Decidability of Robot Manipulation Planning: Three Disks in the Plane

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Abstract. This paper considers the problem of planning collision-free motion of three disks in the plane. One of the three disks, the robot, can autonomously translate in the plane, the other two move only when in contact with the robot. This represents the abstract formulation of a manipulation planning problem. Despite the simplicity of the formulation, the decidability of the problem had remained unproven so far. We prove that the problem is decidable, i.e., there exists an exact algorithm that decides wether a solution exists in finite time.

1 Introduction

The problem of planning collision free motion for a free-flying single-body robot in environments populated by static obstacles has been widely studied in the past decades and can be considered today well understood. In this paper we consider a generalization of this basic problem by allowing the presence of *movable obstacles*, i.e., objects in the environment that the robot can move by "grasping" them, while avoiding collisions with and between all the obstacles.

The problem of motion planning in the presence of movable obstacles was first introduced in [1], the corresponding journal version appearing in [2], where the decidability is proven for the case of discrete grasps. This problem was further generalized in [3] to the so-called manipulation planning problem where the movable obstacles are considered as objects to be moved to reach a goal position. In that paper the authors present an algorithm for the case of discrete placements and grasps. This is the formulation briefly described in Chapter 11 of Latombe's book [4]. Decidability of the problem in the case of continuous grasps and placements was shown in [5] considering one movable object.

While [6] provides an efficient probabilistically complete algorithm in the case of several movable obstacles, the question of the decidability, i.e., the existence of an exact algorithm that decides wether a solution exists in finite time, remained open even in the case of two movable objects as mentioned in [7].

* This work is supported by the EU FP7 ICT-287513 SAPHARI project.

In this paper we prove that the manipulation planning problem for a robot that can freely translate in the plane and two objects that can move only if they are in contact with the robot is decidable. The proof is based on a cell decomposition of the collision-free contact configuration space and on a property (*reduction property*) assessing the equivalence of paths continuously satisfying the contact constraint to manipulation paths along which the objects either translate rigidly with the robot as a single object (*transfer paths*) or remain in a fixed position while the robot translates to a different position (*transit path*). To prove that the reduction property holds for the manipulation model considered in this paper, allowing motion of the objects only through sticky contact with the robot, we make use of the controllability result in [8].

Although somewhat theoretical, the presented result is expected to lay the basis for answering important questions such as characterizing under which conditions motion in contact can be reduced to a manipulation path, how to efficiently construct manipulation graphs related to many different problems (climbing, walking, multi-contact planning) for all of which the present formulation represents an abstraction, how to determine the rate of convergence of probabilistic planners for the manipulation of multiple objects.

The paper is organized as follows. In the next section we formalize the problem after defining the configuration space and its connectivity through manipulation paths. In Sect. 3 we establish the conditions under which motion in contact can be reduced to a manipulation path. Section 4 illustrates the main steps for the construction of the manipulation graph and Sect. 5 concludes the paper. Finally, in the Appendix we propose a constructive geometric proof of the reduction property when the robot is in contact with both obstacles.

2 Problem formulation

Consider the scene depicted in Fig. 1 with two movable rigid objects O_1 and O_2 and one robotic manipulator R, all disk-shaped, translating in a polygonal (or semi algebraic) environment with obstacles. The objects O_1 and O_2 can move only if in contact with R; otherwise, they are considered as fixed obstacles.

2.1 Configuration space

The configuration spaces of the robot and the objects are defined as:

- $\mathcal{C}_R = \boldsymbol{R}^2$, the configuration space of the robot;
- $\mathcal{C}_{O_1} = \boldsymbol{R}^2$, the configuration space of O_1;
- $\mathcal{C}_{O_2} = \boldsymbol{R}^2$, the configuration space of O_2.

The combined configuration space is obtained as $\mathcal{C} = \mathcal{C}_R \times \mathcal{C}_{O_1} \times \mathcal{C}_{O_2} = \boldsymbol{R}^6$. A configuration $\boldsymbol{q} \in \mathcal{C}$ is given by the triplet $\boldsymbol{q} = (\boldsymbol{q}_R, \boldsymbol{q}_{O_1}, \boldsymbol{q}_{O_2})$, where $\boldsymbol{q}_R \in \mathcal{C}_R$, $\boldsymbol{q}_{O_1} \in \mathcal{C}_{O_1}$, $\boldsymbol{q}_{O_2} \in \mathcal{C}_{O_2}$.

The collision-free configuration space $\mathcal{C}_{\text{free}}$ is obtained by removing from $\mathcal{C}$ the set of configurations:

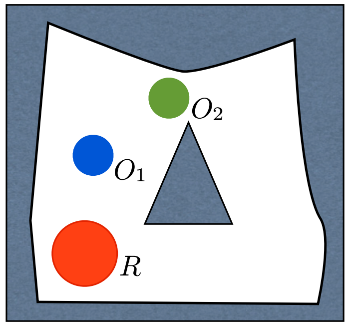

Fig. 1. Scenario of the considered manipulation planning problem. The robot R can autonomously translate in the plane while the movable objects O_1 and O_2 can translate only if in contact with the robot.

