April 14 2015

AUCTION & HONESTY

- Used by Google, eBay, etc.

- SECOND PRICE AUCTION
  (With complete information)
  ⇒ These assumptions can be further relaxed!

★ PLAYERS. $I = \{1, 2, 3, \dots, n\}$

★ OBJECT. An indivisible object to be assigned to a single player $i \in I$.

- $\forall_{i \in I} \ \exists! \, v_i \in \mathbb{R}$ v_i = Player i's "private" valuation of the object.

- WLOG, assume

$$v_1 > v_2 > \cdots > v_n > 0.$$

- Complete Information Version of Vickery / 2nd Price Auction.
  Assume: Every-one knows everyone else's valuations:

$$V: \{v_1, v_2, \dots, v_n\}$$

NOTE BOOK

AUCTION :

⟨1⟩ Players simultaneously submit bids, b_i, $i \in I$.

$$B = \{b_1, b_2, \dots, b_n\}$$

⟨2⟩ The object is assigned to the $\boxed{\text{highest bidder}}$ (with random tie-breaking)

⟨3⟩ The winner pays the $\boxed{\text{second highest bid}}$

Players : $I = \{1, 2, \dots, n\}$

Strategy Profile: $b_i \in B_i = [0, \infty] = \mathbb{R}_+$

$$B = \prod B_i = \mathbb{R}_+^n$$

Utility Function: $u_i : B \to \mathbb{R}$

$$: \langle b_1, b_2, \dots, b_n \rangle \mapsto \begin{cases} v_i - b_j, & \begin{cases} i = \text{highest bidder} \\ j = 2^{nd} \text{ highest bidder} \end{cases} \\ 0 & \text{o.w.} \end{cases}$$

<u>Nash Equilibrium</u>.

$$\begin{cases} b^* = \langle b_1^*, b_2^*, \dots, b_n^* \rangle \text{ s.t.} \\ \forall_{i \in I} \ \forall_{b_i \in \mathbb{R}_+} u_i(\langle b_i^*, b_{-i}^* \rangle) \geq u_i(\langle b_i, b_{-i}^* \rangle) \end{cases}$$

The game has a <u>truthful</u> $\boxed{\text{HONEST}}$ Nash equilibrium.

$$\langle v_1, v_2, \dots, v_n \rangle = b^* = \text{Nash equilibrium.}$$

HONEST BIDDING

LEMMA: In the second price auction, HONEST BIDDING, i.e.

$b^* = \langle v_1, v_2, \ldots, v_n \rangle$, i.e. $b_i^* = v_i$

is a TRUTHFUL Nash Equilibrium.

Proof:

Player 1 receives the object and pays v_2

$u_1(b^*) = v_1 - v_2$,

$\forall_{j \neq 1}\ u_j(b^*) = 0$.

Note 1 Player 1 has no incentive to deviate, since

a) if $b_1 > v_2$, it has no effect on $u_1(b) = v_1 - v_2$

b) if $b_1 < v_2$, it decreases his payoff, $\because u_1(b) = 0$.

[Rationality]

Note 2. Player $j \neq 1$ has no incentive to deviate

a) if $b_j > v_1$, it decreases his payoff

$u_j(b) = v_j - b_j < 0$.

b) if $b_j < v_1$ it has no effect on $u_j(b) = 0$.

$\Rightarrow$ NO PLAYER HAS ANY INCENTIVE TO DEVIATE! $\square$

INCOMPLETE INFORMATION CASE.

- Two additional Nash equilibria:

$$b'^* = \langle v_1, 0, 0, \ldots, 0 \rangle$$
$$b''^* = \langle v_2, v_1, 0, \ldots 0 \rangle$$

- However, it can be shown that HONEST BIDDING → Results in a WEAKLY DOMINANT NASH EQUILIBRIUM.

SIGNALING GAMES

Two players: $\begin{cases} S = \text{Sender} \\ R = \text{Receiver} \end{cases}$

⟨1⟩ Nature selects a $\boxed{\text{type}}$ t_i from
$$T = \{t_1, t_2, \cdots, t_I\}$$
with probability $p(t_i)$.

⟨2⟩ Sender observes t_i and selects a $\boxed{\text{message}}$ m_j from
$$M = \{m_1, m_2, \ldots, m_J\}$$

⟨3⟩ Receiver observes m_j (but not t_i) and takes an $\boxed{\text{action}}$ a_k from
$$A = \{a_1, \ldots, a_K\}$$

$$\text{Payoffs} = \begin{cases} U_S(t_i, m_j, a_k) \\ U_R(t_i, m_j, a_k) \end{cases}$$

◇ POOLING EQUILIBRIUM

All types of senders send the same message

◇ SEPARATING EQUILIBRIUM

All types of senders send different messages.

◇ COMBINATION

Babbling →

Deception, in these Nash equilibria!

