$\boxed{\text{LECTURE \# 11}}$ April 22 2014

STRATEGIC FORM GAMES: $\left\{ \begin{array}{l} \text{Normal-Form} \\ \text{or} \\ \text{Matrix Games.} \end{array} \right.$

(A) NON-COOPERATIVE:
All participants act simultaneously and without knowledge of other player's actions. (or Thoughts/intentions)
↳ REVEALATION PRINCIPLE.

(B) RATIONALITY (or. Common knowledge of Rationality).

$$\left.\begin{array}{l} \text{(i) } \mathcal{J}: \text{ The set of palyers.} \\ \text{(ii) } S_{i\in\mathcal{J}}: \text{ The strategies.} \\ \text{(iii) } u_{i\in\mathcal{J}}: \text{ The payoffs.} \end{array}\right\} \text{Select strategies to optimize payoffs.}$$

(C) INFORMATION
(i) The game form: Captures order of play.

(ii) Information Set: Models asymmetric/incomplete information situations.

$$S = \prod_{i\in\mathcal{J}} S_i \;; \quad S_{-i} = \prod_{j\neq i} S_j \Rightarrow \langle s_i, s_{-i}\rangle \in S_i \times S_{-i} = S$$

BEST RESPONSE:
$$\boxed{B_i(s_{-i}) \in \underset{s_i\in S_i}{\operatorname{argmax}}\; u_i(s_i, s_{-i}).}$$

A strategy of $i \in \mathcal{J}$ that maximizes his utility, provided that all other players have selected s_{-i}.

EQUILIBRIUM:

Everyone should ~~s~~ choose their BEST RESPONSES and not deviate from it: Select s^*

$$\boxed{\forall_{i \in \mathcal{I}} \quad B_i(s^*_{-i}) = s^*_i}$$

Example: PARTNERSHIP GAME

Friend$_1$ \ Friend$_2$	Work Hard	Shirk
(WH) Work Hard	$\overset{*}{(2,2)}$	$(-1,1)$
(S) Shirk	$(1,-1)$	$(0,0)$

$B_1(-, WH) = WH$
$B_2(WH, -) = WH$

$B_1(-, S) = S$
$B_2(S, -) = S$.

DOMINANT STRATEGY:

A strategy $s_i \in S_i$ is DOMINANT for player i, if
$$\left\{ \begin{array}{l} \forall_{s_i' \in S_i} \forall_{s_{-i} \in S_{-i}} \\ u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}). \end{array} \right.$$

$s_i \in S_i$ is STRICTLY DOMINATED
$$\left\{ \begin{array}{l} \exists s_i' \in S_i \ \forall_{s_{-i} \in S_{-i}} \\ u(s_i', s_{-i}) > u_i(s_i, s_{-i}) \end{array} \right.$$

$s_i \in S_i$ is WEAKLY DOMINATED
$$\left\{ \begin{array}{l} \exists s_i' \in S_i \\ \forall_{s_{-i} \in S_{-i}} \ u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \\ \wedge \\ \exists_{s_{-i} \in S_{-i}} \ u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \end{array} \right.$$

Iterated Elimination of Strictly Dominated Strategies

$j \leftarrow 0;$

For $\forall i \in \mathcal{J}$ $S_i^0 \leftarrow S_i;$

$j \leftarrow j+1.$

Loop

For $\forall i \in \mathcal{J}$

$$S_i^j \leftarrow \{ s_i \in S_i^{j-1} \mid \nexists s_i' \in S_i^{j-1} \; \forall s_{-i} \in S_{-i}^{j-1} \; u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \}$$

$$S_i^\infty \leftarrow \bigcap_{k=0}^{\infty} S_i^k;$$

$$S^\infty \leftarrow \prod_i S_i^\infty.$$

Dominant Strategy Equilibrium:

A strategy profile s^* is the dominant strategy equilibrium if for each player $i \in \mathcal{J}$

s_i^* = A dominant strategy.