- $\boldsymbol{q}_R$ such that the robot is in contact with static obstacles or overlaps with either static or movable obstacles;
- $\boldsymbol{q}_{O_1}$ such that O_1 overlaps with the static obstacles, the robot or with O_2 (contact between objects and obstacles is allowed);
- $\boldsymbol{q}_{O_2}$ such that O_2 overlaps with the static obstacles, the robot or with O_1 (contact between objects and obstacles is allowed).

2.2 Configuration space paths and manipulation paths

Paths in $\mathcal{C}$ can be categorized according to the three motion modalities:

- robot free motion: this is a path in the submanifold $\mathcal{C}_R$ with the two objects in fixed positions (obstacles);
- single-contact motion: this is a path in either $\mathcal{C}_R \times \mathcal{C}_{O_1}$ or $\mathcal{C}_R \times \mathcal{C}_{O_2}$ constrained by the condition of contact with one of the objects while the other one is in a fixed position; along the path both the position of the robot and the position of the object relative to the robot can change;
- double-contact motion: this is a path in $\mathcal{C}_R \times \mathcal{C}_{O_1} \times \mathcal{C}_{O_2}$ constrained by the condition that the robot is in contact with both objects; along the path the robot position and the positions of the objects relative to the robot can change.

The above described paths might or might not be feasible for a manipulation system depending on its characteristics. In this work we consider only manipulation by stable grasp. This means that sliding, rolling, pushing are not included in our analysis. Therefore, not all the configuration space paths are feasible in our setting. Feasible motions correspond to paths of two types:

– *transfer paths* along which either the robot grasps one of the two objects and moves rigidly with it (while the other remains in a fixed position) or it

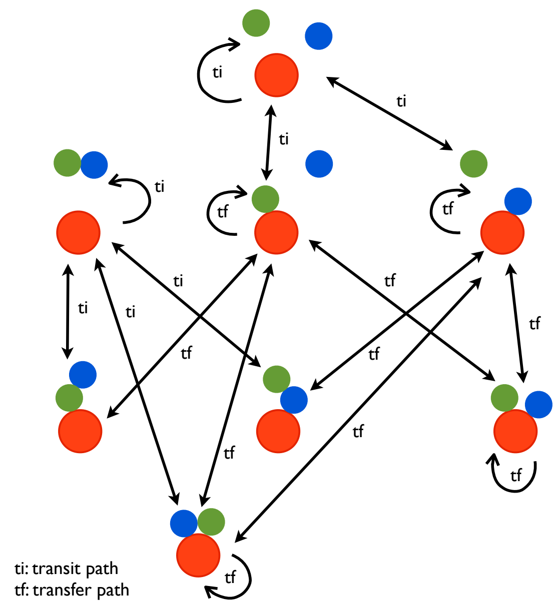

Fig. 2. Structure of the configuration space $\mathcal{C}$ induced by the contact constraints and interconnection of the contact submanifolds through transit and transfer paths.

grasps both the objects and moves rigidly with them; along these paths the relative configurations between robot and objects in contact do not change;

– *transit paths* along which the robot moves alone.

A sequence of transit and transfer paths is called a *manipulation path.*

2.3 Configuration space connectivity through manipulation paths

In this section we illustrate the complete structure of $\mathcal{C}$ in terms of the submanifolds defined by the contact constraints and their interconnection through transit and transfer paths. Figure 2 shows the representative configurations in each manifold.

The embedding configuration space $\mathcal{C}$ has dimension 6 and foliates with the position of the movable objects. In particular, leaves of dimension 2 correspond to fixed positions of the two objects. Transit paths belong to one of these leaves. Manipulation paths across the leaves (the objects change position) require leaving the manifold. A representative configuration in this manifold is shown at the top of Fig. 2.

Configurations on the second row (from top) of Fig. 2 represent the single-contact manifold which has dimension 5, foliates with the absolute position of one object and the relative position of the other with respect to the robot. The leaves of interest for the considered problem have dimension 3 and correspond to fixed positions of the object which is not in contact with the robot. Manipulation paths across the leaves require leaving the manifold.

The double-contact manifold, represented by the configurations on the third row of Fig. 2, has dimension 4 and foliates with the relative position of the contact points. The leaves of interest have dimension 3 and 2 and correspond respectively to one or both the points of contact being fixed. Manipulation paths across the leaves require leaving the manifold.

Finally, the triple-contact manifold has dimension 3, foliates with the position of the contact points and the leaves have dimension 2. Manipulation paths across the leaves require leaving the manifold.

As will be illustrated in section 3.2, the "manipulability" properties associated with these manifolds are actually transversal to this geometric structure and depend on the controllability of the underlying manipulation system.

2.4 The manipulation planning problem

Relying on the definitions and analysis of the previous sections, we can formulate the following problem.

Manipulation Planning Problem. Given an initial configuration $\boldsymbol{q}_s \in \mathcal{C}_{\text{free}}$ and a goal configuration $\boldsymbol{q}_g \in \mathcal{C}_{\text{free}}$, find a sequence of transit and transfer paths joining $\boldsymbol{q}_s$ to $\boldsymbol{q}_g$, if it exists.

