MATHEMATICS' MORTUA MANUS: DISCOVERING DEXTERITY

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[1] Dexterous manipulation, a major subfield of robotics and manufacturing, experienced a mathematical rebirth in the mid 80's, when this nascent field established many beautiful connections to convexity theory and computational geometry. Jack Schwartz played a seminal role in its inception and development. Here, I speculate on where Jack might have liked this field to go in the future.

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Opening

After meeting Jack Schwartz, I promptly made up my mind to abandon theoretical (FOCS/STOC) computer science and embark upon a new career, combining mathematics, computer science and robotics. Jack promised to help. During my first year at Courant, a suspiciously simple-looking but thorny robotics problem kept popping up at our lunch and dinner conversations. Eventually, it led to the discovery of a surprisingly intimate relation between robot grasping and an elegant theorem due to Constantin Carathéodory.

Later, it dawned on me that Jack might have been mentoring me on the art of blending mathematics, computer science and robotics (or for that matter, any other applied field). After that experience, it has never been too difficult to be a "Bud-of-all-trades." But that's only a small part of all I have learned from Jack.

Carathéodory's theorem (belonging to a larger family of Helly-type theorems) [2], [3] is usually stated as follows: *If a point p of $\mathbb{R}^d$ lies in the convex hull of a set X, there is a subset $Y = \{y_1, \ldots, y_{r+1}\}$ of X consisting of $d+1$ or fewer points such that p lies in the convex hull of Y. Equivalently, p lies in an r-simplex with vertices in X, where $r \leq d$.*

Carathéodory proved his theorem in 1907 [4] for the case when X is compact. In 1914 Steinitz expanded Carathéodory's theorem for any sets X in $\mathbb{R}^d$. If one visualizes Carathéodory's theorem in 2 dimensions, it can be seen to state the existence of a triangle consisting of points from X that encloses any point enclosed by X — the theorem can be made constructive. For instance, when X has finitely many points, a triangulation of X's convex-hull will have a triangle containing any point in the convex hull of X. Consider a set $X = \{(0,0),(0,1),(1,0),(1,1)\}$, a subset of $\mathbb{R}^2$. The convex hull of this set is a square. Consider now a

[2] C Carathéodory. Über den Variabilitätsbereich der Koeffizienten von Potenzreihen die gegebene Werte nicht annehmen. *Math. Ann.*, 64:95–115, 1907.

[3] L Danzer, B Grünbaum, and V Klee. Helly's Theorem and its Relatives. *Convexity*, 7: 101–180, 1963.

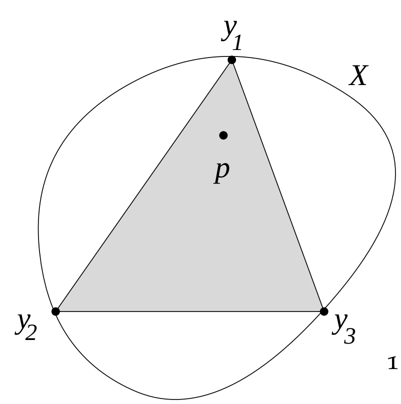

Figure 1: Example of Carathéodory's Theorem for $d = 2$.

[4] C Carathéodory. Über den Variabilitätsbereich der Koeffizienten von Potenzreihen die gegebene Werte nicht annehmen. *Math. Ann.*, 64:95–115, 1907.

point $p = (1/4, 1/4) \in \text{conv } X$. We can then construct a set $\{(0,0), (0,1), (1,0)\} = Y$ ($|Y| = 3$), the convex hull of which is a triangle and encloses p. Another set, containing p in its convex hull, of course is $\{(0,0), (1,1)\} = Y'$ ($|Y'| \leq 3$), but it presents a degenerate example. Similar arguments extend the theorem to higher dimensions.

Carathéodory was born in Berlin in 1873, to a prominent Greek family, closely involved with the Ottoman Empire. After attending a variety of schools in Belgium, Carathéodory finally enrolled as a student of artillery and engineering at the Belgian Military Academy in 1891, where he received extensive technical training in engineering. It covered some antiquated calculus but also courses in mechanics, probability, astronomy, geography, and thermodynamics. His lifelong fascination with descriptive geometry, a core area of engineering mechanics, began at the academy.

When in 1897 an annual Nile flood interrupted his job as an engineer, to kill time, he started studying mathematics: Jordan's *Cours d'Analyze*, Salmon's book on conics, etc. During this process, he became enamored with pure mathematics and decided – to the chagrin of his entire extended aristocratic family – to relinquish engineering. Soon, he was attending lectures in pure mathematics by Schwartz, Fuchs, and Frobenius, and on symbolic logic by Carl Friedrich Stumpf.