Example

(Extended) Prisoner's Dilemma:

Prisoner 1 \ Prisoner 2	Confess	Silence	Suicide
Confess	(-2, -2)	(0, -3)	(-2, -10)
Silence	(-3, 0)	(-1, -1)	(0, -10)
Suicide	(-10, -2)	(-10, 0)	(-10, -10)

(A) Suicide is dominated for both players:
→ Eliminate Suicide.

(B) Next, Silence is dominated for both players:
→ Eliminate Silence

$S^{\infty} = \{(\text{Confess}, \text{Confess})\}$

DOMINANT STRATEGY EQUILIBRIUM.

(Follows from CKR).

Example / Counter Example.

Rock-Paper-Scissors.

P_1 \ P_2	R	P	S
R	(0,0)	(-1,1)	(1,-1)
P	(1,-1)	(0,0)	(-1,1)
S	(-1,1)	(1,-1)	(0,0)

← Zero-Sum Game.

Doesn't have a PURE-STRATEGY NASH EQUILIBRIUM.

MIXED STRATEGIES : (NASH).

Σ_i = Probability measures over the Pure Strategies S_i (for player i).

$\sigma_i \in \Sigma_i$ = Mixed Strategy of player i.

$S_i = \{ s_{i_1}, s_{i_2}, \dots, s_{i_K} \}$ ← i's strategies

$\sigma_i = (p_{i_1}, p_{i_2}, \dots, p_{i_K})$ ← corresponding probabilities.

$p_{ij} = \Pr[s_{i_j} \in S_i \text{ is played}]$ $\begin{cases} p_{ij} \geq 0 \\ \sum p_{ij} = 1. \end{cases}$

$\Sigma = \prod \Sigma_i$ = Mixed Strategy Profiles. {Players randomize independently.

$\sigma \in \Sigma$.

$$u_i(\sigma) = \int_S u_i(s)\, d\sigma(s).$$
$$= \sum p_{ij}\, u_i(s_{ij}, \sigma_{-i})$$

MIXED STRATEGY NASH EQUILIBRIUM.

$\sigma^* \in \Sigma$ = Mixed strategy profile.

σ^* = Mixed Strategy Nash Equilibrium, if

$$\forall_{i \in \mathcal{J}}\ \forall_{\sigma_i \in \Sigma_i} \quad u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*).$$ □

Example
Matching Penny.

Mismatcher \ Matcher	H	T
H	(-1,1)	(1,-1)
T	(1,-1)	(-1,1)

← Unique MS NE $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$

NASH'S THEOREM.

Every finite game has a mixed strategy NASH Eq. □.

KAKUTANI'S FIXED POINT THEOREM.

Let $f: A \rightrightarrows A : x \in A \mapsto f(x) \subseteq A$ be a CORRESPONDENCE satisfying the following conditions:

(1) A = Compact, convex and nonempty
(subset of finite dimensional Euclidean space $\mathbb{R}^d$)

(2) $\forall x \in A \ f(x) \neq \phi$; (3) $\forall_{x \in A}\ f(x)$ = convex

(4) $f(x)$ has a closed graph: $\{x_n, y_n\} \to \{x, y\}$ with $y_n \in f(x_n)$
$\Rightarrow y \in f(x)$.

Then $\exists_{x^* \in A} \quad x^* \in f(x^*)$

f has a fixed point $= x^*$.

□.

Corollary. MS NE = Fixed point f = Best Response.

SIGNALING GAMES.

Two players: $\begin{cases} S = \text{Sender} \\ R = \text{Receiver} \end{cases}$

1) Nature selects a <u>type</u> t_i from $T = \{t_1, \ldots t_I\}$ with prob $p(t_i)$
2) Sender observes t_i and chooses a <u>message</u> m_j from $M = \{m_1, \ldots, m_J\}$
3) Receiver observes m_j (but not t_i) and takes an <u>action</u> a_k from $A = \{a_1, \ldots, a_K\}$

$$\text{Payoffs} = \begin{cases} U_S(t_i, m_j, a_k) \\ U_R(t_i, m_j, a_k) \end{cases}$$

Signaling games have Nash equilibria:

- POOLING EQUILIBRIUM: All types of sender send the same message.
- SEPARATING EQUILIBRIUM: All types of send send different messages.

Combination / Babbling.