 + 2 + 1 = 54$.

   (b) What is $T(n)$ in Θ-land.
   `Solution:` This is low overhead case and $T(n) = n^\alpha$ with $\alpha = \log_2 3$.

7. (20) If multiplication mod p takes 1 second then show, using Island-Hopping, how to calculate x^{1023} quickly. How many seconds did it take? (Hint: What is special about 1023?)
   `Solution:` $1023 = 2^{10} - 1$, in binary it is all ones. Square x nine times getting x^{2^i} for $1 \leq i \leq 9$. Multiply these and x together getting x^{1023}. The last takes nine seconds so a total of eighteen seconds.

8. (20) Find the Huffman code on letters a, b, c, d, e, f with frequencies $.13, .16, .09, .45, .05, .12$ respectively. Give pictures and words clearly indicating the intermediate steps in finding the code.
   `Solution:` Assignment.

9. (5) State the Chinese Remainder Theorem.
   `Solution:` When m, n are relatively prime and a, b arbitrary there is a solution to the system $x \equiv a \bmod m$ and $x \equiv b \bmod n$.

10. (15) We call a positive integer n `SQUAREFREE` if there is no integer $d > 1$ with d^2 dividing n. We call n `NOTSQUAREFREE` if n is not `SQUAREFREE`.

    (a) Argue `NOTSQUAREFREE` is in `NP`.
    `Solution:` Oracle gives d. Verfifier calculates d^2 and divides it into n.

(b) (This is harder!) Argue `SQUAREFREE` is in `NP`.
`Solution:` Oracle gives the prime factorization of n. Verifier checks that the mutiplication comes out to n and, via AKS, that all the "primes" really are primes, and that all of the primes were distinct and to the first power.

11. (5) What was the breakthrough of Agrawal, Kayal and Saxena?
`Solution:` Polynomial Test for Primality.

12. (20) In Depth-First Search each vertex v gets a discovery time $d[v]$ and a finishing time $f[v]$. Let G be a graph and v, w two distinct vertices for which $w \in Adj[v]$.

    (a) Assume $d[v] < d[w]$. Argue that $f[w] < f[v]$.
    `Solution:` During `DFS-VISIT[v]`, w will be found so that `DFS-VISIT[w]` will be a subroutine of `DFS-VISIT[v]` and so will end first.

    (b) Now further assume that G is a DAG (a Directed Acyclic Graph.) Assume $d[w] < d[v]$. Argue that $f[w] < f[v]$.
    `Solution:` As $w \in Adj[v]$ and G is a DAG there is no path from w to v. Thus `DFS-VISIT[w]` does not find v and so is completed before v is discovered, hence before it is finished.

13. (5) What is the difference, if any, between a $\Theta(\lg n)$ algorithm and a $\Theta(\log_{10} n)$ algorithm? (*Short* reason please.)
`Solution:` None. $\lg n$ and $\log_{10} n$ differ by a constant factor and so are the same in Θ-land.