We examine the number of solutions to the equation $x^2 + y^2 = n$ with $x, y \in \mathbb{Z}$. We call $\alpha = x + iy$ the related Gaussian integer. Call two solutions equivalent if you can get from one to the other by adding minus signs and/or flipping $x, y$. For example, $(3, 11)$ and $(11, -3)$ are the same. We actually count the solutions up to equivalence. (While we don’t quite do it, these methods yield the answer for all $n$.)

1. Show (easy!) $(x, y)$ is a solution iff $\alpha \overline{\alpha} = n$. Now write $\alpha \sim \alpha'$ or $\overline{\alpha} \sim \alpha'$. ($\overline{\alpha}$ denotes the complex conjugate and $\alpha \sim \beta$ means $\beta = u\alpha$, $u$ a unit.) Show that two solutions $(x, y), (x', y')$ are equivalent iff their related $\alpha, \alpha'$ have $\alpha \equiv \alpha'$.  

Solution: With $\alpha$ related to $(x, y)$ we have $i\alpha, -\alpha, -i\alpha$ related to $(-y, x), (-x, -y), (y, -x)$, $\overline{\alpha}$ to $(x, -y), i\overline{\alpha}, -i\overline{\alpha}$ to $(y, x), (-x, y), (-y, -x)$.

2. (This problem counts double!) Let $n = p_1 \cdot p_r$ where the $p_i$ are all integer primes and all are of the form $4k + 1$. In $\mathbb{Z}[i]$ write each $p_i = \alpha_i \beta_i$ with $\beta_i = \overline{\alpha_i}$. $\gamma = \gamma_1 \cdots \gamma_r$ with each $\gamma_i \in \{\alpha_i, \beta_i\}$. Note there are $2^r$ choices here.

(a) Setting $\gamma = x + iy$ show that $x^2 + y^2 = n$.

Solution: $\gamma \overline{\gamma} = \prod_i \alpha_i \beta_i = \prod_i p_i = n$.

(b) Show that if $x^2 + y^2 = n$ then there is such a $\gamma = x + iy$.

Solution: Setting $\kappa = x + iy$ we have $\kappa \overline{\kappa} = n$. In $\mathbb{Z}[i]$, $n = \prod_i \alpha_i \overline{\alpha_i}$. By Unique Factorization, $\kappa$ must have precisely one of each $\alpha_i, \overline{\alpha_i}$.

(c) Show that two choices for $\gamma, \gamma'$ give $\gamma \equiv \gamma'$ iff they were either exactly the same choice or exactly the opposite choice.

Solution: By Unique Factorization if $\gamma \sim \gamma'$ they have precisely the same factors. Also, if $\gamma' \sim \overline{\gamma}$ they must have precisely the conjugate factors.

(d) Using the above, find the number of solutions to $x^2 + y^2 = n$.

Solution: The $2^r$ choices split into pairs so there are $2^r - 1$ solutions.

(e) Setting $n = 5 \cdot 13 \cdot 17$ use the above to find the four solutions to $x^2 + y^2 = n$ explicitly.
Solution: We factor

\[ n = (2 + i)(2 - i)(3 + 2i)(3 - 2i)(4 + i)(4 - i) \]

We can assume (going to the complex conjugate if needed) \(2 + i\) is chosen. The four solutions are then

\[
\begin{align*}
(2 + i)(3 + 2i)(4 + i) &= 9 + 32i \\
(2 + i)(3 + 2i)(4 - i) &= 23 + 24i \\
(2 + i)(3 - 2i)(4 + i) &= 33 + 4i \\
(2 + i)(3 - 2i)(4 - i) &= 31 - 12i
\end{align*}
\]

That is,

\[
1105 = 9^2 + 32^2 = 23^2 + 24^2 = 33^2 + 4^2 = 31^2 + 12^2
\]

3. (Just for Fun) Presidential Trivia:

(a) Which president had a great stamp collection?
   Solution: Franklin Delano Roosevelt. He had the State Department forward him interesting stamps.

(b) Which was the fattest president?
   Solution: Taft. Yet, surprisingly, he was an excellent dancer.

(c) Which two presidents died on the same day?
   Solution: John Adams and Thomas Jefferson, precisely on the 50th anniversary of the signing of the Declaration of Independence.

(d) Which presidents were divorced?
   Solution: Ronald Reagan and Donald Trump

4. Here we examine the nature of the ideals of \( \mathbb{Z} [\sqrt{-5}] \) Let \( R \) be the rectangle \( \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{5}\} \).

(a) Let \( \beta \in \mathbb{Z} [\sqrt{-5}], \beta \neq 0 \). The elements of the ideal \( (\beta) \) split the complex plane into equal rectangles. What are the dimensions of these rectangles?
   Solution: The basic rectangle has corners \(0, \beta, \sqrt{-5}\beta, (1 + \sqrt{5})\beta\) and has dimensions \(|\beta| \times \sqrt{5}|\beta|\).
(b) Show that for any \( P = (a, b) \in R \) either \( P \) or \( 2P \) (maybe both) lies less than one away from one of the corners. Here we define \( 2P = (2a \mod 1, 2b \mod \sqrt{5}) \).

Solution: By rectangular symmetry assume \( a \leq 1/2, b \leq \sqrt{5}/2 \). The unit disks around \((0,0), (0,1)\) cover everything (look at the picture!) with \( b \leq \sqrt{3}/2 \). For \( b > \sqrt{3}/2 \) look at \((2a, 2b)\). \( \sqrt{5} > 2b > \sqrt{3} \) so \( 2b \) is within \( \sqrt{5} - \sqrt{3} \) from the top and \( 2a \) is within \( 1/2 \) of one of the sides so \((2a, 2b)\) is within

\[
\sqrt{(\sqrt{5} - \sqrt{3})^2 + (1/2)^2} < 1
\]

of one of the corners.

(c) Let \( I \) be an ideal of \( \mathbb{Z}[\sqrt{-5}] \). Let \( \beta \in I \) be a nonzero element with \( |\beta| \) minimal. Set \( \beta = c + d\sqrt{-5} \) and assume \( c, d \) are both odd. (Other cases could also be done.) Prove that either \( I = (\beta) \) or \( I = (\beta, \beta(1 + \sqrt{-5})/2) \).

Solution: If \( I \neq (\beta) \) there exists an \( \alpha \in I, \alpha \notin (\beta) \). So \( \alpha \) lies in a \(|\beta| \times \sqrt{5}|\beta|\) rectangle. And \( 2\alpha \) can be considered in the rectangle as we can subtract multiples of \( \beta \). But \( \alpha, 2\alpha \in I \) and one (or both) of them lies within \( |\beta| \) of one of the corners, which are in \((\beta)\) and so we get an element \( \gamma \) of \( I \) with \(|\gamma| < |\beta|\). Contradiction? Not quite!! We might have \( 2\alpha \in (\beta) \). In the rectangle, \( \alpha \) could, a priori, be a midpoint of one of the sides or the midpoint of the rectangle. In our particular case with \( a, b \) odd the midpoints of the sides, \( \beta/2 \) and \( \beta\sqrt{-5}/2 \) are not in \( \mathbb{Z}[\sqrt{-5}] \). So we must have \( \alpha \) the midpoint of the rectangle, \( \alpha = \beta(1 + \sqrt{-5})/2 \). And, indeed, this does give an ideal.

Remark: The other parities of \( a, b \) can also be examined and one can get a full description of \textit{all} ideals of \( \mathbb{Z}[\sqrt{-5}] \).