Honors Algebra II
Assignment 10
Due Friday, April 19 in recitation

She guessed at it all, what might live, moving purposefully or drifting aimlessly, under the deep water around her, but she didn’t think too much about any of it. It was enough to be aware of the million permutations possible around her, and take comfort in knowing she would not, and really could not, know much at all.
Dave Eggers, The Circle

You may assume GCT (Galois Correspondence Theorem) throughout but please make clear how you are using it.

1. Let $F \subset L \subset K$ be fields in $C$. Assume that $K : F$ is normal and that $L : F$ is normal. Let $\tau \in \Gamma[K : F]$ and $\sigma \in \Gamma[K : L]$. Let $l \in L$

(a) Argue that $\tau(l) \in L$.
(b) Show that $(\tau \sigma \tau^{-1})(l) = l$.
(c) From the above show that $\Gamma[K : L]$ is a normal subgroup (get out those Algebra I notes!) of $\Gamma[K : F]$.
   (Assume it’s already been shown that it is a subgroup. You only need show the normal part.)

2. Let $\alpha$ be a root of $f(x) = x^3 + x^2 - 2x - 1 \in Q[x]$.

(a) Show $f(x)$ is irreducible over $Q$. Note: You should assume this in what follows.
(b) Find $[Q(\alpha) : Q]$.
(c) Set $\beta = -1/(\alpha + 1)$. Find $\beta$ in the form $a + b\alpha + c\alpha^2$.
(d) Show that $f(\beta) = 0$. (Bit of grunt work here!)
(e) Find $\gamma \in Q(\alpha)$, $\gamma, \beta, \alpha$ distinct, with $f(\gamma) = 0$. (Idea: If $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$ then $\alpha + \beta + \gamma$ is determined.)
(f) Deduce that $Q(\alpha) : Q$ is normal.
(g) Argue that $\Gamma(Q(\alpha) : Q)$ has precisely three elements. Which permutations of $(\alpha, \beta, \gamma)$ do the three automorphisms correspond to? What well known group is $\Gamma(Q(\alpha) : Q)$ isomorphic to? (Hint: There is only one group on three elements!)
3. Let \( f(x) \in \mathbb{Q}[x] \) be irreducible of degree \( n \), let \( \alpha_1, \ldots, \alpha_n \) denote its roots, and let \( K \) be the splitting field \( K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \). Assume \( [K : \mathbb{Q}] = n! \).

(a) Prove \( \Gamma[K : \mathbb{Q}] \cong S_n \) (Note: \( S_n \) is our notation for the full symmetric group on \( n \) letters.)

(b) Using (3a) and Algebra I and the GCT prove that there is a unique \( L \subset K \) with \( [L : \mathbb{Q}] = 2 \).

4. Let \( K : F \) be a normal extension with Galois Group \( G = \Gamma(K : F) \).
   Let \( \sigma \in G \). Set \( L = \{ \alpha \in K : \sigma(\alpha) = \alpha \} \). Prove (use GCT!) that
   \[ [L : F] = \frac{a}{b} \text{ with } a = [K : F], b = o(\sigma) \]
   Note: \( o(\sigma) \) is our notation for the order of \( \sigma \), the least \( m \) such that \( \sigma^m \) is the identity.

5. Let \( K : \mathbb{Q} \) be normal extension. Let \( \tau \) denote complex conjugation.

   (a) Prove that \( K \) is closed under \( \tau \). That is, if \( \alpha = a + b\sqrt{-1} \in K \) then \( \tau(\alpha) = a - b\sqrt{-1} \in K \).

   (b) Let \( L \) be the set of real elements of \( K \). Assume \( K \) contains nonreal elements. Prove \( [K : L] = 2 \).

He rarely copied box scores into the Book, but today it seemed the right thing to do. All those zeroes! He decided for zeroes he’d use red ink. Zero: the absence of number, an incredible idea! Only infinity compared to it, and no batter could hit an infinite number of home runs - no, in a way, the pitchers had it better. Perfection was available to them.

– Robert Coover