Carathéodory came to Göttingen in the summer of 1902, and met Zermelo, Born, Blumenthal, the Youngs (William and Grace), Minkowski, Klein and Hilbert. When he proved the theorem presented earlier, with its centrality in convexity theory, neither he nor his colleagues could foresee any possible application of the theorem — physical or otherwise. The purity (rather absence of any obvious usefulness) seemed to have delighted Carathéodory. Five decades later, when Carathéodory's work began to find applications in economic theories of markets and equilibria, they were dismissed as non-physical (hence artificial) applications, not affecting the utter purity with which Carathéodory had held his theorem.

However, our initial work (started with Schawrtz and Sharir) [5] and its sequels [6], [7], [8] showed how the theorem can be directly related to static problems in classical mechanics and applied for robots to plan "grasping," "work-holding" and "fixturing." With that, alas, whatever purity (imagined or real) Carathéodory might have bestowed on his theorem, seemed to have evaporated irrevocably. On the other hand, this might be seen as yet another example of marvels of mathematics: "The Unreasonable Effectiveness of Mathematics in the Natural Sciences [9]," of which Wigner wrote about so eloquently. "The miracle of the appropriateness of the language of mathematics for the formulation of the laws

[5] B Mishra, JT Schwartz, and M Sharir. On the Existence and Synthesis of Multifinger Positive Grips. *Algorithmica*, 2:541–558, 1987.

[6] B Mishra and N Silver. Some Discussion of Static Gripping and Its Stability. *IEEE Transactions on Systems, Man and Cybernetics*, 19:783–796, 1989.

[7] D Kirkpatrick, B Mishra, and C Yap. Quantitative Steinitz's Theorem with Applications to Multifingered Grasping. *Discrete & Computational Geometry*, 7:295–318, 1992.

[8] M Teichmann. *Grasping and Fixturing: a Geometric Study and an Implementation*. PhD thesis, New York University, New York, 1995.

[9] E Wigner. The Unreasonable Effectiveness of Mathematics in the Natural Sciences. *Communications in Pure and Applied Mathematics*, 13, 1960.

of physics is a wonderful gift, which we neither understand nor deserve. We should be grateful for it..."

Jack seemed to have been rather skeptical of the claims of mathematics' unreasonable effectiveness. "... The intellectual attractiveness of a mathematical argument, as well as the considerable mental labor involved in following it, makes mathematics a powerful tool of intellectual prestidigitation — a glittering deception in which some are entrapped, and some, alas, entrappers," Jack wrote in his 1986 essay entitled *"The Pernicious influence of Mathematics on Science* [10]."

[10] J Schwartz. *The Pernicious Influence of Mathematics on Science*, pages 230–235. Springer, 2006.

In the following few paragraphs, I will outline the intellectual prestidigitation necessary to claim the robot grasping problem as solved — with its elegant reformulation in convexity theory [11] and computational geometry [12], [13]. It is worth pondering how dexterously the assumptions might have been manipulated to bring about this mental entrapment — a reach exceeding our grasp, perhaps. But then, can we rebuild a more realistic theory and algorithms for robot grasping, which would also include hand design as well as kinematics, dynamics and control in their formulations? Few such ideas have been explored preliminarily and tentatively, as in the paradigm of "reactive robotics," a topic to which we will return eventually.

[11] L Danzer, B Grünbaum, and V Klee. Helly's Theorem and its Relatives. *Convexity*, 7: 101–180, 1963.

[12] H Edelsbrunner. *Algorithms in Combinatorial Geometry*. Springer-Verlag, 1987.

[13] J O'Rourke. *Computational Geometry in C*. Cambridge University Press, 1994.

Gripping

Imagine an idealized dextrous hand, consisting of several independently movable force-sensing fingers. These fingers move as points in three-dimensional space. The problem of *grip selection* for an object is to study how to hold that object in equilibrium with point fingers — in the absence of static friction between the surface of the object and the fingers. Since the fingers are assumed to be point fingers, a finger can only apply a force on the object along the surface-normal at the point of contact, directed inward.

When the shape of the object is precisely known, the problem of *grip selection* reduces to that of choosing a set of GRIP POINTS and a set of associated FORCE TARGETS. We may then ask two questions:

- *Can an arbitrary object be gripped with a finite number of fingers?*
- *If so, what are the grip points and the magnitudes of the forces exerted by the fingers (force targets) for such a grip?*

From elementary study of *statics* in classical mechanics, we know how an object in equilibrium can be characterized. We may think of the forces as *polygenic* (the force/torques applied at the fingers are generated by some actuators whose mechanics need not concern us). Equilibrium can be characterized by the *resultant* force and torque equation, as in the classical Newtonian mechanics.

