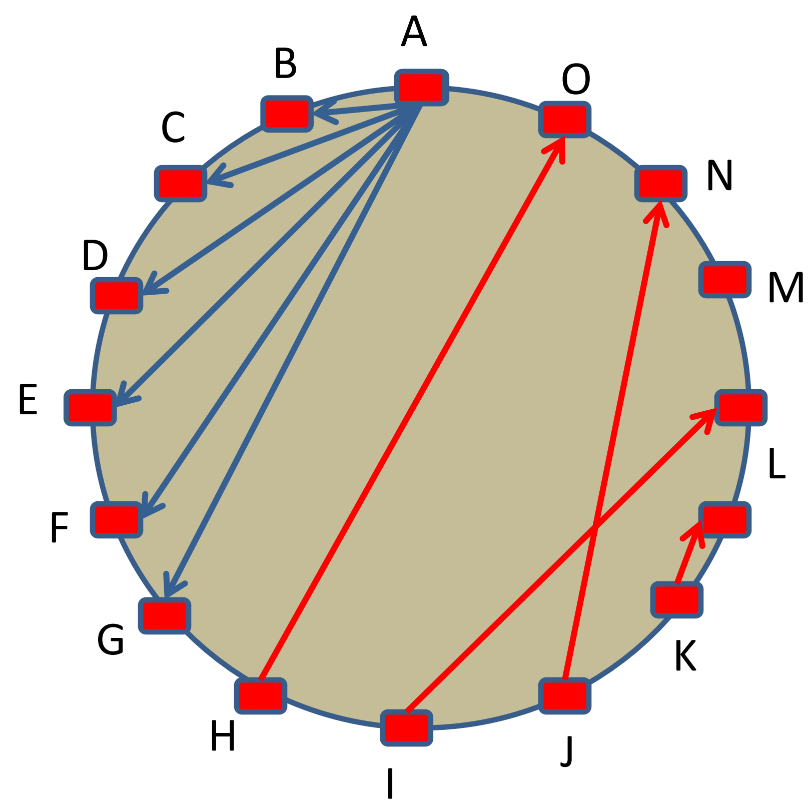
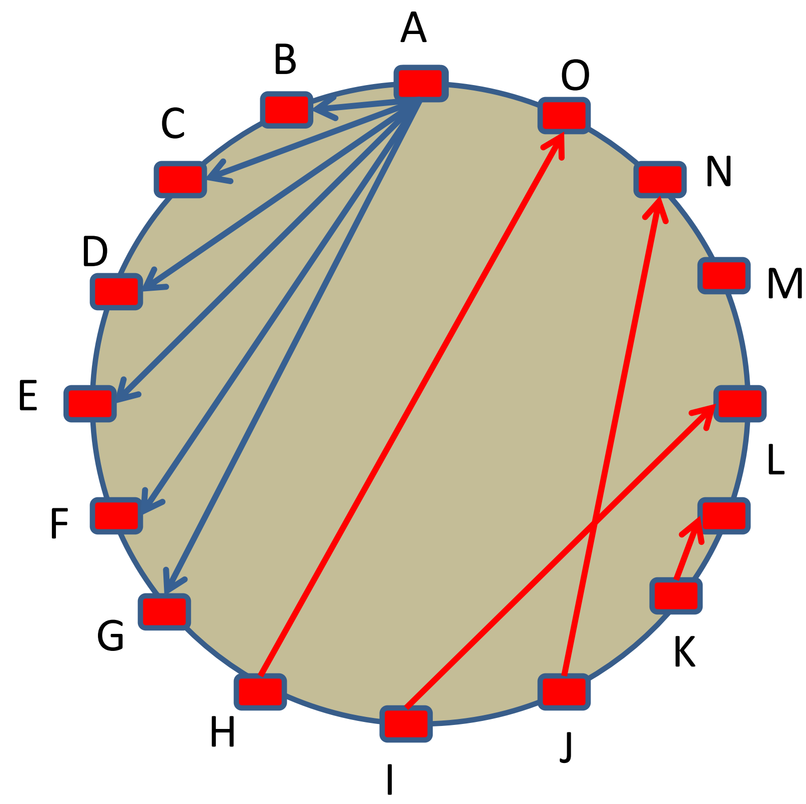
Proof that n/2 + n/4 -1 is optimal for a special case