To prove that this problem is decidable we adopt the same approach as [5]. First we study the problem of reducing the configuration space paths belonging to the contact manifolds represented in Fig. 2 to manipulation paths. Then, we determine a cell decomposition of the contact space. Finally, we complete the proof with the construction of the manipulation graph whose connected components characterize the existence of solutions to the above defined manipulation problem.

The first part of our approach consists, in fact, of answering the following question: is it possible to reduce any collision-free configuration space path describing motion in contact to a (finite) sequence of transit and transfer paths? Answering this question requires studying the local controllability of the manipulation system that is possible to associate with the manipulation model adopted in this paper. The analysis is described in the following section and is based on the result by Goodwine and Burdick [8] providing condition for controllability of kinematic control systems on stratified configuration spaces.

3 Controllability of the manipulation system

To answer the first part of the manipulation planning problem we define here the simple kinematics describing the manipulation system underlying the considered planning problem. This system has a *stratified configuration space* and we use the result in [8] to establish its small-time local controllability.

3.1 Controllability on stratified configuration spaces

We briefly recall here the main definitions and properties of stratified configuration spaces and the stratified controllability property that we prove to hold in our case.

Stratified configuration manifold (Definition 2.2 in [8]): Let M be a manifold (possibly with boundary), and n functions Φ_i: $M \mapsto \mathbb{R}$, $i = 1, \ldots, n$ be such that the level sets $S_i = \Phi_i^{-1}(0) \subset M$ are regular submanifolds of M, for each i, and the intersection of any number of the level sets, $S_{i_1 i_2 \ldots i_m} = \Phi_{i_1}^{-1}(0) \cap \Phi_{i_2}^{-1}(0) \cap \ldots \Phi_{i_m}^{-1}(0)$, $m \leq n$, is also a regular submanifold of M. Then M and the functions Φ_i, define a *stratified configuration space.*

The driftless systems defined on stratified configuration manifolds are described on each stratum, or on strata intersections, by equations of motion characterized by smooth vector fields and the only discontinuities present in the equations of motion are due to transitions on and off of the strata or their intersections.

Stratified controllability (Proposition 4.4 in [8]): if there exists a nested sequence of submanifolds at the configuration x_0

$$x_0 \in S_p \subset S_{p-1} \subset \ldots \subset S_1 \subset S_0 = M,$$

where the subscript is the codimension of the submanifold, such that the associated involutive distributions satisfy

$$\sum_{j=0}^{p} \bar{\Delta}_{S_j} \mid_{x_0} = T_{x_0} M$$

and each $\bar{\Delta}_{S_j}$ has constant rank for some neighborhood $V_j \subset S_j$, of x_0, then the system is stratified controllable from x_0 in M.

Stated differently, if the involutive closures of the distributions associated to each submanifold in the nested sequence intersect *transversely* then the system can flow in any direction in M. The proof of the above proposition provides also additional information particularly relevant in the motion planning context: the set of states reachable up to time T denoted by $R^V(x_0, \leq T)$ contains a neighborhood of x_0 for all neighbors V and all T. With such a T assigned, it is always possible to find a suitable neighborhood by limiting the system to flow in an open set the size of which depends inversely on the codimension of the lowest stratum.

This property is useful in proving that, given a path in contact between the robot and the objects, if the manipulation system is controllable, then it is always possible to approximate this path with a manipulation path contained in a tube with radius equal to the clearance of the contact path to the obstacles.

3.2 Stratified controllability of the manipulation system

For the stated manipulation problem, the ambient manifold M is given by the combined configuration space $\mathcal{C}$ and has dimension 6. The submanifolds are

defined by the contact conditions as described in Sect. 2.3. The lowest stratum is the double contact manifold and has codimension equal to 2. There are two submanifolds of codimension 1 (contact with only one of the two objects) and the sequence will include only one of them.
Denote by $\boldsymbol{x} = (x_R, y_R, x_{O_1}, y_{O_1}, x_{O_2}, y_{O_2})^T$ a configuration of the manipulation system, the equations of motion on each stratum are as follows. Recalling that, in the considered setting, R can only translate in the plane and the objects can be moved when in contact with R with a stable grasp, the equation of motion on each substratum has the form

$$\dot{\boldsymbol{x}} = g_1^{S_i} u_1 + g_2^{S_i} u_2$$

where u_1, u_2 are the inputs for the manipulation system and $g_1^{S_i}, g_2^{S_i}$ are the input vector fields that have a different expression on each substratum. In particular, in $S_0 = \mathcal{C}$ we have

$$g_1^{\mathcal{C}} = (1,\,0,\,0,\,0,\,0,\,0)^T, \quad g_2^{\mathcal{C}} = (0,\,1,\,0,\,0,\,0,\,0)^T.$$

These vector fields describe the motion of the robot alone on a leaf of $\mathcal{C}$ that depends on the position of the objects.

On the single-contact manifold $\mathcal{C}_{c_1}$ they have the expressions

$$g_1^{\mathcal{C}_{c_1}} = (1,\,0,\,1,\,0,\,0,\,0)^T, \quad g_2^{\mathcal{C}_{c_1}} = (0,\,1,\,0,\,1,\,0,\,0)^T$$

and on $\mathcal{C}_{c_2}$

$$g_1^{\mathcal{C}_{c_2}} = (1,\,0,\,0,\,0,\,1,\,0)^T, \quad g_2^{\mathcal{C}_{c_2}} = (0,\,1,\,0,\,0,\,0,\,1)^T.$$

Flowing along these vector fields amounts to moving the object in contact while staying on a leaf that depends on the position of the object that is not touched by the robot. Since both the single-contact manifolds have codimension 1, S_1 will be equal to either one of them in the sequence of nested submanifolds used to show controllability.