Consider a rigid body subject to a set of external polygenic forces $f_1, \ldots, f_k$, applied respectively at the points $p_1, \ldots, p_k$, as in Figure 2. Then the necessary and sufficient condition for the rigid body to be in equilibrium is that *the resultant force and the resultant torque must be null vectors*. In mathematical notations, this condition can be stated as follows:

$$\sum_{i=1}^{k} f_i = 0 \quad \text{and} \quad \sum_{i=1}^{k} p_i \times f_i = 0,$$

where the cross product $\tau = p \times f$ gives a torque [14].

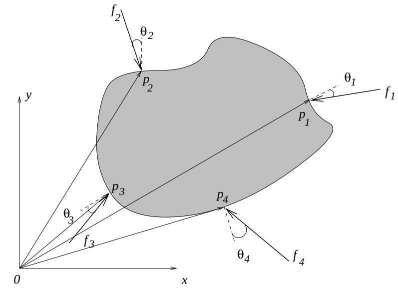

Figure 2: A planar object subject to four forces f_1, f_2, f_3 and f_4.

[14] The cross product $\tau = p \times f$ is defined as

$$\begin{aligned} \tau_x &= p_y f_z - p_z f_y, \\ \tau_y &= p_z f_x - p_x f_z, \quad \text{and} \\ \tau_z &= p_x f_y - p_y f_x. \end{aligned}$$

Thus, in order to hold an object in equilibrium with a multi fingered hand (say, with k fingers), we need to place these fingers at points $p_1, \ldots, p_k$ on the boundary of the objects and apply forces $f_1, \ldots, f_k$ in such a manner that the equilibrium condition is satisfied.

For example, consider a planar rectangular object with four grip points at the mid points of the edges (shown in Figure 3.)

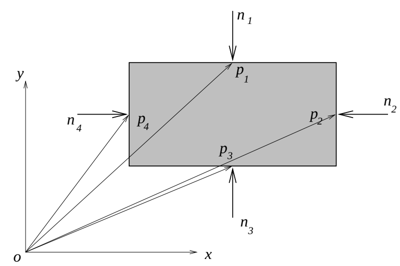

Figure 3: A planar rectangular object with designated grip points $\{p_1, p_2, p_3, p_4\}$.

In this example, let the grip points be denoted as p_1, p_2, p_3 and p_4 and the respective unit surface normals as n_1, n_2, n_3 and n_4. Then we wish to determine if there are four scalar quantities α_1, α_2, α_3 and α_4 such that

$$\begin{aligned} \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3 + \alpha_4 n_4 &= 0 \\ \alpha_1 (p_1 \times n_1) + \alpha_2 (p_2 \times n_2) + \alpha_3 (p_3 \times n_3) + \alpha_4 (p_4 \times n_4) &= 0 \\ \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0, \alpha_4 \geq 0 \quad \text{and} & \quad \text{not all } 0. \end{aligned}$$

Note that, for this example, any choice of $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$ will satisfy the conditions (assuming that at least two of them are nonzero and all of them are nonnegative). In particular, we could have chosen all the α's to be $1/4$!

To make matters little more abstract, we may define a *wrench map*, Γ, taking a point on the boundary of the object B to a point in the d-dimensional wrench space $\mathbb{R}^d$. Note that the term wrench space is used to denote a vector space consisting of all the wrenches. Its dimension d is 1, 3 or 6, depending on whether the object belongs to 1, 2 or 3-dimensional space.

$$\begin{aligned} \Gamma : \partial B &\rightarrow \mathbb{R}^d \\ : p_i &\mapsto (n_i, p_i \times n_i). \end{aligned}$$

Thus the wrench map Γ maps a point $p_i \in \partial B$ on the boundary of the body B to a wrench (a force/torque combination) that would be created if we apply a unit normal force directed inward at the point p_i. Then the feasibility of a positive grip can be expressed in terms of the existence of a solution of the following system of linear equations and inequalities:

$$\begin{aligned}\sum_{i=1}^{k} \alpha_i \Gamma(p_i) &= 0\\ \alpha_i &\geq 0, i = 1, \ldots, k,\\ \sum_{i=1}^{k} \alpha_i &= 1.\end{aligned}$$

The last condition is added only for convenience. Geometrically, we were then asking if some convex combination of the $\Gamma(p_i)$'s would yield the null vector. More compactly,

$$0 \in \text{convex hull } (\Gamma(p_1), \ldots, \Gamma(p_k))?$$

If the answer to the preceding question is yes, then we can hold the object in equilibrium with the given grip points by applying forces whose magnitudes simply correspond to the coefficients used in the convex combination to express the null vector.

How Many ...

One of the simplest problems in grasping theory can be stated as below:

Given: An arbitrary rigid 3-dimensional object B and some number k.

Determine: Whether one can choose k (finite) grip points, $\{p_1, p_2, \ldots, p_k\} \subseteq \partial B$ on the boundary of B such that the object can be grasped (positively) by placing fingers at those grip points.

$$\left(\exists ? \{p_1, \ldots, p_k\} \subseteq \partial B\right) \left[0 \in \text{conv } (\Gamma(p_1), \ldots, \Gamma(p_k))\right].$$

The answer to the problem turns out to be "yes" and the necessary number of fingers is SEVEN (and not five!).