- Some Definitions first.
- Fix A at position 0. This is an arbitrarily chosen convention.
- We classify into two types:
  - *Fixed*: Hints that reference A
  - *Floating*: Hints that do not reference A
- Fixed hints fully determine positions of other nodes. Eg: if hint is "C is 4 places to right of A" nails C to seat #4.
- Floating hints are less obvious. A full seating assignment may be needed to resolve them.
- See figure on right for n=16, blue hints are fixed, red ones are floating.
- Observe that:
  - B, C, D, E, F and G are determined easily.
  - H through O are not determined until all floating hints have been "solved".



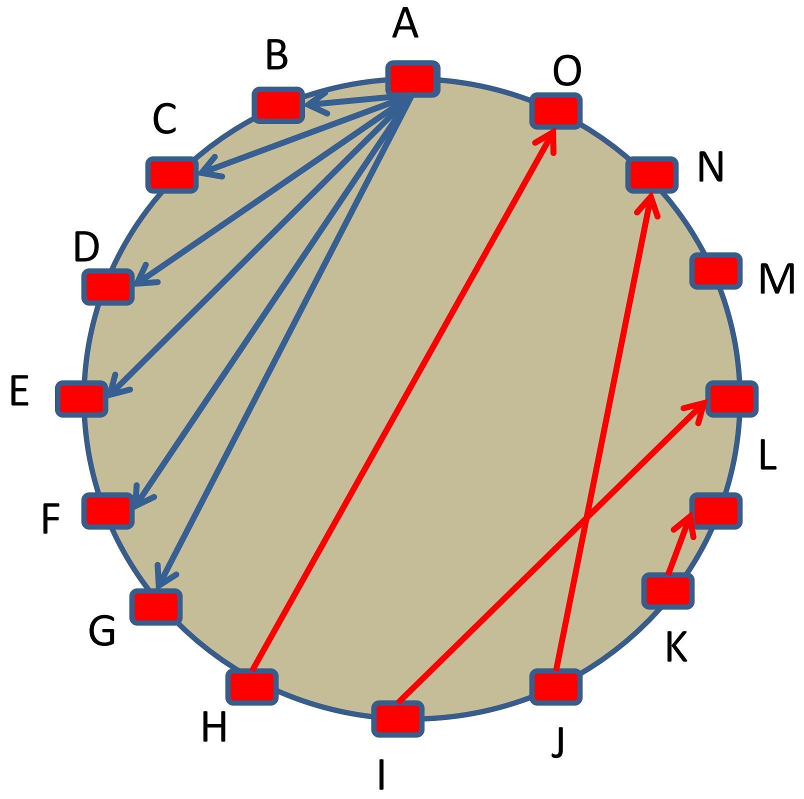
Our Special Case

- We consider the special case wherein:
- All fixed hints are clustered together.
- See picture for an example.
- We use this picture as running example of proof.
- We show that if fixed hints are clustered together, then n/2 + n/4 -1 is tight. Can not do better.
- By "cluster" we mean that the fixed hints point to vertices that are clock-wise or anti-clockwise neighbors of A. Like in the example.
- Thus, the hints in the example do not define a unique seating.
  - n/2+n/4-1 = 11 hints (n=16)
  - We have only 10 hints in this.
- We show by constructing an alternate solution for the floating hints whenever the hints given are not tight (ie, less than n/2 + n/4 -1)



