SOCIAL NETWORKS

LECTURE # 11

Google Graph

$G = (V, E)$ V = Pages.
E = Hyperlinks
$(u, v) \in E$ if u contains a hyperlink to v.

Directed graph determined by hyperlinks.

A = Adjacency Matrix

$a_{ij} \in \{0, 1\}$; $a_{ij} = 1$ iff $(v_i, v_j) \in E$

D = Diagonal Degree Matrix
$d_{ii} = \deg(v_i) = |V_i|$ = # Hyperlinked page

V_i = Neighbours of v_i
= Hyperlinked pages
$= \{ v_j \mid (v_i, v_j) \in E \}$

$$p_{ij} = \frac{a_{ij}}{|V_i|} = \frac{a_{ij}}{d_{ii}} = \boxed{\text{Probability with which Random Surfer moves from } v_i \text{ to } v_j}$$

$$\Leftrightarrow \quad P = D^{-1} A = I + \Delta.$$

RANDOM SURFER:

When the surfer arrives at node v_i, if $V_i \neq \phi$, he chooses a hyperlink to $v_j \in V_i$ uniformly randomly, i.e. with prob $= p_{ij}$

$a_i = 1$ iff $|V_i| = 0$ ← A dangling node.

$\vec{a}$ = Dangling Node Vector $a_i \in \{0,1\}$.

$a_i = 1$ iff $V_i = \phi$.

RANDOM TELEPORTATION.

When the surfer arrives at a dangling node v_i (with $V_i = \phi$), he teleports stochastically to a node $v_j \in V$ uniformly randomly, ie with prob $= \frac{1}{n}$

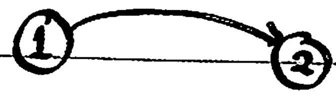
$$\therefore P' = P + a\frac{\mathbb{1}^T}{n}$$

→ Rows of P' ↔

Either same as P if the node is not a dangling node, or $(\frac{1}{n}, \dots, \frac{1}{n})$

48

I

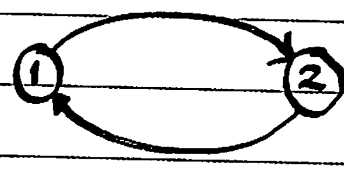


$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = D^{-1}A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad P' = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

$$W = \frac{I + P'}{2} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

II



$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = D^{-1}A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P'$$ (No dangling node)

$$W = \frac{I + P'}{2} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Stochastic States

I $p_0 = (1, 0)$

$\hookrightarrow p_1 = p_0 P' = (0, 1)$

$\hookrightarrow p_2 = p_1 P' = (1/2, 1/2)$

$\hookrightarrow p_3 = p_2 P' = (1/4, 3/4)$

$\hookrightarrow p_4 = p_3 P' = (3/8, 5/8)$

$\hookrightarrow p_5 = p_4 P' = (5/16, 11/16)$

$\hookrightarrow p_6 = p_5 P' = (11/32, 21/32)$

$\dots (1/3, 2/3)$

II $p_0 = (1, 0)$

$\hookrightarrow p_1 = p_0 P' = (0, 1)$

$\hookrightarrow p_2 = p_1 P' = (1, 0)$

$\hookrightarrow p_3 = p_2 P' = (0, 1)$

$\dots$

Modified Random Surfing.

$p_0 = (1, 0)$

$\hookrightarrow p_1 = p_0 W = (1/2, 1/2)$

$\hookrightarrow p_2 = p_1 W = (1/2, 1/2)$

$\dots$

Case I $p_0 = (1, 0) \longrightarrow p_1 = p_0 W = (1/2 \;\; 1/2)$

$\longrightarrow p_2 = p_1 W = (3/8, 5/8) \dots$

α = TREMBLING HAND

$0 \le \alpha \le 1$.

At any node, the surfer's hand trembles with a probability $= \alpha$ and he teleports stochastically to a random node $v_j \in V$.

$$G = \alpha \frac{\mathbb{1}\mathbb{1}^T}{n} + (1-\alpha)\frac{I+P'}{2}$$

Let $W = \frac{I+P'}{2}$

$$= \alpha \frac{\mathbb{1}\mathbb{1}^T}{n} + (1-\alpha) W$$

= GOOGLE MATRIX

THEORY OF MARKOV CHAIN

G = Transition Probability Matrix.

G = Stochastic.

= The rows present probability of transitions

$1 \ge p_{ij} \ge 0 \quad \sum_j p_{ij} = 1.$

Row-Sum $= 1$.

If A, B are stochastic, then their convex combination

$\lambda A + (1-\lambda) B \quad (0 \le \lambda \le 1)$

is also stochastic.

I = Identity Matrix
$P = D^{-1}A$ = Prob. Matrix
$a\frac{\mathbb{1}^T}{n}$ = Dangling Vector Matrix
$\frac{\mathbb{1}\mathbb{1}^T}{n}$ = Trembling Hand Matrix

} All are Stochastic.

$$G = \alpha\frac{\mathbb{1}\mathbb{1}^T}{n} + (1-\alpha)/_2\, P' + (1-\alpha)/_2\, I$$

= Convex Combination of Stochastic Matrices $\longrightarrow$ STOCHASTIC.

G = Irreducible.

Every page is connected to every other page
(Trembling Hand Teleportation).

G = Aperiodic.

$\exists_K \quad G^k > 0 \qquad (e.g., G^1 > 0)$

Self-loops ($G_{ii} > 0$) creates aperiodicity in $W = \frac{I + P'}{2}$.

G = Stochastic Primitive Matrix

$$G = (1-\alpha)\frac{I}{2} + (1-\alpha)\frac{P}{2} + \left\{\frac{(1-\alpha)}{2}a + \alpha\mathbb{1}\right\}\frac{\mathbb{1}^T}{n}$$

Diagonal Matrix

Sparse Row Stochastic Matrix

Dense Rank-One Teleportation Matrix.

Iterative Equation (For updating)

$$p^{(k+1)} = p^{(k)} G$$

until $p^* = p^* G$ $\quad p^* = \pi =$ Fixpoi

p^* = Page Rank Vector
= Stationary Row Vector.

$p^*\mathbb{1} = 1.$

Power Method.

For any starting vector applied to a Stochastic Primitive Matrix G, it converges to a unique positive vector

π s.t $\pi G = \pi$

STATIONARY VECTOR.