Lecture #9

Giant Component.

$G(n, \frac{\lambda}{n})$ - Erdös-Renyi Model.

Two Regimes:

$$p(n) = \frac{\lambda}{n} \begin{cases} \lambda < 1 & \longrightarrow \text{All components of the graph are } \underline{\text{small}}. \\ \text{vs} & \\ \lambda > 1. & \longrightarrow \text{One component of the graph is a unique "giant" component.} \end{cases}$$

A unique Giant Component.

A component that ~~remains~~ connects a constant fraction of individuals in a social network.

Breadth-First Search. { Starting with individual "1".

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BFS(G, v)
  Create a queue Q (:= φ) and a vector V(:= φ);
  Enque(v, Q); Append(v, V);
  while Q ≠ φ loop
     t ← Deque(Q);
     For all e ∈ Adjacent(t, G) loop
         e = (t, u);
         If u ∉ V then
            Enque(u, Q); Append(u, V);
         end if
     end loop
  end loop.
```

S_1 = # nodes in the Erdös-Rényi graph $G(n, \frac{\lambda}{n})$ connected to individual 1.

Note: Expected # of a children for a node
$= n\,p(n) = n\frac{\lambda}{n} = \lambda.$

[Variance $= np(n)(1-p(n)) = \lambda(1-\frac{\lambda}{n})$]

Think of two processes during BFS:

Graph Process → r.v. Z_k^G = # individuals connected to 1. at stage k of the graph.

Branching Process → r.v. Z_k^B = # individual that would be connected to 1 in a pure branching process.

$$Z_k^G < Z_k^B$$ (e.g. due to triadic closure).

$$E[S_1] = \sum_k E[Z_k^G] < \sum_k E[Z_k^B] = \sum_k \lambda^k$$

$$= \begin{cases} \frac{1}{1-\lambda} & \text{if } \lambda < 1. \\ \infty & \text{if } \lambda > 1. \end{cases}$$

THEOREM: Let $p(n) = \frac{\lambda}{n}$ ($\lambda < 1$). Then for all sufficiently large $a > 0$, we have

$$\Pr\left(\max_{1 \le i \le n} |S_i| \ge a \ln n\right) \to 0 \quad \text{as } n \to \infty$$

Proof: Omitted. □.

Assume $\lambda > 1$: Giant component with $p(n) > \frac{1}{n}$.

CLAIM: $Z_K^G \approx Z_K^B$ with $\lambda^k \leq O(\sqrt{n})$. □

CONFLICT: Two of the "friends" at stage k have a common friend at stage k+1.
(TRIADIC CLOSURE)

E [Number of conflicts at stage k+1]

$$< E\left[\binom{Z_k}{2} n\, p^2\right] \leq np^2\, E[Z_k^2]$$

$$= np^2 \left\{ E[Z_k]^2 + Var[Z_k] \right\}$$

$Z_k \sim \text{Poisson}(\lambda^k)$

$E[Z_k] = Var[Z_k] = \lambda^k$

$$= np^2 (\lambda^{2k} + \lambda^k) \leq \frac{\lambda^{2(k+1)}}{n}$$

$\to 0$ if $\lambda^{2k} < n$

or $\lambda^k < \sqrt{n}$.

□

THEOREM:

Let $p(n) = \frac{\lambda}{n}$. $(\lambda > 1)$ in an Erdös-Rényi random graph $G(n, p(n))$. Then there exist some $c > 0$ such that

$$\Pr[\exists \text{ a component of size } > c\sqrt{n} \text{ nodes}] \to 1 \text{ as } n \to \infty.$$

□

Let C_1 and C_2 be two components of size $\sqrt{n}$ or more.

What is the probability of having at least one link connecting them?

$$\Pr[\exists \text{ a link bet'n } C_1 \,\&\, C_2]$$
$$\geq 1 - (1-p(n))^{|C_1| \times |C_2|}$$
$$= 1 - \left(1 - \frac{\lambda}{n}\right)^{c^2 n} = 1 - \left(1 - \frac{\lambda}{n}\right)^{\frac{n}{\lambda} \cdot \lambda c^2}$$
$$= 1 - e^{-c^2 \lambda} > 0$$

A positive constant independent of n.

THEOREM:

Components of size $\leq \sqrt{n}$ connect to each other forming a connected component (GIANT COMPONENT) of size qn for some $0 < q < 1$.

$\square$