Finally on the double-contact manifold $S_2 = \mathcal{C}_{c_1,c_2}$ it is

$$g_1^{\mathcal{C}_{c_1,c_2}} = (1,\,0,\,1,\,0,\,1,\,0)^T, \quad g_2^{\mathcal{C}_{c_1,c_2}} = (0,\,1,\,0,\,1,\,0,\,1)^T.$$

On this stratum the objects move with the robot without changing the points of contact.

It is easy to verify that the stratified controllability proposition holds by choosing as involutive distributions

$$\bar{\Delta}\, S_2 = \text{span}\,(g_1^{\mathcal{C}_{c_1,c_2}} \;\; g_2^{\mathcal{C}_{c_1,c_2}})$$

$$\bar{\Delta}\, S_1 = \text{span}\,(g_1^{\mathcal{C}_{c_1}} \;\; g_2^{\mathcal{C}_{c_1}}) \text{ or } \bar{\Delta}\, S_1 = \text{span}\,(g_1^{\mathcal{C}_{c_2}} \;\; g_2^{\mathcal{C}_{c_2}})$$

$$\bar{\Delta}\, S_0 = \text{span}\,(g_1^{\mathcal{C}} \;\; g_2^{\mathcal{C}}).$$

Note that the codimension of the lowest stratum is given by the number of movable objects and has an impact on the complexity of the considered motion planning problem.

Figure 3 illustrates the stratification of the configuration space induced by the contact constraints. By virtue of the controllability property described above, any continuous path in contact between the robot and one or both objects in each stratum can be approximated by a manipulation path. This is referred to as *reduction property*.

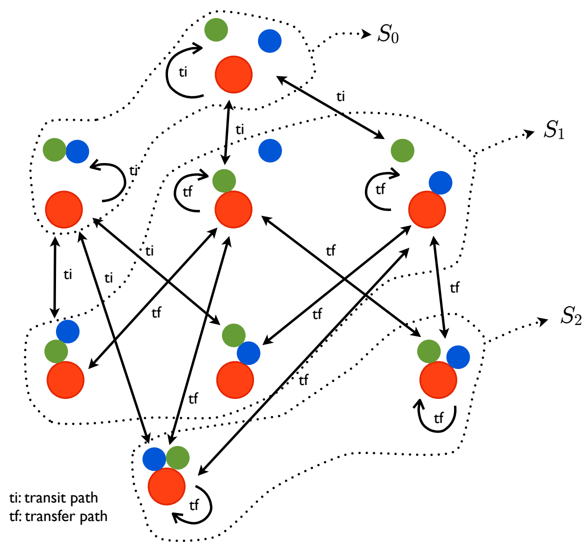

Fig. 3. Stratification of the configuration space induced by the contact constraints.

Having established that any path in the configuration space satisfying the contact constraint between the robot and the objects can be reduced to a manipulation path, we describe in the next section the construction of the manipulation graph.

4 Building the manipulation graph

The reduction property established in the previous section leads to the conclusion that any collision-free path in contact contained in a connected component of a same contact manifold is equivalent to a manipulation path. The key issue that remains is to build a geometric data structure that accounts for the decidability of the manipulation problem.

We propose here an extension of the manipulation graph as it has been introduced in [5] for the case of a single disk to move. In that case a single class

$GRASP$ representing the admissible (i.e., not in collision with static obstacle nor overlapping the object to move) contact configurations between the robot and the object was defined. The nodes of the manipulation graph were then given by the connected components of $GRASP$. The adjacency relation was given by the existence of transit paths between two nodes.[4]

In the case of two movable objects it is necessary to introduce two classes $GRASP_1$ and $GRASP_2$ and to build the manipulation graph over the connected components of $GRASP_1$ and $GRASP_2$.

The class $GRASP_1$ (resp. $GRASP_2$) represents all the configurations in $\mathcal{C}_{\text{free}}$ such that the robot is in contact with the object O_1 (resp. O_2). This means that the position of the object which is not in contact with the robot can change within the class. As a consequence, the reduction property shown in the previous section does not apply on the connected components of $GRASP_1$ and $GRASP_2$, i.e., any path in $GRASP_1$ and $GRASP_2$ cannot be necessarily approximated by a sequence of transit and transfer paths. This is the main difference compared to the case of a single object.

The reduction property holds however inside each leaf of the foliation of $GRASP_1$ (resp. $GRASP_2$) that keeps constant the position of O_2 (resp. O_1): any path inside these leaves can be approximated by a sequence of transit and transfer paths. These are the leaves of dimension 3 in the manifolds defined by the contact constraints schematically represented in Fig. 2.

The key questions are then: (i) how to determine the connected components of $GRASP_1$ and $GRASP_2$, and (ii) how to build a manipulation graph that will account for the existence of a manipulation path.

