1. Assume $3\alpha^5 + 17\alpha^2 + 8 = 0$. Find an explicit integer $r$ such that, setting $\beta = r\alpha$, $\beta$ satisfies a monic quintic – and give the quintic explicitly.

Solution: $r = 3$. So $3(\beta/3)^5 + 17(\beta/3)^2 + 8 = 0$ and, multiplying out, $\beta^5 + 163\beta^3 + 648 = 0$.

2. Let $[K : F] = n$ and let $v_1, \ldots, v_n$ be a basis for $K$ as a vector space over $F$. Let $0 \neq \beta \in K$. Prove that $v_1\beta, \ldots, v_n\beta$ is a basis for $K$ over $F$.

Solution: We first claim $v_1\beta, \ldots, v_n\beta$ forms an independent set. If $\sum c_i(v_i\beta) = 0$ then, dividing by $\beta$, $\sum c_i v_i = 0$ so all $c_i = 0$. But now we have a vector space of dimension $n$ and a set of $n$ vectors $v_1\beta, \ldots, v_n\beta$ which are an independent set and therefore they must be a basis.

3. Let $F \subset \Omega$, $\alpha, \beta \in \Omega$, $[F(\alpha) : F] = m$, $[F(\beta) : F] = n$. Assume $m, n$ are relatively prime. Prove that $[F(\alpha, \beta) : F] = mn$.

Solution: We showed in class that $[F(\alpha, \beta) : F(\alpha)] \leq [F(\beta) : F] = n$ so by the Tower Theorem $F(\alpha, \beta) : F = [F(\alpha) : F]\cdot[F(\alpha, \beta) : F(\alpha)] = mn$. As $F \subset F(\alpha) \subset F(\alpha, \beta)$, $m = [F(\alpha) : F]\cdot[F(ah, \beta) : F]$ and similarly $n = [F(\beta) : F]\cdot[F(ah, \beta) : F]$. As $m, n$ relatively prime $mn = [F(ah, \beta) : F]$.

4. Let $p(x) \in Q[x]$ be irreducible of degree $n$, $n$ odd. Let $\alpha_1, \ldots, \alpha_n$ be the complex roots of $p(x)$. (You can assume these are distinct. We shall prove this in a more general form later in the course.) Show that $p(x)$ is irreducible in $Q(\sqrt{3})[x]$.

Solution: Suppose $p(x)$ has an irreducible factor $q(x)$ of degree $r$. Then $q(\alpha_i) = 0$ for some $i$. So $Q(\sqrt{3})(\alpha_i) : Q(\sqrt{3}) = r$. So $Q(\sqrt{3}, \alpha_i) : Q = 2r$. So $n = Q(\alpha_i) : Q|2r$. But $r < n$ and $n$ is odd, a contradiction.

5. Let $Q = K_0 \subset K_1 \subset \ldots \subset K_t$ be a tower of fields where for each $1 \leq i \leq t$ either

(a) $K_i = K_{i-1}(\beta_i)$ with $\beta_i^2 \in K_{i-1}$
(b) or $K_i = K_{i-1}(\beta_i)$ with $\beta_i^3 \in K_{i-1}$

Give a result about the possible $[K_t : Q]$. Prove that $2^{1/5} \not\in K_t$.

Solution: Each extension $[K_i : K_{i-1}] \in \{1, 2, 3\}$ so by the tower the-
orem \([K_t : K] = 2^a 3^b\) with \(a + b \leq t\). As \(x^5 - 2\) is irreducible (Eisenstein!), were \(2^{1/5} \in K_t\) we would have \(5 = [Q(2^{1/5}) : Q][K_t : K] = 2^a 3^b\), a contradiction.

6. Let \(p(x) = x^6 + ax^4 + bx^2 + c \in \mathbb{Q}[x]\). Let \(\alpha, \beta, \gamma, \delta, \epsilon, \mu\) be the complex roots of \(p(x)\). Prove \([Q(\alpha, \beta, \gamma, \delta, \epsilon, \mu) : Q] \leq 48\). (Hint: The special nature of \(p(x)\) yields a relationship between the roots. Partial credit for other upper bounds.)

**Solution:** The roots (reordering for convenience) can be written \(\pm \alpha, \pm \beta, \pm \gamma\).

\([Q(\alpha) : Q] \leq 6\). Over \(Q(\alpha)\), \(\pm \beta, \pm \gamma\) satisfy the quartic \(p(x)/(x^2 - \alpha^2)\).

So \([Q(\alpha, \beta) : Q(\alpha)] \leq 4\). Over \(Q(\alpha, \beta)\) so \([Q(\alpha, \beta, \gamma) : Q(\alpha \beta)] \leq 2\). By the Tower Theorem \([Q(\alpha, \beta, \gamma) : Q] \leq 6 \cdot 4 \cdot 2 = 48\).