The proof proceeds in three simple steps:

Step 1: Show that

$$0 \in \text{conv } \Gamma(\partial B),$$

where $\Gamma\colon \partial B \to \mathbb{R}^6 : p \mapsto (n, p \times n)$. This is a simple consequence of the fact that an object under uniform pressure remains in equilibrium. The proof of this claim can be given rigorously using the *Divergence theorem of Gauss*.

Step 2: By Carathéodory's theorem

$$\Big(\exists\ \{\Gamma(p_1), \ldots, \Gamma(p_k)\} \subseteq \Gamma(\partial B)\Big)\ \Big[k \leq 7 \text{ and } 0 \in \operatorname{conv}\left(\Gamma(p_1), \ldots, \Gamma(p_k)\right)\Big].$$

Hence there are positive nonnegative scalar quantities $\alpha_1, \ldots,$ α_k such that:

$$\begin{aligned} \alpha_1 n_1 + \cdots + \alpha_k n_k &= 0, \\ \alpha_1(p_1 \times n_1) + \cdots + \alpha_k(p_k \times n_k) &= 0. \end{aligned}$$

Step 3: The positive grip is then selected by choosing as grip points

$$\begin{aligned} \text{Grip Points} &= \{p_1, \ldots, p_k\} \subseteq \partial B, \\ \text{Force Magnitudes} &= \alpha_1, \ldots, \alpha_k, \end{aligned}$$

with k no larger than 7.

Similar arguments in the plane imply that one would need FOUR fingers. The number four is arrived at by taking the dimension of the wrench space and adding one to it, as implied by the Carathéodory's theorem. It is also instructive to examine a set of equilibrium grasps for three planar objects: a rectangle, a triangle and a disc. First consider the grasps for the rectangle. Clearly, the grasps (a) and (d) are not as secure as (g)—a horizontal external force will break the grasp (a) and an external torque about the center of the rectangle will break the grasp (d). In comparison, the grasp (g) is immune to such external disturbances, provided of course that such disturbances are relatively small in magnitude. Similar examination will show that the grasp (h) is the most secure for a triangle. However, in the case of the disc, while the grasps (f) and (i) are better than (c), there is simply no way to resist an external torque about the center irrespective of how many fingers are used.

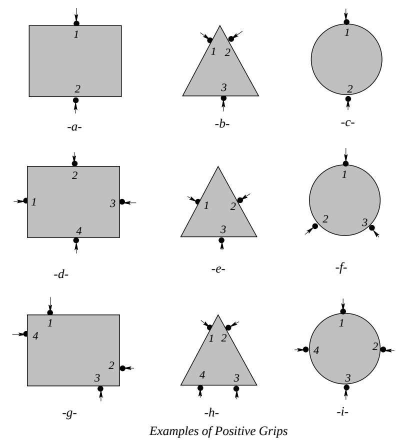

Figure 4: Grasping planar objects.

The kinds of secure grasps described in the preceding paragraph have been characterized as *closure grasps*. Furthermore, exactly those objects that do not allow closure grasps can also be characterized in purely geometric terms, and are referred to as *exceptional objects*. While we shall not go into a detailed description of closure grasps and exceptional objects (see [15]), it should suffice for the present purpose to say that the only planar bounded exceptional object is a disc and the only spatial bounded exceptional object is an object bounded by a surface of revolution[16].

[15] B Mishra, JT Schwartz, and M Sharir. On the Existence and Synthesis of Multifinger Positive Grips. *Algorithmica*, 2:541–558, 1987.

[16] If one allows unbounded objects then in 3-D, we have to include unbounded prisms and helical objects and in 2-D an unbounded strip of constant width. These objects in 3-D describe the so-called Reuleux pairs, studied almost a century ago.

Algorithm

At this point, it is natural for a roboticist to ask how one (a robot) can construct a grasp for a specific object and what sorts of computation this may entail. The answer turns out to be very interesting and shows a close connection of this problem to a classical algorithm, "*the simplex method*," used for solving linear programming problems.

Thus, suppose we have a polyhedral object with n faces. We proceed in a manner not very dissimilar from the ways we proved the existences of such a grasp. We first create a grasp with extremely large number of fingers: about $15n$ grip points, where n is the number of faces of the polyhedron. Next, step by step, we can eliminate one finger in each step while maintaining grasp as long as the number of grip points at the beginning of that step is strictly larger than the lower bound. The algorithm terminates when we are left with appropriate number of grip points (or fewer).

In order to understand the process by which the fingers are eliminated, we shall digress to consider an algorithmic approach to *algebraic manipulation with positive linear combinations*.