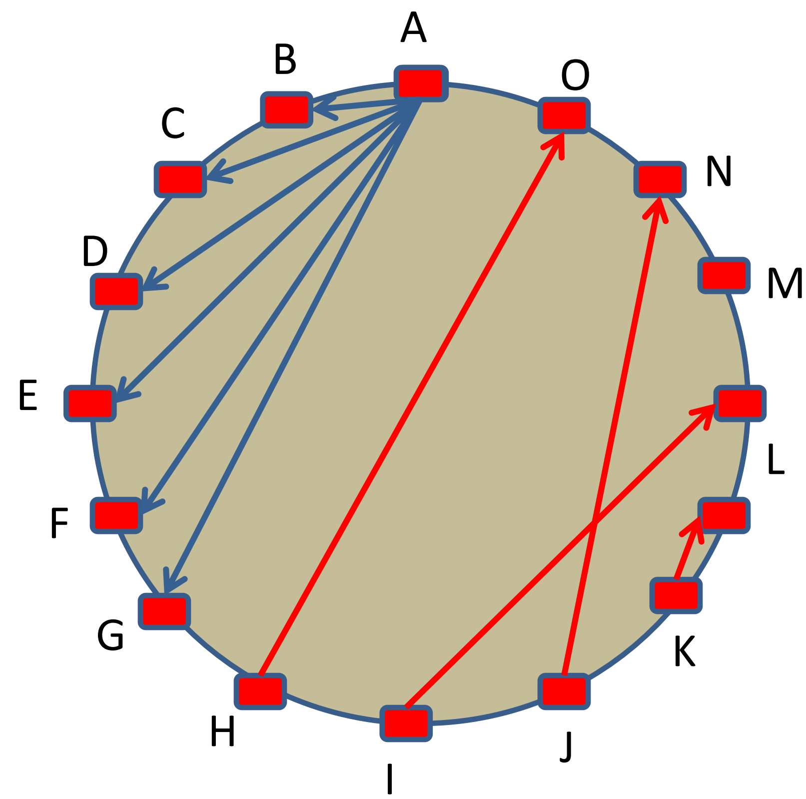
Main Idea

- Basic idea of developing an alternate solution that satisfies all the floating constraints uses two transformations:
  - One. A rotation of the floating hints
  - Two. Reflection of the hints
- We describe these on the example.