The answer to the first question is easy. $GRASP_1$ and $GRASP_2$ are components of the 5-dimensional contact submanifold of $\mathcal{C}_{\text{free}}$. If there exists a cell decomposition of the 6-dimensional space $\mathcal{C}_{\text{free}}$, then this cell decomposition induces by retraction on its boundary a cell decomposition of the 5-dimensional contact space (up to some potential singularities we do not consider in this paper). Then, such a cell decomposition leads to a straightforward characterization of the connected components of $GRASP_1$ and $GRASP_2$. The first question is then reduced to the existence of an algorithm that provides a cell decomposition for the case of three disks moving freely on a plane. It just so happens that Schwartz and Sharir [10] propose a general algorithm for many disks as an extension of their algorithm for two disks[5].

Notice that applying the retraction of the cell decompositions iteratively provides a cell decomposition of the various contact submanifolds, and ultimately a cell decomposition of $GRASP_{O_1,O_2} = GRASP_1 \cap GRASP_2$.

[4] In [9] the authors propose a generalization to the case where the object may be further subjected to some placement constraints. The nodes of the manipulation graph are the various connected components of $Grasp \cap Placement$ space and the adjacency relation is based on the existence of either transit paths or transfer paths.

[5] It should be noted that this extension is not trivial and, to our knowledge, it has never been implemented.

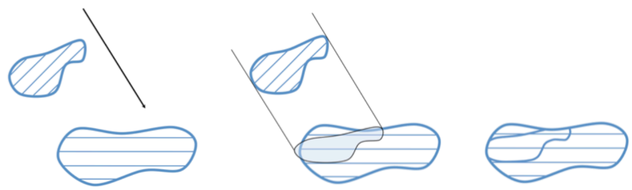

Fig. 4. Schematic illustration of the decomposition induced by projecting a cell onto another.

Building the manipulation graph is the second issue to be addressed. For this purpose, we refine the cell decompositions of the various connected components of $GRASP_1$ and $GRASP_2$ by considering their projections along the three directions of the foliations generated by: (i) transit paths (the robot moves alone), (ii) transfer paths of *type 1* (O_2 does not move), (iii) transfer paths of *type 2* (O_1 does not move). As a result, the projection of a given cell C_1 onto a cell C_2 induces a decomposition of C_2 into several cells C_{2_i} (see Fig. 4). Henceforth, we denote by the letter c all the cells issued from these refinement process.

Consider two points p_1 and p_2 in two cells c_1 and c_2 of $GRASP_1$ and $GRASP_2$ respectively. c_1 and c_2 are 5-dimensional. p_1 and p_2 belong respectively to two 3-dimensional leaves $\mathcal{L}_1$ and $\mathcal{L}_2$.

We consider two cases. Let us first consider the existence of a manipulation path remaining in the contact space. A necessary and sufficient condition for the existence of such a contact path between p_1 and p_2 is that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect a same connected component of $GRASP_{O_1,O_2}$. The existence of the path can be decided by computing a refinement of the cell decomposition of $GRASP_1$ and $GRASP_2$ as follows: consider the merging of the projections of both $GRASP_1$ and $GRASP_2$ cell decomposition along the direction of the respective foliations onto $GRASP_{O_1,O_2}$. It gives rise to a decomposition of $GRASP_{O_1,O_2}$ into many cells. Then refine the initial cell decomposition of $GRASP_1$ and $GRASP_2$ by "lifting" all cells in $GRASP_{O_1,O_2}$ along the foliations. Each elementary cell of $GRASP_{O_1,O_2}$ appears as the basis of two cylinders that contain cells of $GRASP_1$ and $GRASP_2$ respectively. The resulting cells of $GRASP_1$ and $GRASP_2$ constitute the nodes of the manipulation graph.

We then introduce the following adjacency relation: two cells in $GRASP_1$ (resp. $GRASP_2$) are adjacent if and only if they have a common frontier and they belong to a same cylinder. After the general method proposed in [11] it is known that the computation of such a cylindrical decomposition is possible, if not trivial.

For the second case, we consider the existence of a manipulation path between p_1 and p_2 that goes through the free-space. The main idea is the same as for the previous case. It is simpler because we have to consider only the foliation induced

by transit paths. The leaves of the foliation are 2-dimensional. We consider the cell decomposition of $GRASP_1$ and $GRASP_2$ after addressing the first case above. We add an edge between two cells c_1 and c_2 belonging respectively to $GRASP_1$ and $GRASP_2$ if and only if the the projection of c_1 onto c_2 along the foliation by transit path is not empty.
We have then the following
Theorem: There exists a manipulation path between two configurations in the free space if and only if these configurations retract on two cells belonging to the same connected component of the manipulation graph.

The proof follows the same principle as the proof in [5] and [9]

5 Conclusion

We have shown in this paper that for the manipulation planning problem for three disks (one robot and two movable objects) in the plane it is possible to construct an exact representation of the admissible (i.e., collision-free and satisfying the contact constraints) configuration space in the form of a manipulation graph to be search for a solution.

To prove the result, we have preliminarily generalized the so called *reduction property* to the case of double contact. Then, using the cell decomposition proposed by Schwartz and Sharir [10] and a specific analysis of the structure of the configuration space, we have illustrated the fundamental steps for the construction of the manipulation graph the connectivity of which accounts for the existence of a manipulation path.

