Honors Algebra II V63.0349
Assignment 6
Due, Friday, Mar 8 in Recitation

Working with Paul Erdős was like taking a walk in the hills. Every time when I thought that we had achieved our goal and deserved a rest, Paul pointed to the top of another hill and off we would go.
– Fan Chung

1. Assume $3\alpha^5 + 17\alpha^2 + 8 = 0$. Find an explicit integer $r$ such that, setting $\beta = r\alpha$, $\beta$ satisfies a monic quintic – and give the quintic explicitly.

2. Let $[K : F] = n$. Let $v_1, \ldots, v_n$ be a basis for $K$ as a vector space over $F$. Let $0 \neq \beta \in K$. Prove that $v_1 \beta, \ldots, v_n \beta$ is a basis for $K$ over $F$. (Idea: As the number is correct you only need show either linear independence or spanning.)

3. Let $F \subset \Omega$, $\alpha, \beta \in \Omega$, $[F(\alpha) : F] = m$, $[F(\beta) : F] = n$. Assume $m, n$ are relatively prime. Prove that $[F(\alpha, \beta) : F] = mn$.

4. Let $p(x) \in \mathbb{Q}[x]$ be irreducible of degree $n$, with $n$ odd. Let $\alpha_1, \ldots, \alpha_n$ be the complex roots of $p(x)$. (You can assume these are distinct. We shall prove this in a more general form later in the course.) Show that $p(x)$ is irreducible in $\mathbb{Q}(\sqrt[3]{3})[x]$.

5. Let $Q = K_0 \subset K_1 \subset \ldots \subset K_t$ be a tower of fields where for each $1 \leq i \leq t$ either
   (a) $K_i = K_{i-1}(\beta_i)$ with $\beta_i^2 \in K_{i-1}$
   (b) or $K_i = K_{i-1}(\beta_i)$ with $\beta_i^3 \in K_{i-1}$

   (That is, extensions are by square roots and/or cube roots.) Give a result about the possible $[K_t : Q]$. Prove that $2^{1/5} \not\in K_t$.

6. Let $p(x) = x^6 + ax^4 + bx^2 + c \in \mathbb{Q}[x]$. Let $\alpha, \beta, \gamma, \delta, \epsilon, \mu$ be the complex roots of $p(x)$. Prove $[Q(\alpha, \beta, \gamma, \delta, \epsilon, \mu) : Q] \leq 48$. (Hint: The special nature of $p(x)$ yields a relationship between the roots.)

Now, every time I witness a strong person, I want to know: What darkness did you conquer in your story? Mountains do not rise without earthquakes.
Katherine MacKenett