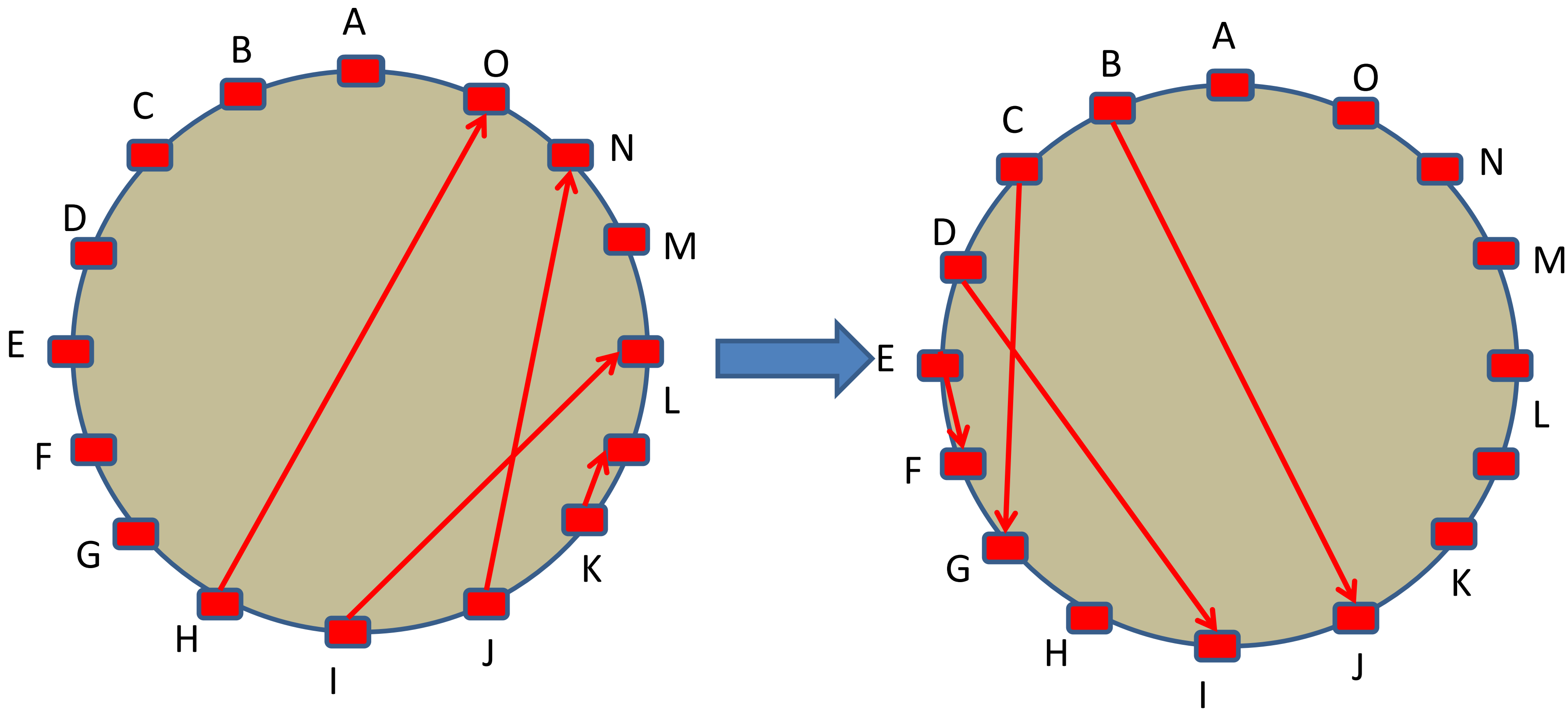


Rotation of floating hints

- General construction.
- Assume k fixed hints. They determine first (k+1) positions ('A' included)
- In example, k=6, so 7 vertices are determined.
- Ignore these fixed hints.
- Rotate floating hints by k positions. See next slide.



