Honors Algebra II
Assignment 11
Due Friday, April 26 in recitation

REMINDER: FINAL IS ON
MAY 11, 10:10 -11:50 in wwh 317

I doubt sometimes whether a quiet and unagitated life would have suited me – yet I sometimes long for it. – Byron

1. Let $K : Q$ be a normal extension, set $G = \Gamma[K : Q]$. Let $H$ be a subgroup of $G$. Let $a = |H|$, $s = |G|$. Let $\alpha \in K$. Set $\gamma = \sum_{\sigma \in H} \sigma(\alpha)$.
   (a) Show that $\tau(\gamma) = \gamma$ for all $\tau \in H$.
   (b) Show that $H \subseteq Q(\gamma)^*$.
   (c) Deduce (using GCT) an upper bound on $[Q(\gamma) : Q]$.

2. Let $G = (\mathbb{Z}_4 \times \mathbb{Z}_6, +)$
   (a) Find two subgroups $H \subset G$ with precisely 12 elements.
   (b) Find a third subgroups $H \subset G$ with precisely 12 elements.
   (c) (*) Prove there aren't any more $H$.
   (d) Let $K : Q$ be normal with $\Gamma(K : Q)$ isomorphic to $G$ above. Show (using GCT) that $K$ has precisely three distinct nontrivial square roots. (We count $\sqrt{a}$ and $\sqrt{q^2a}$ as the same.)

3. Set $\epsilon = e^{2\pi i/35}$ and $K = Q(\epsilon)$.
   (a) Find an injective homomorphism from $\Gamma[K : Q]$ to $G$ as in (2).
   (b) Let $\sigma \in \Gamma[K : Q]$. Show that there are at most 24 possible values of $\sigma(\epsilon)$.
   (c) Now assume $[K : Q] = 24$. (This is true though not easy to prove.) Show $\Gamma[K : Q] \cong G$.

4. Let $K$ be the splitting field of $x^p - 2$ ($p \geq 3$ prime) over $Q$. Set $\beta = 2^{1/p}$ (the real root), $\epsilon = e^{2\pi i/p}$.
   (a) Show $K = Q(\beta, \epsilon)$.
   (b) Give $[Q(\epsilon) : Q]$ and $[Q(\beta) : Q]$. 
(c) Given (4b) find $[K : Q]$.

(d) Let $\sigma \in \Gamma[K : Q]$.
   i. What are the possible values of $\sigma(\beta)$.
   ii. What are the possible values of $\sigma(\epsilon)$.
   iii. Using (4c) and the GCT to show that all pairs of values $\sigma(\beta), \sigma(\epsilon)$ do indeed give a $\sigma \in \Gamma[K : Q]$.

5. Let $f(x) = x^4 + ax^2 + bx + c \in Q[x]$ with roots $\alpha, \beta, \gamma, \delta$. (The zero $x^3$ is a convenience.) Set $\kappa = \alpha \beta + \gamma \delta$, $\lambda = \alpha \gamma + \beta \delta$, $\mu = \alpha \delta + \beta \gamma$. Set $g(x) = (x - \kappa)(x - \lambda)(x - \mu) = x^3 + dx^2 + ex + f$.

   (a) Prove $g(x) \in Q[x]$. (Use Symmetric Polynomials!)
   (b) Find $d, e$ as explicit polynomials in $a, b, c$. ($f$ is also doable.)

6. Let $[K : Q] = 3^t$ be a normal extension. Let $\alpha \in K$. Prove that $\alpha$ is expressible using square roots and cube roots.

I cannot live without people. – Pope Francis