Social Networks
Solution to Homework 3

Q1. [5] Exercise #1
In *G(n, p)* the probability of a vertex having degree *k* is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

Show by direct calculation that the expected degree is *np*. Where is the mode of the binomial distribution? [Mode is the point at which the probability is maximum.] Compute directly the variance of the distribution.

Answer:

$$E(k) = \sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

Assume $s = k-1, m = n-1,$

$$E(s+1) = np \sum_{s=0}^{m} \frac{m!}{s!\,(m-s)!} p^s (1-p)^{m-s} = np(p+1-p)^m = np$$

Mode is $\lfloor (n+1)p \rfloor$

$Var(k) = n(n-1)p^2 + np - (np)^2 = n^2p^2 - np^2 + np - n^2p^2$

$= np - np^2 = np(1-p)$

Q2. [5]

In *G(n, 1/n)* what is the probability that there is a vertex of n degree *logn*? Give an exact formula; also derive simple approximations.

Answer:

$$P(logn) = \binom{n}{logn} \left(\frac{1}{n}\right)^{logn} \left(1-\frac{1}{n}\right)^{n-logn} = \binom{n}{logn} \frac{e^{-1}}{(n-1)logn}$$

Q3. [10]
What is the expected number of triangles and squares (3-cycles & 4-cycles) in *G(n, d/n)*? What is the expected number of 4-cliques in *G(n, d/n)*?

Answer:

In the graph G(n, d/n), there are $\binom{n}{3}$ possible sets of 3 vertices, and the probability of being a triangle is (d/n)^3. The expected number of triangles is $\binom{n}{3}\left(\frac{d}{n}\right)^3$.

Similarly, the expected number of squares is $\binom{n}{4}\left(\frac{d}{n}\right)^4$. The expected number of 4-cliques is $\binom{n}{4}\left(\frac{d}{n}\right)^6$