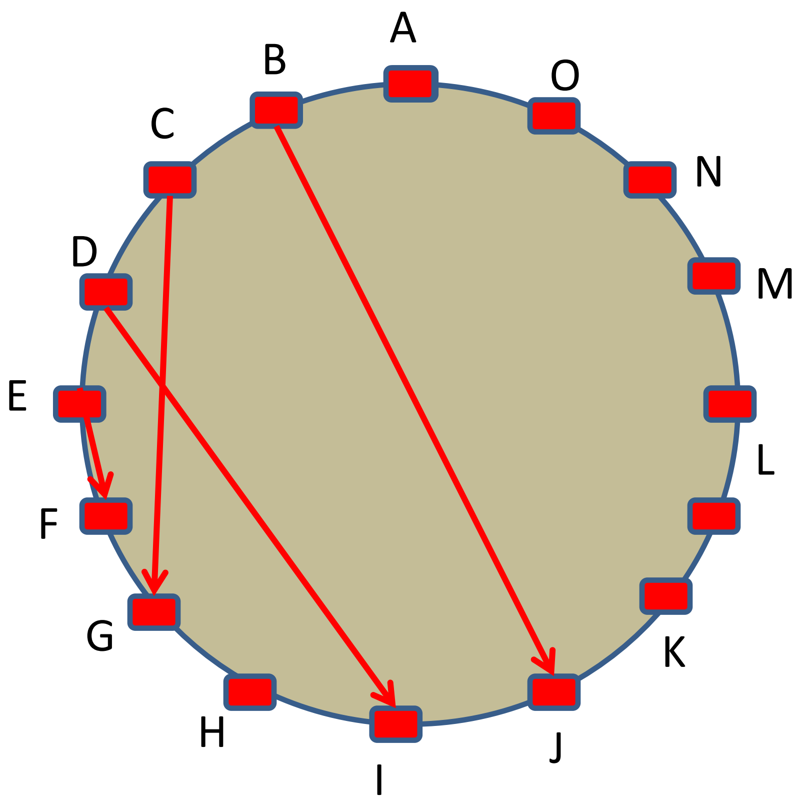
Rotation Example



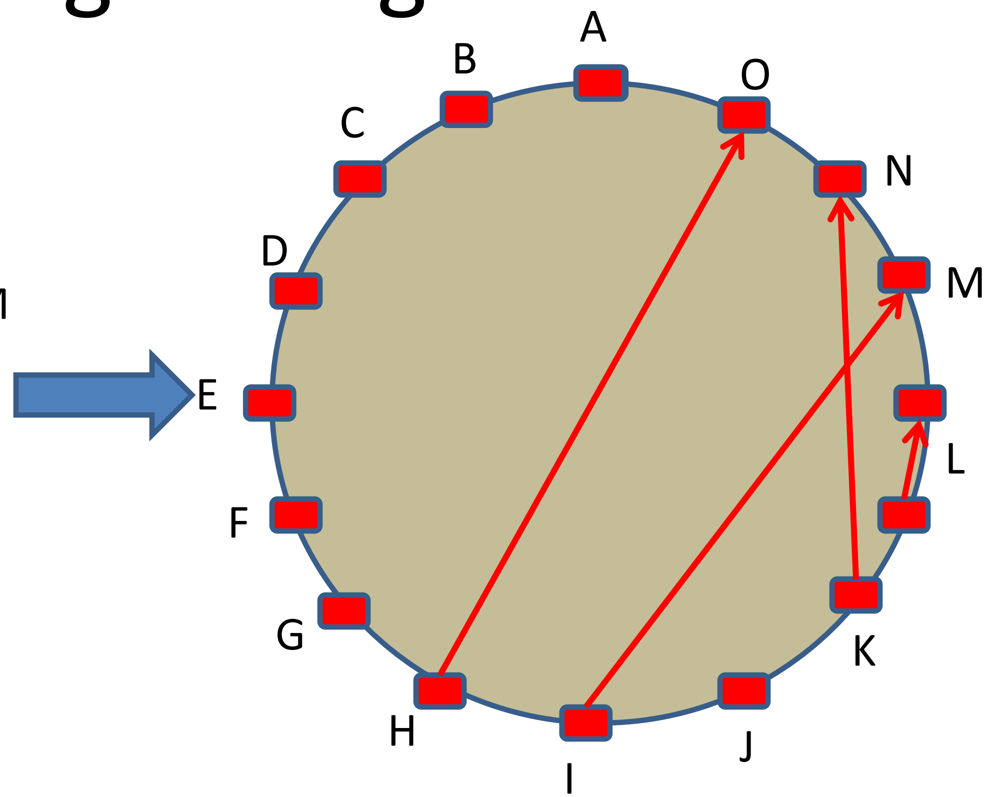
Before Rotation

After Rotation
(by 6 positions clockwise)