Given: A set of vectors $\{V_1, V_2, \ldots, V_l\} \subseteq \mathbb{R}^d$ and $V \in \mathbb{R}^d$ such that

$$\begin{aligned} \alpha_1 V_1 + \cdots + \alpha_l V_l &= \alpha V \\ & \alpha_i \geq 0, \alpha > 0, V \neq 0. \end{aligned}$$

Find: A subset $m \leq d$ vectors

$$\{V_{i_1}, V_{i_2}, \ldots, V_{i_m}\} \subseteq \{V_1, \ldots V_l\} \quad \text{and} \quad \alpha' > 0$$

such that

$$\begin{aligned} \alpha'_1 V_{i_1} + \cdots + \alpha'_m V_{i_m} &= \alpha' V \\ & \alpha'_i \geq 0, (\alpha' > 0, V \neq 0). \end{aligned}$$

Reduction Algorithm

if $l \leq d$ then HALT;

else repeat

Choose d vectors from $\{V_1, \ldots, V_l\}$
(Say, the first d): $\{V_1, \ldots, V_d\}$
There are two cases to consider, depending on whether the vectors $V_1, \ldots, V_d$ are *linearly dependent* or not.

Case 1: $V_1, \ldots, V_d$ **are linearly dependent.**

We can write

$$\beta_1 V_1 + \cdots + \beta_d V_d = 0,$$

not all $\beta_i = 0$.
Assume that at least one $\beta_i < 0$ (otherwise, replace each β_i by $-\beta_i$ in the equation to satisfy the condition.)
Let

$$\gamma = \min_{\beta_i < 0} (\alpha_i / \beta_i) < 0.$$

(For specificity, we may assume $\gamma = \alpha_1/\beta_1$.)
Put $\alpha'_i = \alpha_i - \gamma\beta_i$ for $1 \leq i \leq d$.
Hence by adding the equation ($\sum_{i=1}^{l} \alpha_i V_i = \alpha V$) to ($-\gamma \sum_{i=1}^{d} \beta_i V_i = 0$), we get

$$\alpha'_2 V_2 + \cdots + \alpha'_d V_d + \alpha_{d+1} V_{d+1} + \cdots + \alpha_l V_l = \alpha V,$$

and by construction $\alpha'_2, \ldots, \alpha'_d \geq 0$.

Case 2: $V_1, \ldots, V_d$ **are linearly independent.**
We can write

$$\beta_1 V_1 + \cdots + \beta_d V_d = V.$$

Assume that at least one $\beta_i < 0$ (otherwise, we have nothing more to do!)
Let

$$\gamma = \min_{\beta_i < 0} (\alpha_i / \beta_i) < 0.$$

(For specificity, we may assume $\gamma = \alpha_1/\beta_1$.)
Put $\alpha'_i = \alpha_i - \gamma\beta_i$ for $1 \leq i \leq d$, and $\alpha' = \alpha - \gamma > 0$.
Hence by adding the equation ($\sum_{i=1}^{l} \alpha_i V_i = \alpha V$) to ($-\gamma \sum_{i=1}^{d} \beta_i V_i = -\gamma V$), we get

$$\alpha'_2 V_2 + \cdots + \alpha'_d V_d + \alpha_{d+1} V_{d+1} + \cdots + \alpha_l V_l = \alpha' V,$$

and by construction $\alpha'_2, \ldots, \alpha'_d \geq 0$.

In algorithmic terminology, we can prove that "the reduction algorithm has a time complexity of $O(ld^3)$." In our grasping application, d will turn out to be a constant ($= 6$) and l no more than $15n$.

Let us get back to our original question about grasping a polyhedron B with n faces. As hinted earlier, we shall start with a closure grasp of B using no more than $15n$ grip points. Assume that B is provided with a triangulation of each face, and

$$t_1, t_2, \ldots, t_N$$

is the set of triangles partitioning ∂B. For each triangle t_i, choose three non-collinear grip points p_{i_1}, p_{i_2} and $p_{i_3} \in t_i$ such that

$(p_{i_1} + p_{i_2} + p_{i_3})/3$ is the centroid of t_i. In totality they will give us the initial $3N$ grip points. Using Euler's formula and some simple combinatorics, one can show that $N \leq 5n - 12$ and the total number of grip points is no more than $15n - 36$ (see [17]).

