I think my life would be incomplete if I knew everything, if there were no mystery. On the other hand, it would also be incomplete if it were all mystery. So I am willing to recognize the limits of what I know but I’m also glad that mathematics gives me a chance to surround something and say “Yes, I do have some idea of truth.”
Donald Knuth

1. Let \( \epsilon = e^{2\pi i/18} \). Now assume (and this is true) that \( [Q(\epsilon) : Q] = 6 \).
   Let \( p(x) \in Q[x] \) be the minimal polynomial for \( \epsilon \).
   (a) What is the degree of \( p(x) \)?
   (b) Give all the roots of \( p(x) \). (Warning: 18 is not prime!)
   (c) Describe all \( \sigma \in \Gamma[Q(\epsilon) : Q] \) in a nice way. Give a table of all products in \( \Gamma[Q(\epsilon) : Q] \)
   (d) What well known group is \( \Gamma[Q(\epsilon) : Q] \) isomorphic to? Give the isomorphism explicitly.

Comment: We shall later find \( \Gamma[Q(\epsilon) : Q] \) with \( \epsilon = e^{2\pi i/n} \) for arbitrary \( n \).

2. Let \( K = Q(\sqrt{a_1}, \ldots, \sqrt{a_s}) \) with \( a_1, \ldots, a_s \in Q \).
   (a) Give an upper bound on \( [K : Q] \).
   (b) Let \( \sigma \in \Gamma[K : Q] \). Prove \( \sigma^2 = e \).
   (c) Let \( \sigma, \tau \in \Gamma[K : Q] \). Prove \( \sigma \tau = \tau \sigma \).

Caution: The actual \( \Gamma[K : Q] \) can depend in a complicated way on the choice of the \( a \)s. For example: \( a_1 = 2, a_2 = 3, a_3 = 5 \) and \( a_1 = 2, a_2 = 3, a_3 = 6 \) give different \( \Gamma \).

3. Let \( p(x) \in Q[x] \) be an irreducible cubic with roots \( \alpha, \beta, \gamma \in C \). Let \( K = Q(\alpha, \beta, \gamma) \). Suppose \( \sqrt{a} \in K \) where \( a \in Q \) and \( \sqrt{a} \notin Q \). Set \( L = Q(\sqrt{a}) \).
   (a) Prove \( p(x) \) is still irreducible when considered in \( L[x] \). (Hint: When cubics reduce they have a root.)

4. Set $\alpha = 2^{1/4}$, $\beta = i\alpha$, $\gamma = -\alpha$, $\delta = -i\alpha$. Set $p(x) = x^4 - 2$. Set $K = Q(\alpha, \beta, \gamma, \delta)$. Set $L = Q(i)$. Set $\Gamma = \Gamma[K : Q]$, the Galois Group of $K$ over $Q$.

(a) Show that $p(x)$ is irreducible in $Q[x]$.
(b) Give the factorization of $p(x)$ into irreducible factors in $K[x]$.
(c) Show that $p(x)$ is irreducible in $L[x]$. (As $p(x)$ is quartic it is not enough to look for factors as, a priori, it could be the product of two quadratics. But given your factorization for problem 4b any factorization over the smaller field $L$ must come from joining together factors from 4b. Show that none of them work.)
(d) Show $K = Q(\alpha, i)$.
(e) Show that $[K : Q] = 8$ and give a basis for $K$ over $Q$.
(f) Show that actually $|\Gamma| \leq 8$. (Idea: $\sigma(\alpha)$ determines $\sigma(\gamma)$.)
(g) List the eight possible permutations of $\alpha, \beta, \gamma, \delta$ that could come from a $\sigma \in \Gamma$.
(h) Actually $\Gamma$ is given by the eight permutations that you just found. Given this, show that $\Gamma$ is not Abelian.

There is a theory which states that if ever anybody discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.

Douglas Adams