Now Reflect this around the vertical axis running through A

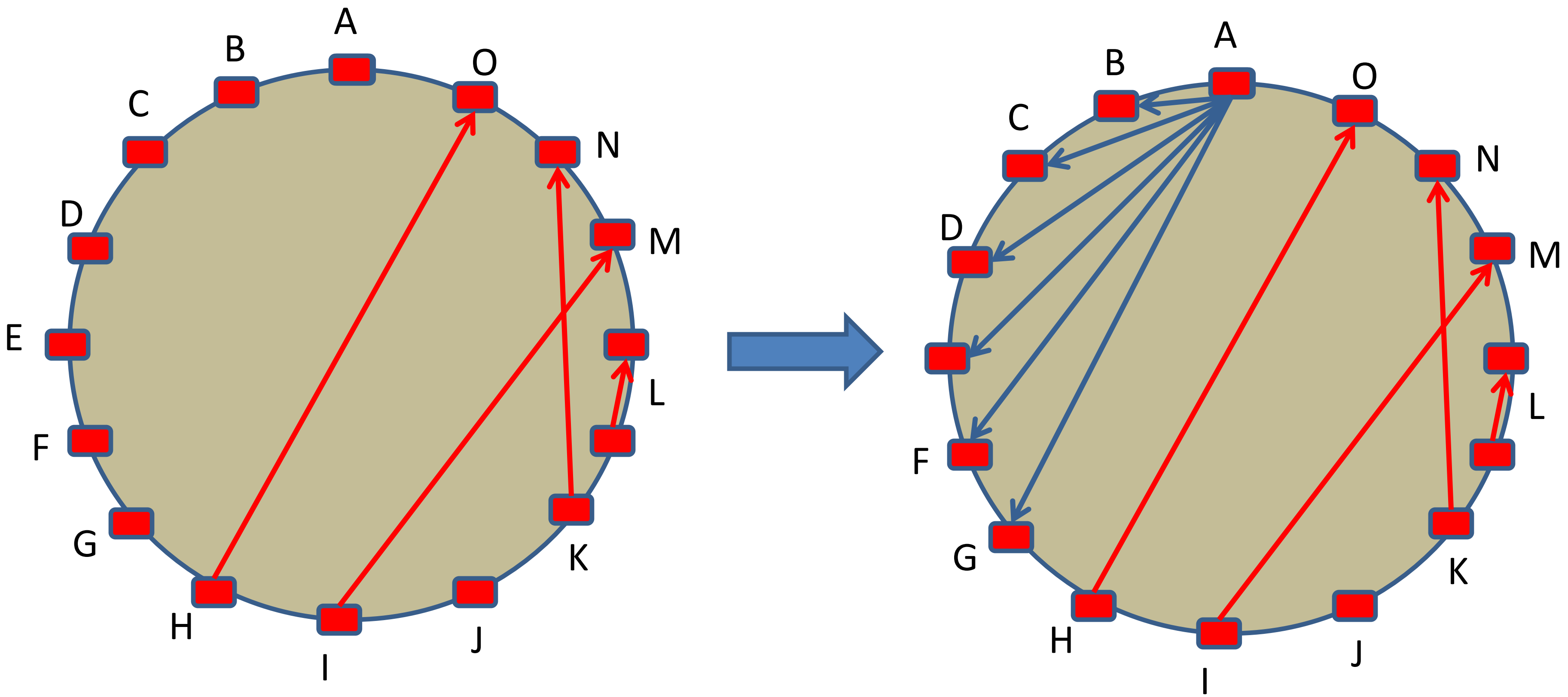


Before Reflection

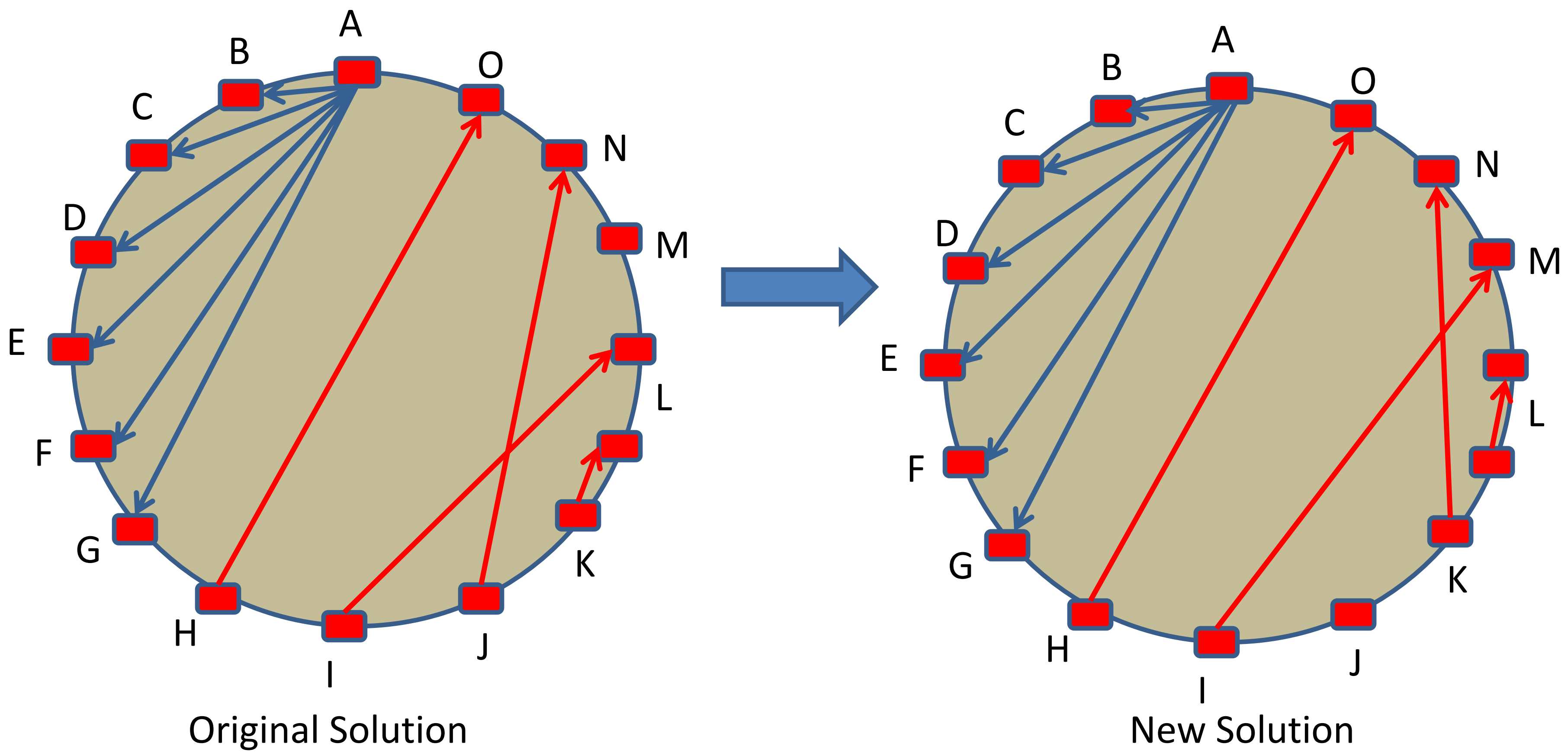


After Reflection

Add the fixed hints back in



We have demonstrated a new solution! ➔ Not Unique



What did we do?

- Under the given assumptions:
  - Clustered fixed hints
  - k < n/2-1 fixed hints (in the example k=6 whereas n/2-1=7)
  - Start with one solution
  - By rotation by k positions followed by a reflection, we revealed a new solution that satisfies all floating hints
  - Thus we showed solution is not unique. ➔ means k + n/4 hints not enough where k < n/2-1