Now, it can be shown that if one chooses p_{i_j}'s, $1 \leq i \leq N$, $j = 1$, 2, 3, as the grip points then they give rise to a closure grasp. In particular, we can see [18] (by using linear algebraic manipulations) that

$$\frac{\text{Area}\,(t_1)}{3}\Gamma(p_{1_1}) + \frac{\text{Area}\,(t_1)}{3}\Gamma(p_{1_2}) + \frac{\text{Area}\,(t_1)}{3}\Gamma(p_{1_3}) + \cdots + \frac{\text{Area}\,(t_N)}{3}\Gamma(p_{N_1}) + \frac{\text{Area}\,(t_N)}{3}\Gamma(p_{N_2}) + \frac{\text{Area}\,(t_N)}{3}\Gamma(p_{N_3}) = 0,$$

and that

$$\text{pos}\left(\Gamma(p_{1_1}), \Gamma(p_{1_2}), \ldots, \Gamma(p_{N_3})\right) = \mathbb{R}^6.$$

Henceforth, rewriting these grip points as $\{p_1, p_2, \ldots, p_l\}$, and the "area terms" as magnitude of coefficients: $\alpha_1, \alpha_2, \ldots, \alpha_l$, we have

$$\alpha_1\Gamma(p_1) + \alpha_2\Gamma(p_2) + \cdots + \alpha_l\Gamma(p_l) = 0, \quad (1)$$

where $\alpha_i > 0$. Furthermore, since

$$\text{lin}\left(\Gamma(p_1), \Gamma(p_2), \ldots, \Gamma(p_l)\right) = \mathbb{R}^6,$$

without loss of generality, assume that the first six wrenches are linearly independent, thus spanning the entire wrench space, i.e.,

$$\text{lin}\left(\Gamma(p_1), \ldots, \Gamma(p_6)\right) = \mathbb{R}^6.$$

Synthesizing a Equilibrium Grasp with Seven Fingers Let us now see how we can go from here to get a simple equilibrium grasp with no more than seven fingers. Note first that we can rewrite our equation 1 (for l-fingered grip) as

$$\frac{\alpha_1}{\alpha_l}\Gamma(p_1) + \cdots + \frac{\alpha_{l-1}}{\alpha_l}\Gamma(p_{l-1}) = -\Gamma(p_l),$$

where $\alpha_i > 0$ and $\Gamma(p_i) \in \mathbb{R}^6$. Now, we can use the "Reduction Algorithm" to find

$$\{p_{i_1}, p_{i_2}, \ldots, p_{i_m}\} \subseteq \{p_1, \ldots, p_{l-1}\}$$

satisfying the conditions below:

$$\alpha'_1\Gamma(p_{i_1}) + \cdots + \alpha'_m\Gamma(p_{i_m}) = -\alpha'\Gamma(p_l),$$

[17] B Mishra, JT Schwartz, and M Sharir. On the Existence and Synthesis of Multifinger Positive Grips. *Algorithmica*, 2:541–558, 1987.

[18] B Mishra, JT Schwartz, and M Sharir. On the Existence and Synthesis of Multifinger Positive Grips. *Algorithmica*, 2:541–558, 1987.

and $m \leq 6$. Thus we have

$$\alpha'_1\Gamma(p_{i_1}) + \cdots + \alpha'_m\Gamma(p_{i_m}) + \alpha'\Gamma(p_l) = 0,$$

with $\alpha'_1 \geq 0, \ldots, \alpha'_m \geq 0$ and $\alpha' > 0$. Of course, this is our equilibrium grasp using no more than $m+1 \leq 7$ fingers, placed at grip points $p_{i_1}, \ldots, p_{i_m}, p_l$ with associated force magnitudes $\alpha'_1, \ldots, \alpha'_m, \alpha'$.

As analyzed earlier, our grasping algorithm could be shown to take $O(n)$ time with a constant in the complexity growing as $O(d^3)$.

A few years later, in 1990, Papadimitriou and his colleagues revisited the problem [19], and proved (without appealing to Carathéodory-like theorems) similar bounds on number of fingers. They also showed how to turn the algorithmic problem into a linear programming problem in certain special cases (e.g., planar convex obects or non-convex objects with bounded number of concave angles). As Megiddo had shown that these linear programming problems have linear time solutions, when the dimension d is treated as a constant, Papadimitriou et al. had also demonstrated that grasping could be done in linear time – at least, for certain special geometries.

[19] X Markenscoff, L Ni, and CH Papadimitriou. The Geometry of Grasping. *International Journal of Robotics Research*, 9:61–74, 1990.

Thus, it seemed that any comprehensible formulation of the grasping problem would unavoidably appeal to convexity theory (to Carathéodory's dismay). However, the complexity of the algorithms (thus applicability) depended crucially on the exact formulation – going the Megiddo [20] route meant that the algorithm would have an $O(d2^{O(d)}n)$ time complexity. Big Ouch!

[20] N Megiddo. Linear programming in linear time when the dimension is fixed. *J. ACM*, 31 (1):114–127, 1984.

However, except for few such theoretical quibbles, the grasping problem had been more or less solved and with *panache blanc* – or so we thought.

