Now, every time I witness a strong person, I want to know: What darkness did you conquer in your story? Mountains do not rise without earthquakes.

Katherine MacKenett

1. (10) Suppose Dijkstra’s Algorithm with source vertex $s$ has been completed, giving a parent $\pi(v)$ and a distance $d(v)$ for each $v$. Now you want to go from $s$ to input $v$ via the shortest path. Give a (short!) algorithm that returns that $w$ which is the first node on the optimal path from $s$ to $v$.

2. (15) Consider a “Dumb Prim” on the complete graph on $n$ vertices with weight function $w(x, y)$. In Dumb Prim given the set $S$ we check all edges $x, y$ with $x \in S, y \not\in S$. With $\{x, y\}$ being the edge of minimal weight of that type we add $y$ to $S$ and $\{x, y\}$ to the MST. Analyze, in $\Theta$-land, how long Dumb Prim would take.

3. (20) Consider an array $A$ of length $n$ where $A[i]$ are integers, $0 \leq A[i] < n^2$.
   (a) (5) How long would Counting Sort take?
   (b) (5) How long would Merge Sort take?
   (c) (5) How long would Radix Sort using base 2 (that is, binary) take?
   (d) (5) How long would Radix Sort using base $n$ take?

Give brief reasons for your answers!

4. (5) Aashish is asked by Tata to keep track of its 700000 employees. He sets up a hash table of size $10^7$. He uses a random number generator to place each record in a position – and if it is filled he keeps trying until he finds one that is empty. His bosses are happy with him for a couple of days but then he is summarily sacked. What was wrong (briefly) with his solution?

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In the U.S. we would say “fired”
5. (20) Let $G$ be a weighted directed graph with all weights positive. Set $s$ have adjacency list $a, b, c$ and assume $w(s, a) = 30, w(s, b) = 20, w(s, c) = 50$.

(a) (15) Prove that the path from $s$ to $b$ of minimal weight is the arc $(s, b)$. (A good picture will help – but quoting an algorithm will not be a proof.)

(b) (5) Give a simple example where the path from $s$ to $a$ of minimal weight is not the arc $(s, a)$.

6. (15) Call a positive integer $n$ lejia if there it has a prime divisor $p$ of the form $p = 4k + 3$. Let $LEJIA$ be be the language (or set) of all lejia integers. Let $NOTLEJIA$ be the language (or set) all all integers which are not lejia.

(a) (5) Show that $LEJIA \in NP$.

(b) (10) (This is harder!) Show that $NOTLEJIA \in NP$.

In both cases describe the certificate supplied by the Oracle and the protocol that the Verifier will use. You may use Agarwal/Kayal/Saxena but be explicit in how it is used.

7. (20) Using the Extended-Euclid Algorithm find $d = \gcd(100, 31)$ and $x, y$ with $100x + 31y = d$. Show all work. (Simply “seeing” the solution does not receive credit!)

8. (10) Consider a MAX-HEAP $A$ with heapsize 1023 and all values $A[i]$ different.

(a) (5) Give all possible $i$ such that $A[i]$ is the smallest value.

(b) (5) Reset $A[1]$ to $A[1023]$ and heapsize to 1022. What is the most number of exchanges when $HEAPIFY[A, 1]$ is now called?

9. (5) Which is faster (or are they both the same) when $n$ is large: A $\Theta(n^2)$ algorithm or a $\Theta(n \log^4 n)$ algorithm. (Brief reason please!)

10. (10) Let $\epsilon = e^{2\pi i / n} = \cos(2\pi/n) + i \sin(2\pi/n)$. Let $p(x)$ be a polynomial of degree less than $n$. Write $p(0)$ as a simple expression in terms of $p(1), p(\epsilon), \ldots, p(\epsilon^{n-1})$. 
11. (10) Let $G$ be a DAG [Directed Acyclic Graph]. Suppose $w \in Adj[v]$. Consider a running of DFS. Suppose that when $w$ turns Grey that $v$ is White. What will be the color of $v$ when $w$ turns Black? (Reason, please! A good picture will help.)

12. (10) Assume modulo $m$ multiplication of two numbers can be done in one nanosecond\(^2\). Let $s$ be a positive integer, at most $10^{100}$, given in binary form. Show how $3^s \mod m$ can be computed in less than a microsecond.

13. (10) While stuck in quarantine Shaojun finds a method to find the median of $n$ numbers with $8n$ comparisons. Shaojun creates ShaoSort, finding the median, using it as a pivot, and then recursively applying ShaoSort to the two sides. Give a recursive equation for $T(n)$, the number of total comparisons used in ShaoSort. Use the Master Theorem (brief reason, please!) to solve the equation in $\Theta$-land. (Note: Ignore floors and ceilings and $-1$ terms.)

14. (10) In Kruskal’s Algorithm suppose $\{x, y\}$ is the next edge to be considered. Describe in clear words how it is determined if the edge is added to the MST. Illustrate with two nice (simple!) pictures of cases when the edge is and is not accepted.

Far and away the best prize that life offers is the chance to work hard at work worth doing.
Theodore Roosevelt, 1903

\(^2\text{nanosecond} = 10^{-9} \text{ seconds}; \text{microsecond} = 10^{-6} \text{ seconds}\)