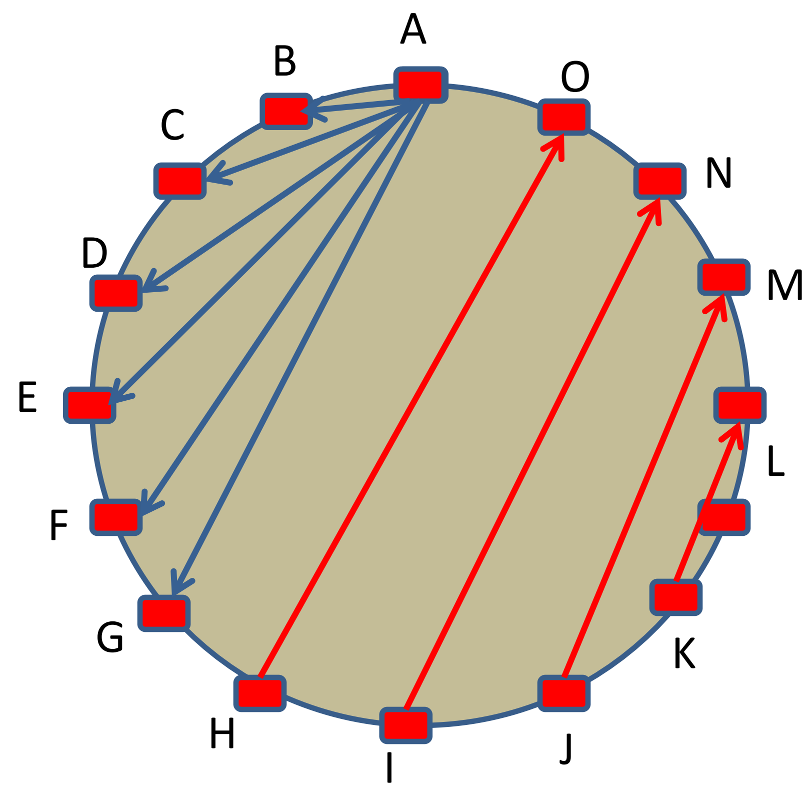
What happens when k=n/2-1

- When we have k=n/2-1 fixed hints
- Then if rotation + reflection generates a different solution then one of the below has to hold:
  - Two or more hints are pairwise identical in terms of hint distances ➔ indistinguishable and thus not unique
  - Or must look like this example:
  - This solution is a "fixed point"
  - When rotated + reflected, it
  gives rise to itself

- This "fixed point" set of hints need
n/4 of them.
(because their hint distances go in
Steps of 2.)
In example, they go: 2 (k,l)
4 (j,m), 6 (I,n) and 8 (h, o)

- Any fewer floating hints (< n/4) will not be
unique.

- Any more floating hints (> n/4) will lead to duplicate
hint distances (due to steps-of-2 hint distances) and thus
no unique solution.

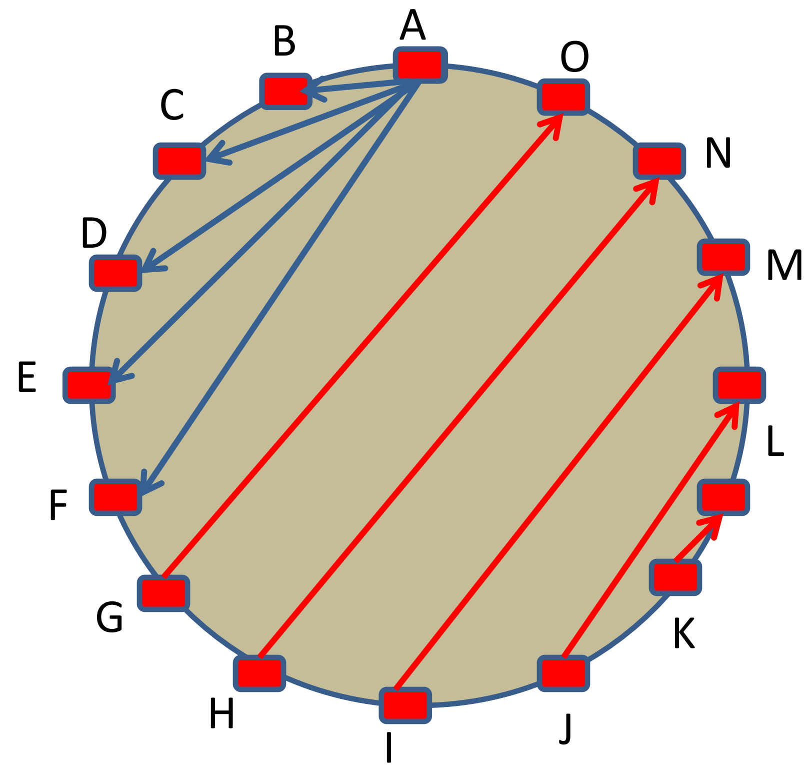


Conclusion

- Classify hints as fixed and floating
- Assume k fixed hints that are clustered
- Show by rotation and reflection of floating hints that a new solution can be obtained if $k < n/2-1$
- When $k = n/2-1$ then only unique solution is the one which is invariant under rotation + reflection
- This invariant requires exactly $n/4$ hints
- Thus $n/2-1+n/4$ is tight when the fixed hints are clustered.

Appendix-1

- > n/4 floating hints in fixed-point configuration lead to duplicate hint distances
- Example: here five floating hint distances (1, 3, 5, 7, 9)
- 9 is indistinguishable from 7 (7 going other way around on the circle is 9 away)



Appendix-2

- < n/4 floating hints in fixed-point configuration are too few hints
- Example: here 3 floating hint distances (3, 5,7)
- 3 vertices completely free – can not resolve uniquely