NETFLIX SIGNALLING GAME:

(with additional players

- Verifiers (VERA)
- Recommenders (REKHA).

$$\text{SENDER (User)} \xrightarrow{M} \text{RECEIVER (Netflix)}$$

$D = \{u_1, u_2, .., u_m\}$ identities / unknown preferences

$A = \{v_1, v_2, \dots, v_n\}$ items / movies.

$M \equiv A$.

⟨1⟩ MINIMALLY INFORMATIVE SENDERS.

$D = \{u_1, \ldots, u_m\}$ $u_i \in \mathbb{R}^k$ = unknown features.

$M = A = \{v_1, v_2, \ldots, v_n\}$ $v_j \in \mathbb{R}^k$ = unknown features.

$U_S(u_i, v_j, v_j) = \langle u_i, v_j \rangle \in \mathbb{R}_+$

$U_R(u_i, v_j, v_j) = \text{const} \in \mathbb{R}_+$

(2) NONMARKOVIAN RECOMMENDERS.

Infer $\hat{U}_S : D \times M \times A \to \mathbb{R}_+$

Data-Science Problem: Subsets:
Labeled $\subseteq$ Watched $\subseteq D \times M \times A$
Use Labeled $\oplus$ Watched to statistically infer $\hat{U}_S$

Information Asymmetry $\to$?

Loss Function: $\sum_{\alpha \in \text{Labeled}} \| U_S(\alpha) - \hat{U}_S(\alpha) \|_2$ L_2-Norm.

Action v_j is recommended to u_i
iff $\hat{U}_S(u_i, v_j, v_j) \geq \theta$
and $\langle u_i, v_j, v_j \rangle \notin$ Watched.

⟨3⟩ NON-OBLIVIOUS VERIFIERS

$$\left\{ \begin{array}{l} \text{Function of } \hat{U}_s(\cdot, v_j, v_j) \\ \quad [\text{General Opinion w.r.t. item } v_j] \end{array} \right.$$

$\alpha(u_i, V_j)$ = Pr. that u_i will reject all $v_j \in V_j$.

NETFLIX GAME MATRIX

$$N = \{\perp \cup \mathbb{R}_+\}^{m \times n} \quad m \times n \text{ Matrix}$$

$$N_{i,j} = \begin{cases} \in \mathbb{R}_+ & \text{if } (u_i, v_j, v_j) \text{ is labeled} \\ \perp & \text{if unlabeled.} \end{cases}$$

$$\begin{array}{c|c} & \begin{array}{cccc} v_1 & \cdots & & v_n \end{array} \\ \begin{array}{c} u_1 \\ \vdots \\ \\ \\ u_m \end{array} & \left\{ \begin{array}{cccc} \perp & \perp & \cdots & \perp \\ \perp & 2 & \cdots & 5 \\ 6 & 3 & \cdots & \perp \\ \vdots & \vdots & \ddots & \vdots \\ \perp & 4 & \cdots \perp & \perp \end{array} \right\} \end{array} = N \approx UDV^T = \hat{U}_s$$

$U = \mathbb{R}^{m \times k}$

$V = \mathbb{R}^{n \times k}$

$D = \mathbb{R}^{k \times k} \rightarrow$ Diagonal

UDV^T = $m \times n$ matrix of rank k minimizes a loss-function.

SINGULAR VALUE DECOMPOSITION

N = $m \times n$ matrix

m points in n-dimensional space.

Sender View } ⟨1⟩ Each point represents a user
⟨2⟩ The row vector $\in \mathbb{R}^n$ is his utility function.

Receiver View } There is a different matrix
$$P \in \mathbb{R}^{n \times m}$$
which represents the receiver's view about how much utility each item can extract.

$$N \neq P^T \quad \text{unless} \quad U_S = U_R$$

Item Distance:
$$d_i(v_p, v_q) = \| N(\cdot\, p) - N(\cdot\, q) \|$$

User Distance:
$$d_u(u_p, u_q) = \| P(\cdot\, p) - P(\cdot\, q) \|.$$

Singular Value Decomposition of N

N = Netflix Matrix.

$\tilde{N} = UDV^T$ $\quad \tilde{N}$ = rank-k.

U and V are orthonormal $\quad \langle u_i, u_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$

$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$

D = Diagonal
with positive entries.

$k \ll n \ll m$

We want $\|\tilde{N} - N\| < \epsilon_K$.

$$VD^{-1}U^T \times UDV^T = VD^{-1}DV^T = VV^T = I$$
$$UDV^T \times VD^{-1}U^T = UDD^{-1}U^T = UU^T = I.$$
$$\Rightarrow \quad VD^{-1}U^T = \text{Inverse of } \tilde{N}$$

Columns of U : Left Singular Vectors of $\tilde{N}$
$\in \mathbb{R}^m \Rightarrow$ INDEPENDENT FEATURE SET for U

Columns of V = Right Singular Vectors of $\tilde{N}$
$\in \mathbb{R}^n \Rightarrow$ INDEPENDENT FEATURE SET for V.