Future work includes studying the case of an arbitrary number of movable objects and the adaptation of the result to more realistic manipulation systems. Different manipulation models, possibly including pushing and sliding are also a potential interesting evolution of this work.

Appendix

In this section we propose a constructive geometric proof of the reduction property for paths in configuration space constrained by contact between the robot and both objects. Preliminary to this proof is the conceptual illustration of the contact manifolds.

Single-contact manifold

Paths corresponding to motion in contact with only one object lie in a 5-dimensional manifold immersed in $\mathcal{C}_{\text{free}}$ that foliates with the position of the obstacle that is not in contact. On each leaf the reduction property in [5] can be applied to transform any path in contact into a sequence of transfer and transit paths. In principle, there exist two identical spaces of this kind, one for each object, and they are transversal to each other. We call these spaces $\mathcal{C}_{c_1}$ and $\mathcal{C}_{c_2}$. Figure 5 provides a conceptual illustration of $\mathcal{C}_{c_1}$ and the paths in $\mathcal{C}_{c_1}$ and $\mathcal{C}_{c_1} \cap \mathcal{C}_{c_2}$ represented in $\mathcal{C}_{c_1}$.

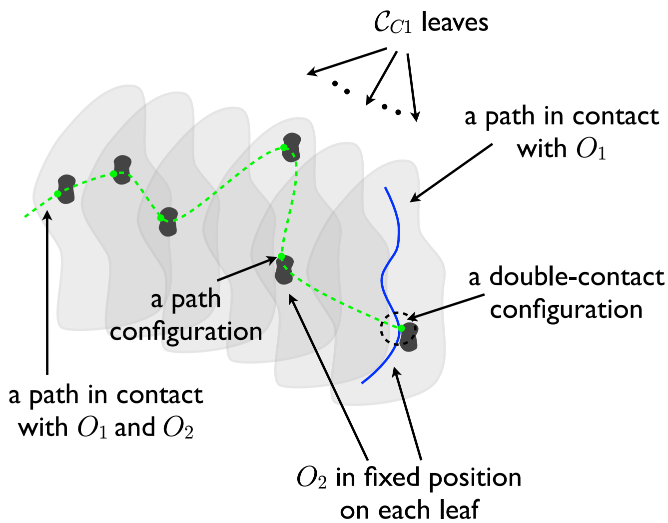

Fig. 5. Illustration of $\mathcal{C}_{c_1}$ and the paths in contact: $\dim(\mathcal{C}_{c_1}) = 5$ while the dimension of its leaves is 3. Each leaf is a replication of the configuration space of R in contact with O_1, the only difference between leaves being the position of O_2. A path in contact with both O_1 and O_2 is transversal to the leaves spanning $\mathcal{C}_{c_1}$. The manifold of contact configurations between R and O_2 has the same structure but is transversal to the space illustrated in the figure.

Double-contact manifold

Paths of the robot in contact with both objects belong to the 4-dimensional manifold $\mathcal{C}_{c_1,c_2} = \mathcal{C}_{c_1} \cap \mathcal{C}_{c_2}$ at the intersection between $\mathcal{C}_{c_1}$ and $\mathcal{C}_{c_2}$. A path in contact with both objects is represented by the green dashed path in Fig. 6 as a path "across" the foliation of one of the single-contact manifolds.

We start with the following conjecture: Because of the foliations of $\mathcal{C}_{c_1}$ and $\mathcal{C}_{c_2}$, any path in this manifold should be equivalent to a sequence of transfer paths with two contacts and paths in either $\mathcal{C}_{c_1}$ or $\mathcal{C}_{c_2}$. Figure 6 shows an example of such a decomposition: the green dashed path in contact with both objects can be reduced to the sequence composed by the black dotted path and the blue continuous path. Along the black dotted path both objects are in contact and the contact points do not change along the path. The path terminates where one of the object has reached the desired position. The blue path is a single-contact path lying on a leaf of one of the single contact manifolds. We know that the reduction property applies to paths in contact lying on either of these two manifolds, therefore, we only need to show that the green dashed path is equivalent to the sequence of black and blue paths. Figure 7 illustrates the property through an

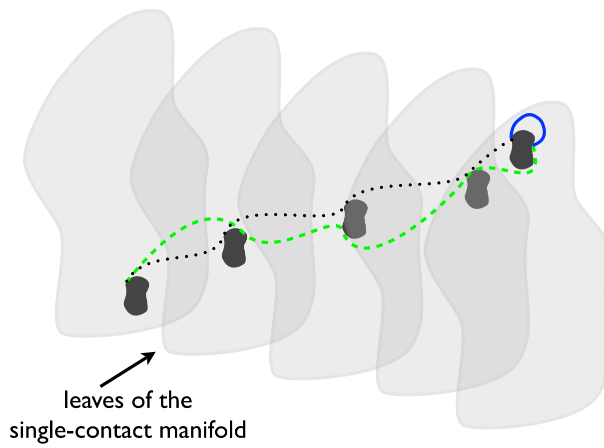

Fig. 6. Illustration of the "reduction property" to be proven: is the dashed path equivalent to the sequence of blue and green path?