Groping

In an article [21] appearing about a decade later, it was lamented that, "Notwithstanding the great effort spent, and the [impressive] technological and theoretical results achieved by the robotics community in building and controlling dexterous robot hands, the number of applications in the real-world and the performance of such devices in operative conditions should be frankly acknowledged as not yet satisfactory. In particular, the high degree of sophistication in the mechanical design prevented so far dexterous robotics hand to succeed in applications where factors such as reliability, weight, small size, or cost, are at a premium. One figure partially representing such complexity is the number of actuators, which ranges between 9 and 32 for hands considered above.

[21] A. Bicchi. Hands for dextrous manipulation and robust grasping: a difficult road towards simplicity. *IEEE Trans. on Robotics and Automation*, 16(6):652–662, December 2000.

Further reduction of hardware complexity, even below the theoretically minimum number of 9, is certainly one of the avenues for overcoming this impasse." Thus, while the elegant theory we had developed gave many insights into how to create a field of dexterous manipulation, the industrial (or elsewhere) applications of robot hands have never really embraced the needed complexity. Instead, simple parallel-jaw grippers still rules the manufacturing world.

What could be done? How can we connect the mathematical theories with applications. Jack [22] had worried that, "Related to this deficiency of mathematics ... is the simple-mindedness of mathematics – its willingness to elaborate upon any idea, however absurd; to dress scientific brilliancies and scientific absurdities alike in the impressive uniform of formulae and theorems. Unfortunately however, an absurdity in uniform is far more persuasive than an absurdity unclad." We may wish to return to the various underlying assumptions of the grasping theories to separate the ones that are apt from those that are absurd.

[22] J Schwartz. *The Pernicious Influence of Mathematics on Science*, pages 230–235. Springer, 2006.

Setting aside the issues of finger properties (friction, softness, compliance, etc.) [23], object properties (degrees of freedom, deformability, elasticity, etc.), closure grasps [24], grasp quality [25], [26], grasp stability, robustness, gaiting, grasp planning, hand kinematics, dynamics and control, one may just focus on one issue: why simple hands have done so much better. For instance, a parallel-jaw gripper works well only with objects with antipodal grip points and of simple geometry (e.g., $2\frac{1}{2}$-dimensional), and yet it is ubiquitous.

[23] B Mishra and M Teichmann. *The Power of Friction: Quantifying the 'Goodness' of Frictional Grasps*, pages 311–320. A.K. Peters, Wellesley, MA, 1997.

[24] B Mishra and N Silver. Some Discussion of Static Gripping and Its Stability. *IEEE Transactions on Systems, Man and Cybernetics*, 19:783–796, 1989.

[25] D Kirkpatrick, B Mishra, and C Yap. Quantitative Steinitz's Theorem with Applications to Multifingered Grasping. *Discrete & Computational Geometry*, 7:295–318, 1992.

In a recent publication, Matt Mason and colleagues [27] asked, "While complex hands offer the promise of generality, simple hands are more practical for most robotic and telerobotic manipulation tasks, and will remain so for the foreseeable future. This raises the question: how do generality and simplicity trade off in the design of robot hands?" Their answer was to focus on using "knowledge of stable grasp poses as a cue for object localization." Yet, a different approach is to integrate the hand design, grasp control algorithm and grasp selection into one framework – as done in our work on "Reactive Robotics."

[26] B Mishra. *Grasp Metrics: Optimality and Complexity*, pages 137–166. A.K. Peters, Wellesley, MA, 1995.

[27] M Mason, SS Srinivasa, and AS Vazquez. Generality and Simple Hands. In *International Symposium on Robotics Research*, 2009.

With my students and colleagues, we invented a clever parallel-jaw reactive gripper, and showed how to drive its grasp control algorithm by a set of discrete rules, that simply translate certain boolean conditions determined by the sensors into immediate ("reactive") actions of the actuators. The gripper could very quickly grasp any convex object in just two antipodal points and enjoys many robustness properties.

Similar ideas can be extended to three-finger-hands, such as the commercially-available Barrett hand [28], which is stiff, not

[28] B Technolgies. The Barrett Hand. `http://www.barrett.com/robot/products-hand.html`.

frictionless and has 7 DOFs (degrees of freedom), four of which are active. The key idea is to use the local geometry of the object to find a set of grip points, while solving the local motion planing problem of getting the fingers to their grasping positions.The object is assumed to have a smooth boundary and be convex. The grasping algorithm can be shown to be "non-disturbing," (i.e., it does not affect the object's location or motion, until it is grasped.)

The gripper consists of 3 fingers, simplified by the constraint to have their end-points move in a plane. The fingers move arbitrarily, but their order (around the triangle they form) remains fixed. The "reactive 3-finger hand" searches for three grip points by following the object boundary until some geometric condition is satisfied. Each finger is equipped with simple sensors that allow them to follow the object's contour and can determine the angle of the object boundary (it is close to). The sensors that may be considered are: (1) an omni-directional distance sensor (measuring distance to the object in any direction), and (2) an angle sensor (measuring angle of the object boundary at the closest point). Such sensors can be easily built using a pair of simple IR reflective sensors.

