Fields to Groups and Back Again II

Let us fix some finite extension \( F \subset K \) of subfields of \( C \) and set \( G \) to be the Galois Group \( \Gamma(K : F) \). However, we now assume \( K \) is a Normal Extension of \( F \). Recall that we have already defined the map \( * \) from intermediate fields to subgroups and the map \( \dagger \) from subgroups to intermediate fields.

**Theorem 0.1** Let \( F \subset K \) be subfields of \( C \) with \( K \) a Normal extension of \( F \) and set \( G \) to be the Galois Group \( \Gamma(K : F) \). Then for any intermediate field \( L \)

\[
(L^*)\dagger = L
\]

**Proof**: We already know \( L \subset (L^*)\dagger \). Now suppose \( \beta \in K \) and \( \beta \notin L \). Our goal is to show \( \beta \notin (L^*)\dagger \). Recall that as \( K \) is a normal extension of \( F \), \( K \) is a normal extension of \( L \).

Let \( p(x) \) be the minimal polynomial for \( \beta \in L[x] \) and let \( \beta_1 \) be another root of \( p(x) \). As \( K \) is a normal extension of \( L \), \( \beta_1 \in K \). Thus there is an isomorphism \( \sigma : L(\beta) \to L(\beta_1) \) which fixed \( L \) and has \( \sigma(\beta) = \beta_1 \). Applying the Full Isomorphism Extension Theorem we extend \( \sigma \) to an isomorphism \( \sigma^{++} \) with domain \( K \). But as \( \sigma^{++} \) fixes \( L \) and \( K \) is normal over \( L \), the range of \( \sigma^{++} \) must be \( K \). That is, \( \sigma^{++} \) is an automorphism of \( K \) which fixes all \( \alpha \in L \) but does not fix \( \beta \). So \( \beta \notin (L^*)\dagger \). End of Proof.