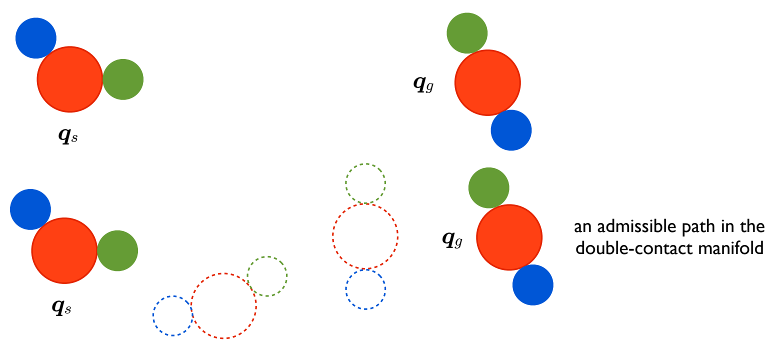

Fig. 7. Any path in the double-contact manifold is admissible to go from $\boldsymbol{q}_s$ to $\boldsymbol{q}_g$ but not any path is a manipulation path.

example: given the initial and the final configurations, respectively $\boldsymbol{q}_s$ and $\boldsymbol{q}_g$, any path in the double-contact manifold is admissible. Figure 8 shows how to reduce it to a sequence of transfer and transit paths. A formal proof to this *Generalized Reduction Property* follows.

Generalized reduction property

Generalized Reduction Property: Any two configurations belonging to the same connected component of the double-contact manifold can be connected by a manipulation path.

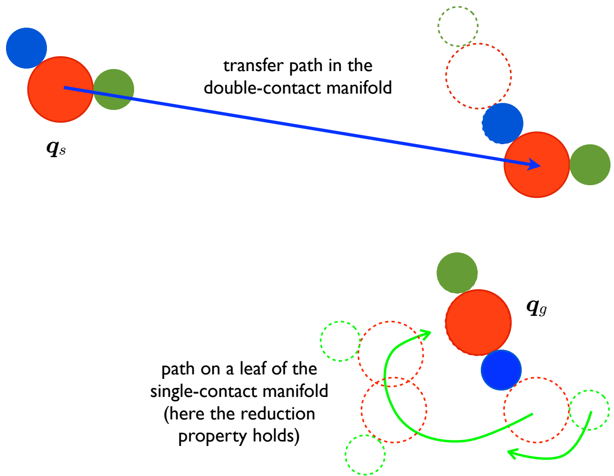

Fig. 8. Can any path in the double-contact manifold be "reduced" to a sequence of transfer and transit paths as in the figure?

Proof. It is a direct generalization of the reduction property proof in [5]. Let $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ be two configurations in the double-contact manifold connected by a collision-free path in $\mathcal{C}_{c_1,c_2}$. Note that, since the robot is not allowed to move in contact with static obstacles, this path is actually contained in the subset $\tilde{\mathcal{C}}_{c_1,c_2}$ of $\mathcal{C}_{c_1,c_2}$ of all configurations such that the robot is not in contact with any static obstacle. This is an open set in $\mathcal{C}_R$ but might not be in $\mathcal{C}$.

Denoted the collision-free path as $\boldsymbol{p} : [0,1] \to \tilde{\mathcal{C}}_{c_1,c_2}$, with $\boldsymbol{p}(0) = \boldsymbol{q}_a$ and $\boldsymbol{p}(1) = \boldsymbol{q}_b$, some preliminary definitions are in order:

$\boldsymbol{p}_R$: projection of $\boldsymbol{p}$ on $\mathcal{C}_R$;
$\boldsymbol{p}_{O_1}$: projection of $\boldsymbol{p}$ on $\mathcal{C}_{O_1}$;
$\boldsymbol{p}_{O_2}$: projection of $\boldsymbol{p}$ on $\mathcal{C}_{O_2}$;
$\boldsymbol{p}_{R-O_1}$: contact configuration relative to object O_1 on $\boldsymbol{p}$;
$\boldsymbol{p}_{R-O_2}$: contact configuration relative to object O_2 on $\boldsymbol{p}$.

Assume that the objects can neither be in contact with obstacles nor in contact between themselves (quite unrealistic, to be removed later) and let $\boldsymbol{q} = \boldsymbol{p}(s)$, $s \in [0,1]$, be a configuration on the path. Due to the non-contact hypothesis, it is always possible to define an open ball B_1 in the collision-free single-contact configuration space $\mathcal{C}_{c_1,\mathrm{free}}$, centered on the contact configuration $\boldsymbol{p}_{R-O_1}(s)$[6] and without considering O_2. Its projection D_{ϵ_1} in $\mathcal{C}_R$ is homeomorphic to a disk of radius $\epsilon_1 > 0$. The object O_1 will not collide with obstacles as long as it is in contact with $R \in D_{\epsilon_1}$. In the same way there exists a ball B_2 in the collision-free

[6] This is a point in $\mathcal{C}_{c_1}$.

single-contact configuration space $\mathcal{C}_{c_2,\text{free}}$, centered on the contact configuration $\boldsymbol{p}_{R-O_2}(s)$. Its projection D_{ϵ_2} in $\mathcal{C}_R$ is a disk of radius $\epsilon_2 > 0$.