The key idea behind the grasping algorithm is for the hand to discover "reactively" a locally minimal area triangle that encloses the object. The grip points can be determined from this triangle via a theorem of Klee [29]: *if T has a locally minimum area among all triangles containing a convex body B, then the midpoints of each side of T touches B.* It can also been shown that [30] if the midpoint of an edge *e* of a triangle does not touch the object then *e* can be perturbed such that its midpoint after perturbation lies inside the original triangle. This perturbation reduces the triangle area. Thus, the grasping algorithm has two phases:

- **Phase 1**: Find a triangle that contains the object, by, say, closing the fingers along three concurrent lines spaced at equal angles (120^o) from each other, until they come to close proximity of the object boundary. If the triangle is not "bounded," the hand can fix it by a small perturbing rotation.

- **Phase 2**: Find a locally minimal triangle enclosing the object. The basic step requires a finger to do the following: the finger divides its triangle edge into two segments and moves in the direction of the larger segment. Consequently, both the ratio of the larger segment to smaller segment for each edge will be reduced; so will the area of the enclosing triangle. It may seem that each finger has to move one at a time synchronously, but that is really not necessary, if certain care is taken as one approaches convergence to a grasp.

[29] V Klee. *Facet Centroids and Volume Minimization*. 1986.

[30] J O'Rourke, A Aggarwal, S Maddila, and M Baldwin. An optimal algorithm for finding minimal enclosing triangles. *J. Algorithms*, 7(2):258–269, 1986. ISSN 0196-6774. doi: http://dx.doi.org/10.1016/0196-6774(86)90007-6.

The reactive hand works (like an analog computer) by minimizing a potential function defined by the triangle area, and from this an appropriate notion of stability and robustness can be derived. Once the triangle is determined, since the lines through edge mid-points and perpendicular to the corresponding edges are concurrent at the point which is at the center of the circumscribing circle of the triangle, we can use these midpoints to get a planar (force-closure) grasp without relying on friction. If the object and finger-tips have some static friction (which is true of Barrett hands) then the resulting grasp also has a planar torque closure. More details can be found in [31], [32].

[31] B Mishra and M Teichmann. Reactive Algorithms for 2 and 3 Finger Grasping. In *Proceedings of the 1994 International Workshop on Intelligent Robots and Systems, IRS 94*, 1994a.

[32] B Mishra and M Teichmann. Reactive Algorithms for Grasping Using a Modified Parallel Jaw Gripper. In *Proceedings of the 1994 IEEE International Conference on Robotics and Automation , ICRA 94*, 1994b.

What is more interesting is the way the reactive gripper may be anthropomorphized: The corresponding reactive algorithm will appear to have three fingers groping around an object blindly (as they have not performed any a priori computation on a model of the object) until deciding on a grip. What separates groping from gripping? Isn't groping just an analog computation performed by the finger sensors and actuators to solve an optimization problem (namely, the minimal point of a potential function, determined by the area of an enclosing triangle)? So then what exactly is a computation in robotics? How does a robotic algorithm separate sensing, planning and actuation?

I wish I knew how Jack might have thought about these questions...

Closing

Over the last year, I have realized how much we all miss Jack, his polymathic and eclectic conversational topics and gentle mentoring. With Jack's death, Courant seems to have lost a significant part of its basic character.

Soon after my arrival at Courant in 1985, Jack had walked me over to the intersection of Mercer and fourth, and given me my first and the shortest tour of Manhattan: At the time, there was a Swensen's right across from Courant, a music place called Bottom Line on fourth and a Yeshiva in the opposite corner which still stands. He pointed out that without going too far I could now have food, religion, music and mathematics – that was all the Manhattan I needed. Jack never ever mentioned religion after that.

Later Jack took it upon himself to introduce me to all sorts of exotic food and information: Alexander's campaign route through Bactria and Parthia to India, explained over Matzo ball soup in Second Avenue Deli; Ferdowsi's Shahnameh and its significance to Persian culture over some ultra-hot vindaloo in Curry in a Hurry and how to design a balloon robot over many many servings of twice-cooked pork. He also told me that he considered himself a gourmet diner who likes to try the best and the tastiest from every cuisine – that too was his style in science and mathematics.

Jack seemed to have found interesting mathematics in almost everything: how to move a piano, how to grasp a greasy pig, how to manage personal relationships, how to trade in foreign exchange markets, how to visualize a genome, how to write music to be read by a computer, how to use cartoons to explain special theory, how to become immortal and a zillion other things like that.

With Jack, everything led to voracious gourmet feasting. Jack always skipped his appetizers, and never lingered on for the desserts.

[33]

[33] The paper has improved considerably following many insightful suggestions from several colleagues: most notably, S. Kleinberg and E. Schonberg of NYU, M. Mason of CMU and M. Wigler of CSHL.

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