The universe is not only queerer than we suppose but queerer than we can suppose.
– J.B.S. Haldane

We examine the number of solutions to the equation \( x^2 + y^2 = n \) with \( x, y \in \mathbb{Z} \). We call \( \alpha = x + iy \) the related Gaussian integer. Call two solutions equivalent if you can get from one to the other by adding minus signs and/or flipping \( x, y \). For example, \((3, 11)\) and \((11, -3)\) are the same. We actually count the solutions up to equivalence. (While we don’t quite do it, these methods yield the answer for all \( n \).)

1. Show (easy!) \((x, y)\) is a solution iff \( \alpha \bar{\alpha} = n \). Now write \( \alpha \equiv \alpha' \) if either \( \alpha \sim \alpha' \) or \( \bar{\alpha} \sim \alpha' \). (\( \bar{\alpha} \) denotes the complex conjugate and \( \alpha \sim \beta \) means \( \beta = u\alpha \), \( u \) a unit.) Show that two solutions \((x, y), (x', y')\) are equivalent iff their related \( \alpha, \alpha' \) have \( \alpha \equiv \alpha' \).

2. (This problem counts double!) Let \( n = p_1 \cdot p_r \) where the \( p_i \) are all integer primes and all are of the form \( 4k + 1 \). In \( \mathbb{Z}[i] \) write each \( p_i = \alpha_i \beta_i \) with \( \beta_i = \bar{\alpha}_i \). \( \gamma = \gamma_1 \cdots \gamma_r \) with each \( \gamma_i \in \{\alpha_i, \beta_i\} \). Note there are \( 2^r \) choices here.
   
   (a) Setting \( \gamma = x + iy \) show that \( x^2 + y^2 = n \).
   
   (b) Show that if \( x^2 + y^2 = n \) then there is such a \( \gamma = x + iy \).
   
   (c) Show that two choices for \( \gamma, \gamma' \) give \( \gamma \equiv \gamma' \) iff they were either exactly the same choice or exactly the opposite choice.
   
   (d) Using the above, find the number of solutions to \( x^2 + y^2 \).
   
   (e) Setting \( n = 5 \cdot 13 \cdot 17 \) use the above to find the four solutions to \( x^2 + y^2 = n \) explicitly.

3. (Just for Fun) Presidential Trivia:

   (a) Which president had a great stamp collection?
   
   (b) Which was the fattest president?
   
   (c) Which two presidents died on the same day?
   
   (d) Which presidents were divorced?
4. Here we examine the nature of the ideals of $\mathbb{Z}[\sqrt{-5}]$. Let $R$ be the rectangle $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{5}\}$.

(a) Let $\beta \in \mathbb{Z}[\sqrt{-5}], \beta \neq 0$. The elements of the ideal $(\beta)$ split the complex plane into equal rectangles. What are the dimensions of these rectangles?

(b) Show that for any $P = (a, b) \in R$ either $P$ or $2P$ (maybe both) lies less than one away from one of the corners. Here we define $2P = (2a \mod 1, 2b \mod \sqrt{5})$. (A good picture will help!)

(c) Let $I$ be an ideal of $\mathbb{Z}[\sqrt{-5}]$. Let $\beta \in I$ be a nonzero element with $|\beta|$ minimal. Set $\beta = c + d\sqrt{-5}$ and assume $c, d$ are both odd. (Other cases could also be done.) Prove that either $I = (\beta)$ or $I = (\beta, \beta(1 + \sqrt{-5})/2)$.

Well, you see, Haresh Chacha, its like this. First you have ten, that’s just ten, that is, ten to the first power. Then you have a hundred, which is ten times ten, which makes it ten to the second power. Then you have a thousand which is ten to the third power. Then you have ten thousand, which is ten to the fourth power - but this is where the problem begins, don’t you see? We don’t have a special word for that, and we really should.

... But you know, said Haresh, I think there is a special word for ten thousand. The Chinese tanners of Calcutta once told me that they used the number ten-thousand as a standard unit of counting. What they call it I can’t remember ... Bhaskar was electrified. But Haresh Chacha you must find that number for me, he said. You must find out what they call it. I have to know, he said, his eyes burning with mystical fire and his small frog-like features taking on an astonishing radiance.

– from A Suitable Boy by Vikran Seth