Denote by $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Due to the continuity of $\boldsymbol{p}$, there exists an $\eta_R > 0$ such that

$$\forall \tau \in]s - \eta_R, s + \eta_R[,\ \boldsymbol{p}_R(\tau) \in D_{\epsilon/2},$$

an $\eta_1 > 0$ such that

$$\forall \tau \in]s-\eta_1, s+\eta_1[, ||(\boldsymbol{p}_R(\tau) - \boldsymbol{p}_{O_1}(\tau)) - (\boldsymbol{p}_R(s) - \boldsymbol{p}_{O_1}(s))|| < \epsilon/4,$$

and an $\eta_2 > 0$ such that

$$\forall \tau \in]s-\eta_2, s+\eta_2[, ||(\boldsymbol{p}_R(\tau) - \boldsymbol{p}_{O_2}(\tau)) - (\boldsymbol{p}_R(s) - \boldsymbol{p}_{O_2}(s))|| < \epsilon/4.$$

Denote by $\eta_3 = \min\{\eta_1, \eta_2\}$, and conclude that

$$\forall \tau \in]s-\eta_3, s+\eta_3[, ||(\boldsymbol{p}_{O_2}(\tau) - \boldsymbol{p}_{O_1}(\tau)) - (\boldsymbol{p}_{O_2}(s) - \boldsymbol{p}_{O_1}(s))|| < \epsilon/2.$$

Consider now $\eta = \min\{\eta_R, \eta_3\}$ and two configurations along the path: $\boldsymbol{q}_1 = \boldsymbol{p}(\tau_1)$ and $\boldsymbol{q}_2 = \boldsymbol{p}(\tau_2)$, with $\tau_1 < \tau_2$ and both in the interval $]s - \eta, s + \eta[$.

The path $\boldsymbol{p}(\tau) = (\boldsymbol{p}_R(\tau), \boldsymbol{p}_{O_1}(\tau), \boldsymbol{p}_{O_2}(\tau)), \tau \in [\tau_1, \tau_2]$ that transfers of O_1 in double-contact can be written as

$$\begin{aligned}
\boldsymbol{p}_R(\tau) &= \boldsymbol{p}_{O_1}(\tau) + (\boldsymbol{p}_R(\tau_1) - \boldsymbol{p}_{O_1}(\tau_1)) \\
\boldsymbol{p}_{O_1}(\tau) &= \boldsymbol{p}_{O_1}(\tau) \\
\boldsymbol{p}_{O_2}(\tau) &= \boldsymbol{p}_{O_1}(\tau) + (\boldsymbol{p}_{O_2}(\tau_1) - \boldsymbol{p}_{O_1}(\tau_1)),
\end{aligned}$$

and the transfer path $\boldsymbol{p}(\tau) = (\boldsymbol{p}_R(\tau), \boldsymbol{p}_{O_1}(\tau), \boldsymbol{p}_{O_2}(\tau)), \tau \in [\tau_1, \tau_2]$ of O_2 in single-contact to its goal position

$$\begin{aligned}
\boldsymbol{p}_R(\tau) &= (\boldsymbol{p}_{O_1}(\tau_2) + (\boldsymbol{p}_{O_2}(\tau) - \boldsymbol{p}_{O_1}(\tau)) + (\boldsymbol{p}_R(\tau_1) - \boldsymbol{p}_{O_2}(\tau_1)) \\
\boldsymbol{p}_{O_1}(\tau) &= \boldsymbol{p}_{O_1}(\tau_2) \\
\boldsymbol{p}_{O_2}(\tau) &= \boldsymbol{p}_{O_1}(\tau_2) + (\boldsymbol{p}_{O_2}(\tau) - \boldsymbol{p}_{O_1}(\tau))).
\end{aligned}$$

Finally, transit path $\boldsymbol{p}(\tau) = (\boldsymbol{p}_R(\tau), \boldsymbol{p}_{O_1}(\tau), \boldsymbol{p}_{O_2}(\tau)), \tau \in [\tau_1, \tau_2]$ of R to its goal:

$$\begin{aligned}
\boldsymbol{p}_R(\tau) &= (\boldsymbol{p}_{O_2}(\tau_2) + (\boldsymbol{p}_R(\tau) - \boldsymbol{p}_{O_2}(\tau)) \\
\boldsymbol{p}_{O_1}(\tau) &= \boldsymbol{p}_{O_1}(\tau_2) \\
\boldsymbol{p}_{O_2}(\tau) &= \boldsymbol{p}_{O_2}(\tau_2).
\end{aligned}$$

As a result of the choice of η these paths should all be feasible, i.e., collision-free. A symmetric argument can be provided if O_2 is transferred first to its goal position.$\diamond$

This proof could be completed by considering the case of objects-obstacles and object-objects contacts, but omitted due to lack of space. The critical point in this case is that double-contact motion could not be allowed because it would not possible to define an open disk in either of the two one-contact manifolds. It is then necessary to prove that a motion in double contact can be reduced to a sequence of motions in single contact. To achieve this reduction it is sufficient to break both contacts and move back to one of the two single-contact manifolds where the reduction property holds. It is, in fact, possible to show that there always exists a set of "escape" directions allowing the robot to un-grasp both obstacles.

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